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A novel accelerated extragradient algorithm to solve pseudomonotone variational inequalities

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Abstract In this paper, we propose a new inertial iterative method to solve classical variational inequalities with pseudomonotone and Lipschitz continuous operators in the setting of a real Hilbert space. The proposed iterative scheme is basically analogous to the extragradient method used to solve the problems of variational inequalities in real Hilbert spaces. The strong convergence of the proposed algorithm is set up with the prior knowledge of Lipschitz's constant of an operator. Finally, several computational experiments are listed to show the applicability and efficiency of the proposed algorithm.

Mathematics Subject Classification 65Y05 · 65K15 · 68W10 · 47H05 · 47H10

1 Introduction

This article studies the iterative method that is used to estimate the solution of *variational inequality problem* (shortly, VIP) in the setting of a real Hilbert space. Let $\mathcal{X}$ be a real Hilbert space and $\mathcal{D}$ be a non-empty, closed, and convex subset of $\mathcal{X}$. Let $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ to be an operator. The problem (VIP) for $\mathcal{S}$ on $\mathcal{D}$ is given as follows [15,22]:

$$\text{Find } u^* \in \mathcal{D} \text{ such that } \left\langle \mathcal{S}(u^*), y - u^* \right\rangle \geq 0, \quad \forall\, y \in \mathcal{D}. \tag{VIP}$$

Let us consider that Π is the solution set of the problem (VIP). This idea of variational inequalities includes different disciplines such as partial differential equations, optimization, optimal control, mechanics, mathematical programming, and finance (see [6,11–14,18,24]). This problem is an important topic in the physical

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sciences, and a considerable amount of discussion has been given to many authors who have dedicated themselves to studying not only the theory of existence and the stability of solutions but also the iterative method used to solve the problem.

Korpelevich [16] and Antipin [2] have established the following extragradient method. Their method consists of the following: Let $u_0 \in \mathcal{D}$ and $0 < \tau < \frac{1}{L}$ such that

$$\begin{cases} y_n = P_{\mathcal{D}}[u_n - \tau \mathcal{S}(u_n)], \\ u_{n+1} = P_{\mathcal{D}}[u_n - \tau \mathcal{S}(y_n)]. \end{cases} \tag{1}$$

On the other hand, many projection methods are used to figure out the numerical solution of variational inequalities. Many researchers have suggested various forms of projection techniques to solve the problem (VIP) (see for details [10,16,19,25,26,28–32,34,36]). Almost all the methods for solving the problem (VIP) are based on the projection method, which is computed on the feasible set $\mathcal{D}$. It is important to note that the above well-established method has two significant flaws, the first being the fixed constant step size, which involves the knowledge or approximation of the Lipschitz constant of the respective operator and is only weakly convergent in the Hilbert spaces. From the computational point of view, it might be problematic to use a fixed step size, and hence the convergence rate and appropriateness of the method could be affected.

The main contribution of this study is to develop an inertial-type method used to improve the convergence rate of the sequence. Previously, such approaches have been developed based on the oscillator equation with damping and conservative force restoration. This second-order dynamical structure is called a strong friction ball and was originally studied by Polyak in [20]. Primarily, the functionality of the inertial-type method is that it will use the two prior iterations to execute the next iteration.

Therefore, a natural question is raised:

"Is it possible to introduce new strongly convergent inertial extragradient-like method to solve the problem (VIP)"?

In this study, we give a positive answer to the above question, i.e., the gradient method indeed generates a strongly convergent iterative sequence by letting a fixed and variable step size rule. In this paper, we study a different method to obtain strong convergence and introduce a new iterative method for solving variational inequalities involving pseudomonotone and the Lipschitz operator in a real Hilbert space. Our method is inspired by one projection method [16] and the inertial technique in [20]. At each iteration, the method only needs to compute one projection onto the feasible set. Under some suitable conditions imposed on control parameters, the iterative sequences generated by our method converge strongly to some solution of the considered problem. We also provide numerical examples to illustrate the computational effectiveness of the new method over some existing methods.

The paper is organized in the following way. In Sect. 2, we review some concepts and preliminary results used in the paper. Section 3 deals with the description of the method and proves its convergence theorems. Finally, Sect. 4 presents some numerical results to illustrate the convergence and effectiveness of the proposed method.

2 Background

In this section of the manuscript, we have written a number of important identities and relevant lemmas and definitions.

For all $u, y \in \mathcal{X}$, we have

$$\|u + y\|^2 = \|u\|^2 + 2\langle u, y\rangle + \|y\|^2.$$

A *metric projection* $P_{\mathcal{D}}(y_1)$ of $y_1 \in \mathcal{X}$ is defined by

$$P_{\mathcal{D}}(y_1) = \arg\min\{\|y_1 - y_2\| : y_2 \in \mathcal{D}\}.$$

First, we list some of the important features of projection operator.

Lemma 2.1 [3] *Suppose that $P_{\mathcal{D}} : \mathcal{X} \to \mathcal{D}$ is a metric projection. Then, the following conditions were satisfied.*

(i) $y_3 = P_{\mathcal{D}}(y_1)$ *if and only if*

$$\langle y_1 - y_3, y_2 - y_3 \rangle \leq 0, \quad \forall\, y_2 \in \mathcal{D}.$$

(ii)

$$\|y_1 - P_{\mathcal{D}}(y_2)\|^2 + \|P_{\mathcal{D}}(y_2) - y_2\|^2 \leq \|y_1 - y_2\|^2, \quad y_1 \in \mathcal{D}, y_2 \in \mathcal{X}.$$

(iii)

$$\|y_1 - P_{\mathcal{D}}(y_1)\| \leq \|y_1 - y_2\|, \quad y_2 \in \mathcal{D}, y_1 \in \mathcal{X}.$$

Lemma 2.2 [35] *Assume that* $\{p_n\} \subset [0, +\infty)$ *is a sequence satisfies the following inequality:*

$$p_{n+1} \leq (1 - q_n) p_n + q_n r_n, \quad \forall\, n \in \mathbb{N}.$$

Furthermore, $\{q_n\} \subset (0, 1)$ *and* $\{r_n\} \subset \mathbb{R}$ *be two sequences such that*

$$\lim_{n \to +\infty} q_n = 0, \quad \sum_{n=1}^{+\infty} q_n = +\infty \quad \text{and} \quad \limsup_{n \to +\infty} r_n \leq 0.$$

Then, $\lim_{n \to +\infty} p_n = 0$.

Lemma 2.3 [17] *Assume that* $\{p_n\}$ *is a sequence of real numbers such that there exists a subsequence* $\{n_i\}$ *of* $\{n\}$ *such that*

$$p_{n_i} < p_{n_{i+1}} \quad \forall\, i \in \mathbb{N}.$$

Then, there is a non decreasing sequence $m_k \subset \mathbb{N}$ *such that* $m_k \to +\infty$ *as* $k \to +\infty$, *and meet the following requirements for numbers* $k \in \mathbb{N}$:

$$p_{m_k} \leq p_{m_{k+1}} \quad \text{and} \quad p_k \leq p_{m_{k+1}}.$$

Indeed, $m_k = \max\{j \leq k : p_j \leq p_{j+1}\}$.

Next, we list some of the important identities that were used to prove the convergence analysis.

Lemma 2.4 [3] *For any* $y_1, y_2 \in \mathcal{X}$ *and* $\ell \in \mathbb{R}$. *Then, the following inequalities hold:*

(i)

$$\|\ell y_1 + (1 - \ell) y_2\|^2 = \ell \|y_1\|^2 + (1 - \ell)\|y_2\|^2 - \ell(1 - \ell)\|y_1 - y_2\|^2.$$

(ii)

$$\|y_1 + y_2\|^2 \leq \|y_1\|^2 + 2\langle y_2, y_1 + y_2 \rangle.$$

Lemma 2.5 [23] *Assume that* $\mathcal{S} : \mathcal{D} \to \mathcal{X}$ *is a pseudomonotone and continuous operator. Then,* u^* *is a solution to the problem* (VIP) *if and only if* u^* *is a solution to the following problem:*

$$\text{Find } u \in \mathcal{D} \text{ such that } \langle \mathcal{S}(y), y - u \rangle \geq 0, \quad \forall\, y \in \mathcal{D}.$$

Algorithm 1 (Inertial Strongly Convergent Iterative Method)

Step 0: Select $u_0, u_1 \in \mathcal{D}$, $\alpha > 0$ and $0 < \tau < \frac{1}{L}$. Moreover, choose $\{\psi_n\} \subset (0, 1)$ satisfies the following conditions:

$$\lim_{n\to+\infty} \psi_n = 0 \text{ and } \sum_{n=1}^{+\infty} \psi_n = +\infty.$$

Step 1: Compute

$$t_n = u_n + \alpha_n(u_n - u_{n-1}) - \psi_n\big[u_n + \alpha_n(u_n - u_{n-1})\big],$$

where α_n such that

$$0 \le \alpha_n \le \hat{\alpha_n} \quad \text{and} \quad \hat{\alpha_n} = \begin{cases} \min\left\{\frac{\alpha}{2}, \frac{\epsilon_n}{\|u_n - u_{n-1}\|}\right\} & \text{if} \quad u_n \neq u_{n-1}, \\ \frac{\alpha}{2} & \text{else,} \end{cases} \tag{2}$$

where $\epsilon_n = \circ(\psi_n)$ is a positive sequence, i.e., $\lim_{n\to+\infty} \frac{\epsilon_n}{\psi_n} = 0$.

Step 2: Compute

$$y_n = P_{\mathcal{D}}(t_n - \tau\mathcal{S}(t_n)).$$

If $t_n = y_n$, then STOP and y_n is a solution. Otherwise, go to next step.

Step 3: Compute

$$u_{n+1} = P_{\mathcal{D}}(t_n - \tau\mathcal{S}(y_n)).$$

Set $n = n + 1$ and go back to **Step 1**.

3 Main results

To investigate the convergence analysis, it is considered that the following requirements are satisfied:
$(\mathcal{S}1)$ A solution set of problem (VIP) is denoted by Π and it is non-empty;
$(\mathcal{S}2)$ An operator $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ is said to be pseudomonotone if

$$\big\langle \mathcal{S}(y_1), y_2 - y_1 \big\rangle \ge 0 \Longrightarrow \big\langle \mathcal{S}(y_2), y_1 - y_2 \big\rangle \le 0, \quad \forall\, y_1, y_2 \in \mathcal{D};$$

$(\mathcal{S}3)$ An operator $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ is said to be *Lipschitz continuous* with constant $L > 0$ if there exists $L > 0$ such that

$$\|\mathcal{S}(y_1) - \mathcal{S}(y_2)\| \le L\|y_1 - y_2\|, \quad \forall\, y_1, y_2 \in \mathcal{D};$$

$(\mathcal{S}4)$ An operator $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ is said to be *sequentially weakly continuous* if $\{\mathcal{S}(u_n)\}$ converges weakly to $\mathcal{S}(u)$ for every sequence $\{u_n\}$ converges weakly to u.
The main algorithm is given the following form:

Lemma 3.1 *Assume that* $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ *satisfies the conditions* $(\mathcal{S}1)$*–*$(\mathcal{S}4)$*. For a given* $u^* \in \Pi \neq \emptyset$*, we have*

$$\|u_{n+1} - u^*\|^2 \le \|t_n - u^*\|^2 - (1 - \tau L)\|t_n - y_n\|^2 - (1 - \tau L)\|u_{n+1} - y_n\|^2.$$

Proof First consider the following

$$\begin{aligned} \big\|u_{n+1} - u^*\big\|^2 &= \big\|P_{\mathcal{D}}[t_n - \tau\mathcal{S}(y_n)] - u^*\big\|^2 \\ &= \big\|P_{\mathcal{D}}[t_n - \tau\mathcal{S}(y_n)] + [t_n - \tau\mathcal{S}(y_n)] - [t_n - \tau\mathcal{S}(y_n)] - u^*\big\|^2 \\ &= \big\|[t_n - \tau\mathcal{S}(y_n)] - u^*\big\|^2 + \big\|P_{\mathcal{D}}[t_n - \tau\mathcal{S}(y_n)] - [t_n - \tau\mathcal{S}(y_n)]\big\|^2 \\ &\quad + 2\big\langle P_{\mathcal{D}}[t_n - \tau\mathcal{S}(y_n)] - [t_n - \tau\mathcal{S}(y_n)], [t_n - \tau\mathcal{S}(y_n)] - u^*\big\rangle. \end{aligned} \tag{3}$$

It is given that $u^* \in \Pi \subset \mathcal{D}$ such that

$$\begin{aligned} &\big\|P_{\mathcal{D}}[t_n - \tau\mathcal{S}(y_n)] - [t_n - \tau\mathcal{S}(y_n)]\big\|^2 \\ &\quad + \big\langle P_{\mathcal{D}}[t_n - \tau\mathcal{S}(y_n)] - [t_n - \tau\mathcal{S}(y_n)], [t_n - \tau\mathcal{S}(y_n)] - u^*\big\rangle \\ &= \big\langle [t_n - \tau\mathcal{S}(y_n)] - P_{\mathcal{D}}[t_n - \tau\mathcal{S}(y_n)], u^* - P_{\mathcal{D}}[t_n - \tau\mathcal{S}(y_n)]\big\rangle \le 0, \end{aligned} \tag{4}$$

which implies that

$$\begin{aligned}&\big\langle P_{\mathcal{D}}[t_n-\tau\mathcal{S}(y_n)]-[t_n-\tau\mathcal{S}(y_n)],[t_n-\tau\mathcal{S}(y_n)]-u^*\big\rangle\\&\quad\le-\big\|P_{\mathcal{D}}[t_n-\tau\mathcal{S}(y_n)]-[t_n-\tau\mathcal{S}(y_n)]\big\|^2.\end{aligned}\tag{5}$$

By the use of expressions (3) and (5), we obtain

$$\begin{aligned}\|u_{n+1}-u^*\|^2&\le\big\|t_n-\tau\mathcal{S}(y_n)-u^*\big\|^2-\big\|P_{\mathcal{D}}[t_n-\tau\mathcal{S}(y_n)]-[t_n-\tau\mathcal{S}(y_n)]\big\|^2\\&\le\|t_n-u^*\|^2-\|t_n-u_{n+1}\|^2+2\tau\big\langle\mathcal{S}(y_n),u^*-u_{n+1}\big\rangle.\end{aligned}\tag{6}$$

By given that u^* is the solution of problem (VIP), we have

$$\langle\mathcal{S}(u^*),y-u^*\rangle\ge0,\quad\text{for all }y\in\mathcal{D}.$$

Due to the pseudomonotonicity of $\mathcal{S}$ on $\mathcal{D}$, we get

$$\langle\mathcal{S}(y),y-u^*\rangle\ge0,\quad\text{for all }y\in\mathcal{D}.$$

By substituting $y=y_n\in\mathcal{D}$, we get

$$\langle\mathcal{S}(y_n),y_n-u^*\rangle\ge0.$$

Thus, we have

$$\big\langle\mathcal{S}(y_n),u^*-u_{n+1}\big\rangle=\big\langle\mathcal{S}(y_n),u^*-y_n\big\rangle+\big\langle\mathcal{S}(y_n),y_n-u_{n+1}\big\rangle\le\big\langle\mathcal{S}(y_n),y_n-u_{n+1}\big\rangle.\tag{7}$$

By the use of expressions (6) and (7), we obtain

$$\begin{aligned}\|u_{n+1}-u^*\|^2&\le\|t_n-u^*\|^2-\|t_n-u_{n+1}\|^2+2\tau\big\langle\mathcal{S}(y_n),y_n-u_{n+1}\big\rangle\\&\le\|t_n-u^*\|^2-\|t_n-y_n+y_n-u_{n+1}\|^2+2\tau\big\langle\mathcal{S}(y_n),y_n-u_{n+1}\big\rangle\\&\le\|t_n-u^*\|^2-\|t_n-y_n\|^2-\|y_n-u_{n+1}\|^2+2\big\langle t_n-\tau\mathcal{S}(y_n)-y_n,u_{n+1}-y_n\big\rangle.\end{aligned}\tag{8}$$

It is given that $u_{n+1}=P_{\mathcal{D}}[t_n-\tau\mathcal{S}(y_n)]$, we have

$$\begin{aligned}&2\big\langle t_n-\tau\mathcal{S}(y_n)-y_n,u_{n+1}-y_n\big\rangle\\&\quad=2\big\langle t_n-\tau\mathcal{S}(t_n)-y_n,u_{n+1}-y_n\big\rangle+2\tau\big\langle\mathcal{S}(t_n)-\mathcal{S}(y_n),u_{n+1}-y_n\big\rangle\\&\quad\le2\tau L\|t_n-y_n\|\|u_{n+1}-y_n\|\le\tau L\|t_n-y_n\|^2+\tau L\|u_{n+1}-y_n\|^2.\end{aligned}\tag{9}$$

Combining expressions (8) and (9), we obtain

$$\|u_{n+1}-u^*\|^2\le\|t_n-u^*\|^2-(1-\tau L)\|t_n-y_n\|^2-(1-\tau L)\|u_{n+1}-y_n\|^2.\tag{10}$$

□

Theorem 3.2 *Let $\{u_n\}$ be a sequence generated by Algorithm 1 and satisfies the conditions ($\mathcal{S}1$)-($\mathcal{S}4$). Moreover, choose $\{\psi_n\}\subset(0,1)$ meet the conditions, i.e.,*

$$\lim_{n\to+\infty}\psi_n=0\quad\textit{and}\quad\sum_{n=1}^{+\infty}\psi_n=+\infty.$$

Then, $\{u_n\}$ strongly converges to $u^*\in\Pi$. Moreover, $P_{\Pi}(0)=u^*$.

Proof It is given in expression (2) that

$$\lim_{n\to+\infty}\frac{\alpha_n}{\psi_n}\left\|u_n-u_{n-1}\right\| \le \lim_{n\to+\infty}\frac{\epsilon_n}{\psi_n}\left\|u_n-u_{n-1}\right\| = 0. \tag{11}$$

By the use of definition of $\{t_n\}$ and inequality (11), we obtain

$$\begin{aligned}\left\|t_n-u^*\right\| &= \left\|u_n+\alpha_n(u_n-u_{n-1})-\psi_n u_n-\alpha_n\psi_n(u_n-u_{n-1})-u^*\right\|\\ &= \left\|(1-\psi_n)(u_n-u^*)+(1-\psi_n)\alpha_n(u_n-u_{n-1})-\psi_n u^*\right\| \qquad (12)\\ &\le (1-\psi_n)\left\|u_n-u^*\right\|+(1-\psi_n)\alpha_n\left\|u_n-u_{n-1}\right\|+\psi_n\left\|u^*\right\|\\ &\le (1-\psi_n)\|u_n-u^*\|+\psi_n M_1, \qquad (13)\end{aligned}$$

where

$$(1-\psi_n)\frac{\alpha_n}{\psi_n}\left\|u_n-u_{n-1}\right\|+\left\|u^*\right\| \le M_1.$$

By the use of Lemma 3.1, we obtain

$$\|u_{n+1}-u^*\|^2 \le \|t_n-u^*\|^2, \quad \forall n\in\mathbb{N}. \tag{14}$$

Combining (13) with (14), we obtain

$$\begin{aligned}\|u_{n+1}-u^*\| &\le (1-\psi_n)\|u_n-u^*\|+\psi_n M_1\\ &\le \max\left\{\|u_n-u^*\|, M_1\right\}\\ &\;\;\vdots\\ &\le \max\left\{\|u_0-u^*\|, M_1\right\}. \qquad (15)\end{aligned}$$

Thus, we conclude that the $\{u_n\}$ is bounded sequence. Indeed, by (13) we have

$$\begin{aligned}\left\|t_n-u^*\right\|^2 &\le (1-\psi_n)^2\|u_n-u^*\|^2+\psi_n^2M_1^2+2M_1\psi_n(1-\psi_n)\|u_n-u^*\|\\ &\le \|u_n-u^*\|^2+\psi_n\left[\psi_n M_1^2+2M_1(1-\psi_n)\|u_n-u^*\|\right]\\ &\le \|u_n-u^*\|^2+\psi_n M_2, \qquad (16)\end{aligned}$$

for some $M_2>0$. Combining the expressions (10) with (16), we have

$$\begin{aligned}\|u_{n+1}-u^*\|^2 &\le \|u_n-u^*\|^2+\psi_n M_2\\ &\quad -(1-\tau L)\|t_n-y_n\|^2-(1-\tau L)\|u_{n+1}-y_n\|^2. \qquad (17)\end{aligned}$$

Due to the Lipschitz-continuity and pseudomonotonicity of $\mathcal{S}$ implies that the solution set Π is a closed and convex set. It is given that $u^*=P_\Pi(0)$ and using Lemma 2.1(ii), we have

$$\langle 0-u^*, y-u^*\rangle \le 0, \quad \forall y\in\Pi. \tag{18}$$

The remainder of the facts shall be split into the following two parts:
Case 1: Now consider that a fixed number $N_1\in\mathbb{N}$ such that

$$\|u_{n+1}-u^*\| \le \|u_n-u^*\|, \quad \forall n\ge N_1. \tag{19}$$

Thus, above implies that $\lim_{n\to+\infty}\|u_n-u^*\|$ exists and let $\lim_{n\to+\infty}\|u_n-u^*\|=l$, for some $l\ge 0$. From the expression (17), we have

$$\begin{aligned}&(1-\tau L)\|t_n-y_n\|^2+(1-\tau L)\|u_{n+1}-y_n\|^2\\ &\quad\le \|u_n-u^*\|^2+\psi_n M_2-\|u_{n+1}-u^*\|^2. \qquad (20)\end{aligned}$$

Due to existence of a limit of sequence $\|u_n-u^*\|$ and $\psi_n\to 0$, we infer that

$$\|t_n-y_n\|\to 0 \quad \text{and} \quad \|u_{n+1}-y_n\|\to 0 \quad \text{as } n\to+\infty. \tag{21}$$

By the use of expression (21), we have

$$\lim_{n\to+\infty}\|t_n-u_{n+1}\|\leq\lim_{n\to+\infty}\|t_n-y_n\|+\lim_{n\to+\infty}\|y_n-u_{n+1}\|=0. \tag{22}$$

Next, we will evaluate

$$\begin{aligned}\|t_n-u_n\|&=\|u_n+\alpha_n(u_n-u_{n-1})-\psi_n\big[u_n+\alpha_n(u_n-u_{n-1})\big]-u_n\|\\&\leq\alpha_n\|u_n-u_{n-1}\|+\psi_n\|u_n\|+\alpha_n\psi_n\|u_n-u_{n-1}\|\\&=\psi_n\frac{\alpha_n}{\psi_n}\|u_n-u_{n-1}\|+\psi_n\|u_n\|+\psi_n^2\frac{\alpha_n}{\psi_n}\|u_n-u_{n-1}\|\longrightarrow 0.\end{aligned} \tag{23}$$

The above provides that

$$\lim_{n\to+\infty}\|u_n-u_{n+1}\|\leq\lim_{n\to+\infty}\|u_n-t_n\|+\lim_{n\to+\infty}\|t_n-u_{n+1}\|=0. \tag{24}$$

The above explanation guarantees that the sequences $\{t_n\}$ and $\{y_n\}$ are also bounded. By the use of reflexivity of $\mathcal{X}$ and the boundedness of $\{u_n\}$ guarantees that there exits a subsequence $\{u_{n_k}\}$ in order that $\{u_{n_k}\}\rightharpoonup\hat{u}\in\mathcal{X}$ as $k\to+\infty$. Next, we have to prove that $\hat{u}\in\Pi$. This is provided that $y_{n_k}=P_{\mathcal{D}}[t_{n_k}-\tau\mathcal{S}(t_{n_k})]$ that is equivalent to

$$\langle t_{n_k}-\tau\mathcal{S}(t_{n_k})-y_{n_k},\,y-y_{n_k}\rangle\leq 0,\quad\forall\,y\in\mathcal{D}. \tag{25}$$

The above inequality implies that

$$\langle t_{n_k}-y_{n_k},\,y-y_{n_k}\rangle\leq\tau\langle\mathcal{S}(t_{n_k}),\,y-y_{n_k}\rangle,\quad\forall\,y\in\mathcal{D}. \tag{26}$$

Thus, we shall obtain

$$\frac{1}{\tau}\langle t_{n_k}-y_{n_k},\,y-y_{n_k}\rangle+\langle\mathcal{S}(t_{n_k}),\,y_{n_k}-t_{n_k}\rangle\leq\langle\mathcal{S}(t_{n_k}),\,y-t_{n_k}\rangle,\quad\forall\,y\in\mathcal{D}. \tag{27}$$

Due to boundedness of the sequence $\{t_{n_k}\}$ implies that $\{\mathcal{S}(t_{n_k})\}$ is also bounded. By the use of $\lim_{k\to\infty}\|t_{n_k}-y_{n_k}\|=0$ and $k\to\infty$ in (27), we obtain

$$\liminf_{k\to\infty}\langle\mathcal{S}(t_{n_k}),\,y-t_{n_k}\rangle\geq 0,\quad\forall\,y\in\mathcal{D}. \tag{28}$$

Moreover, we have

$$\begin{aligned}&\langle\mathcal{S}(y_{n_k}),\,y-y_{n_k}\rangle\\&\quad=\langle\mathcal{S}(y_{n_k})-\mathcal{S}(t_{n_k}),\,y-t_{n_k}\rangle+\langle\mathcal{S}(t_{n_k}),\,y-t_{n_k}\rangle+\langle\mathcal{S}(y_{n_k}),\,t_{n_k}-y_{n_k}\rangle.\end{aligned} \tag{29}$$

Since $\lim_{k\to\infty}\|t_{n_k}-y_{n_k}\|=0$ and $\mathcal{S}$ is L-Lipschitz continuous on $\mathcal{X}$ implies that

$$\lim_{k\to\infty}\|\mathcal{S}(t_{n_k})-\mathcal{S}(y_{n_k})\|=0. \tag{30}$$

which together with (29) and (30), we obtain

$$\liminf_{k\to\infty}\langle\mathcal{S}(y_{n_k}),\,y-y_{n_k}\rangle\geq 0,\quad\forall\,y\in\mathcal{D}. \tag{31}$$

Let consider a sequence of positive numbers $\{\epsilon_k\}$ that is decreasing and converge to zero. For each k, we denote m_k by the smallest positive integer such that

$$\langle\mathcal{S}(t_{n_i}),\,y-t_{n_i}\rangle+\epsilon_k\geq 0,\quad\forall\,i\geq m_k. \tag{32}$$

Due to $\{\epsilon_k\}$ is decreasing and $\{m_k\}$ is increasing.
Case I: If there is a $t_{n_{m_{k_j}}}$ subsequence of $t_{n_{m_k}}$ such that $\mathcal{S}(t_{n_{m_{k_j}}})=0$ $(\forall j)$. Let $j\to\infty$, we obtain

$$\langle\mathcal{S}(\hat{u}),\,y-\hat{u}\rangle=\lim_{j\to\infty}\langle\mathcal{S}(t_{n_{m_{k_j}}}),\,y-\hat{u}\rangle=0. \tag{33}$$

Hence $\hat{u}\in\mathcal{D}$, therefore we obtain $\hat{u}\in\Pi$.

Case II: If there exits $N_0 \in \mathbb{N}$ such that for all $n_{m_k} \geq N_0$, $\mathcal{S}(t_{n_{m_k}}) \neq 0$. Consider that

$$\Xi_{n_{m_k}} = \frac{\mathcal{S}(t_{n_{m_k}})}{\|\mathcal{S}(t_{n_{m_k}})\|^2}, \quad \forall\, n_{m_k} \geq N_0. \tag{34}$$

Due to the above definition, we obtain

$$\langle \mathcal{S}(t_{n_{m_k}}), \Xi_{n_{m_k}} \rangle = 1, \quad \forall\, n_{m_k} \geq N_0. \tag{35}$$

Moreover, expressions (32) and (35), for all $n_{m_k} \geq N_0$, we have

$$\langle \mathcal{S}(t_{n_{m_k}}),\ y + \epsilon_k \Xi_{n_{m_k}} - t_{n_{m_k}} \rangle \geq 0. \tag{36}$$

Due to the pseudomonotonicity of $\mathcal{S}$ for $n_{m_k} \geq N_0$,

$$\langle \mathcal{S}(y + \epsilon_k \Xi_{n_{m_k}}),\ y + \epsilon_k \Xi_{n_{m_k}} - t_{n_{m_k}} \rangle \geq 0. \tag{37}$$

For all $n_{m_k} \geq N_0$, we have

$$\langle \mathcal{S}(y), y - t_{n_{m_k}} \rangle \geq \langle \mathcal{S}(y) - \mathcal{S}(y + \epsilon_k \Xi_{n_{m_k}}),\ y + \epsilon_k \Xi_{n_{m_k}} - t_{n_{m_k}} \rangle - \epsilon_k \langle \mathcal{S}(y), \Xi_{n_{m_k}} \rangle. \tag{38}$$

Due to $\{t_{n_k}\}$ weakly converges to $\hat{u} \in \mathcal{D}$ through $\mathcal{S}$ is sequentially weakly continuous on the set $\mathcal{D}$, we get $\{\mathcal{S}(t_{n_k})\}$ weakly converges to $\mathcal{S}(\hat{u})$. Suppose that $\mathcal{S}(\hat{u}) \neq 0$, we have

$$\|\mathcal{S}(\hat{u})\| \leq \liminf_{k \to \infty} \|\mathcal{S}(t_{n_k})\|. \tag{39}$$

Since $\{t_{n_{m_k}}\} \subset \{t_{n_k}\}$ and $\lim_{k\to\infty} \epsilon_k = 0$, we have

$$0 \leq \lim_{k \to \infty} \|\epsilon_k \Xi_{n_{m_k}}\| = \lim_{k \to \infty} \frac{\epsilon_k}{\|\mathcal{S}(t_{n_{m_k}})\|} \leq \frac{0}{\|\mathcal{S}(\hat{u})\|} = 0. \tag{40}$$

Next, consider $k \to \infty$ in (38), we obtain

$$\langle \mathcal{S}(y), y - \hat{u} \rangle \geq 0, \quad \forall\, y \in \mathcal{D}. \tag{41}$$

By the use of Minty Lemma 2.5, we infer $\hat{u} \in \Pi$. Next, we have

$$\limsup_{n \to +\infty} \langle u^*, u^* - u_n \rangle = \lim_{k \to +\infty} \langle u^*, u^* - u_{n_k} \rangle = \langle u^*, u^* - \hat{u} \rangle \leq 0. \tag{42}$$

By the use of $\lim_{n\to+\infty} \|u_{n+1} - u_n\| = 0$. Therefore, (42) implies that

$$\begin{aligned} &\limsup_{n \to +\infty} \langle u^*, u^* - u_{n+1} \rangle \\ &\quad \leq \limsup_{n \to +\infty} \langle u^*, u^* - u_n \rangle + \limsup_{n \to +\infty} \langle u^*, u_n - u_{n+1} \rangle \leq 0. \end{aligned} \tag{43}$$

Consider the expression (12), we have

$$\begin{aligned} &\|t_n - u^*\|^2 \\ &= \|u_n + \alpha_n(u_n - u_{n-1}) - \psi_n u_n - \alpha_n \psi_n (u_n - u_{n-1}) - u^*\|^2 \\ &= \|(1 - \psi_n)(u_n - u^*) + (1 - \psi_n)\alpha_n(u_n - u_{n-1}) - \psi_n u^*\|^2 \\ &\leq \|(1 - \psi_n)(u_n - u^*) + (1 - \psi_n)\alpha_n(u_n - u_{n-1})\|^2 + 2\psi_n \langle -u^*, t_n - u^* \rangle \\ &= (1 - \psi_n)^2 \|u_n - u^*\|^2 + (1 - \psi_n)^2 \alpha_n^2 \|u_n - u_{n-1}\|^2 \\ &\quad + 2\alpha_n (1 - \psi_n)^2 \|u_n - u^*\| \|u_n - u_{n-1}\| + 2\psi_n \langle -u^*, t_n - u_{n+1} \rangle + 2\psi_n \langle -u^*, u_{n+1} - u^* \rangle \\ &\leq (1 - \psi_n) \|u_n - u^*\|^2 + \alpha_n^2 \|u_n - u_{n-1}\|^2 + 2\alpha_n (1 - \psi_n) \|u_n - u^*\| \|u_n - u_{n-1}\| \\ &\quad + 2\psi_n \|u^*\| \|t_n - u_{n+1}\| + 2\psi_n \langle -u^*, u_{n+1} - u^* \rangle \end{aligned}$$

$$= (1-\psi_n)\|u_n - u^*\|^2 + \psi_n\Big[\alpha_n\|u_n - u_{n-1}\|\frac{\alpha_n}{\psi_n}\|u_n - u_{n-1}\| \\ + 2(1-\psi_n)\|u_n - u^*\|\frac{\alpha_n}{\psi_n}\|u_n - u_{n-1}\| + 2\|u^*\|\|t_n - u_{n+1}\| + 2\langle u^*, u^* - u_{n+1}\rangle\Big]. \tag{44}$$

From expressions (14) and (44), we obtain

$$\|u_{n+1} - u^*\|^2 \\ \le (1-\psi_n)\|u_n - u^*\|^2 + \psi_n\Big[\alpha_n\|u_n - u_{n-1}\|\frac{\alpha_n}{\psi_n}\|u_n - u_{n-1}\| \\ + 2(1-\psi_n)\|u_n - u^*\|\frac{\alpha_n}{\psi_n}\|u_n - u_{n-1}\| + 2\|u^*\|\|t_n - u_{n+1}\| + 2\langle u^*, u^* - u_{n+1}\rangle\Big]. \tag{45}$$

By the use of (22), (43), (45) and applying Lemma 2.2, conclude that $\lim_{n\to+\infty}\|u_n - u^*\| = 0$.
Case 2: Suppose there is one $\{n_i\}$ subsequence of $\{n\}$ such that

$$\|u_{n_i} - u^*\| \le \|u_{n_{i+1}} - u^*\|, \quad \forall i \in \mathbb{N}.$$

Using Lemma 2.3, there exists a sequence $\{m_k\} \subset \mathbb{N}$ as $\{m_k\} \to +\infty$ such that

$$\|u_{m_k} - u^*\| \le \|u_{m_{k+1}} - u^*\| \quad \text{and} \quad \|u_k - u^*\| \le \|u_{m_{k+1}} - u^*\|, \quad \text{for all } k \in \mathbb{N}. \tag{46}$$

As in Case 1, the relation (20) gives that

$$(1-\tau L)\|t_{m_k} - y_{m_k}\|^2 + (1-\tau L)\|u_{m_k+1} - y_{m_k}\|^2 \\ \le \|u_{m_k} - u^*\|^2 + \psi_{m_k}M_2 - \|u_{m_k+1} - u^*\|^2. \tag{47}$$

Due to $\psi_{m_k} \to 0$, we deduce the following:

$$\lim_{k\to+\infty}\|t_{m_k} - y_{m_k}\| = \lim_{k\to+\infty}\|u_{m_k+1} - y_{m_k}\| = 0. \tag{48}$$

It follows that

$$\lim_{k\to+\infty}\|u_{m_{k+1}} - t_{m_k}\| \le \lim_{k\to+\infty}\|u_{m_{k+1}} - y_{m_k}\| + \lim_{k\to+\infty}\|y_{m_k} - t_{m_k}\| = 0. \tag{49}$$

Next, evaluate

$$\|t_{m_k} - u_{m_k}\| = \|u_{m_k} + \alpha_{m_k}(u_{m_k} - u_{m_k-1}) - \psi_{m_k}\big[u_{m_k} + \alpha_{m_k}(u_{m_k} - u_{m_k-1})\big] - u_{m_k}\| \\ \le \alpha_{m_k}\|u_{m_k} - u_{m_k-1}\| + \psi_{m_k}\|u_{m_k}\| + \alpha_{m_k}\psi_{m_k}\|u_{m_k} - u_{m_k-1}\| \\ = \psi_{m_k}\frac{\alpha_{m_k}}{\psi_{m_k}}\|u_{m_k} - u_{m_k-1}\| + \psi_{m_k}\|u_{m_k}\| + \psi_{m_k}^2\frac{\alpha_{m_k}}{\psi_{m_k}}\|u_{m_k} - u_{m_k-1}\| \longrightarrow 0. \tag{50}$$

This follows that

$$\lim_{k\to+\infty}\|u_{m_k} - u_{m_k+1}\| \le \lim_{k\to+\infty}\|u_{m_k} - t_{m_k}\| + \lim_{k\to+\infty}\|t_{m_k} - u_{m_k+1}\| = 0. \tag{51}$$

Using the same explanation as in the Case 1, such that

$$\limsup_{k\to+\infty}\langle u^*, u^* - u_{m_k+1}\rangle \le 0. \tag{52}$$

Using the expressions (45) and (46), we get

$$\|u_{m_k+1} - u^*\|^2 \\ \le (1-\psi_{m_k})\|u_{m_k} - u^*\|^2 + \psi_{m_k}\Big[\alpha_{m_k}\|u_{m_k} - u_{m_k-1}\|\frac{\alpha_{m_k}}{\psi_{m_k}}\|u_{m_k} - u_{m_k-1}\| \\ + 2(1-\psi_{m_k})\|u_{m_k} - u^*\|\frac{\alpha_{m_k}}{\psi_{m_k}}\|u_{m_k} - u_{m_k-1}\| + 2\|u^*\|\|t_{m_k} - u_{m_k+1}\| + 2\langle u^*, u^* - u_{m_k+1}\rangle\Big]$$

$$\leq (1-\psi_{m_k})\|u_{m_{k+1}}-u^*\|^2+\psi_{m_k}\Big[\alpha_{m_k}\|u_{m_k}-u_{m_k-1}\|\frac{\alpha_{m_k}}{\psi_{m_k}}\|u_{m_k}-u_{m_k-1}\|$$
$$+2(1-\psi_{m_k})\|u_{m_k}-u^*\|\frac{\alpha_{m_k}}{\psi_{m_k}}\|u_{m_k}-u_{m_k-1}\|+2\|u^*\|\|t_{m_k}-u_{m_k+1}\|+2\langle u^*,u^*-u_{m_k+1}\rangle\Big]. \quad (53)$$

Thus, above implies that

$$\|u_{m_k+1}-u^*\|^2$$
$$\leq \Big[\alpha_{m_k}\|u_{m_k}-u_{m_k-1}\|\frac{\alpha_{m_k}}{\psi_{m_k}}\|u_{m_k}-u_{m_k-1}\|$$
$$+2(1-\psi_{m_k})\|u_{m_k}-u^*\|\frac{\alpha_{m_k}}{\psi_{m_k}}\|u_{m_k}-u_{m_k-1}\|+2\|u^*\|\|t_{m_k}-u_{m_k+1}\|+2\langle u^*,u^*-u_{m_k+1}\rangle\Big]. \quad (54)$$

Since $\psi_{m_k}\to 0$, and $\|u_{m_k}-u^*\|$ is a bounded. Therefore, expressions (52) and (54) implies that

$$\|u_{m_k+1}-u^*\|^2\to 0, \quad \text{as } k\to+\infty. \quad (55)$$

This means that

$$\lim_{n\to+\infty}\|u_k-u^*\|^2\leq \lim_{n\to+\infty}\|u_{m_k+1}-u^*\|^2\leq 0. \quad (56)$$

As a consequence $u_n\to u^*$. This is going to conclude the proof of the theorem. □

The second projection in Algorithm 1 is substituted with half-space to minimize computing cost, as inspired by Algorithm 4.1 in the paper [4]. The second main algorithm is written as follows:

Algorithm 2 (Inertial Strongly Convergent Iterative Method)

Step 0: Select $u_0, u_1\in\mathcal{D}$, $\alpha>0$ and $0<\tau<\frac{1}{L}$. Moreover, choose $\{\psi_n\}\subset(0,1)$ fulfills the following criteria: $\lim_{n\to+\infty}\psi_n=0$ and $\sum_{n=1}^{+\infty}\psi_n=+\infty$.
Step 1: Compute

$$t_n=u_n+\alpha_n(u_n-u_{n-1})-\psi_n\big[u_n+\alpha_n(u_n-u_{n-1})\big],$$

where α_n such that

$$0\leq\alpha_n\leq\hat{\alpha_n} \quad\text{and}\quad \hat{\alpha_n}=\begin{cases}\min\left\{\frac{\alpha}{2},\frac{\epsilon_n}{\|u_n-u_{n-1}\|}\right\} & \text{if } u_n\neq u_{n-1},\\ \frac{\alpha}{2} & \text{else,}\end{cases}$$

where $\epsilon_n=\circ(\psi_n)$ is a positive sequence, i.e., $\lim_{n\to+\infty}\frac{\epsilon_n}{\psi_n}=0$.
Step 2: Compute

$$y_n=P_{\mathcal{D}}(t_n-\tau\mathcal{S}(t_n)).$$

If $t_n=y_n$, then STOP and y_n is a solution.
Step 3: Compute

$$u_{n+1}=P_{\mathcal{X}_n}(t_n-\tau\mathcal{S}(y_n)),$$

where $\mathcal{X}_n=\{z\in\mathcal{X}:\langle t_n-\tau\mathcal{S}(t_n)-y_n, z-y_n\rangle\leq 0\}$. Set $n=n+1$ and go back to **Step 1**.

---

4 Numerical illustrations

This section discusses two numerical tests to explain the efficacy of the proposed algorithms. All these numerical studies give a detailed understanding of how better control parameters can be chosen. Each of them shows the advantages of the proposed methods relative to the existing ones in the literature.

Example 4.1 First consider the HpHard problem which is taken from [7]. This example has been considered by many people for experimental test (see, [5,8,21]). A operator $\mathcal{S}:\mathbb{R}^m\to\mathbb{R}^m$ is defined by

$$\mathcal{S}(u)=Mu+q$$

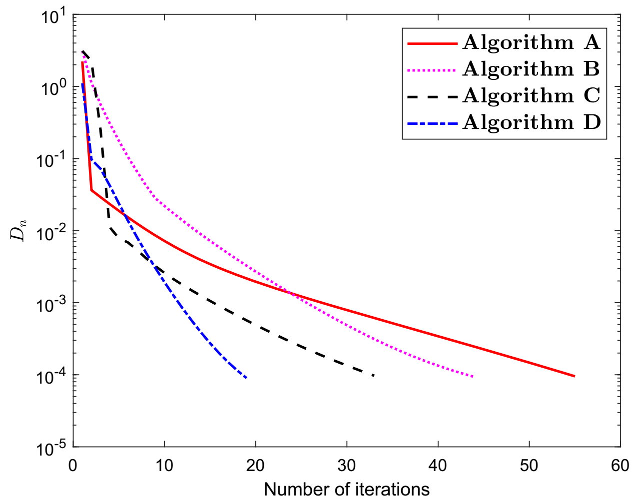

Fig. 1 Numerical illustration of Algorithm 1 with Algorithm 1 in [16] and Algorithm 3.1 in [1] and Algorithm 3.1 in [27] while $m = 5$

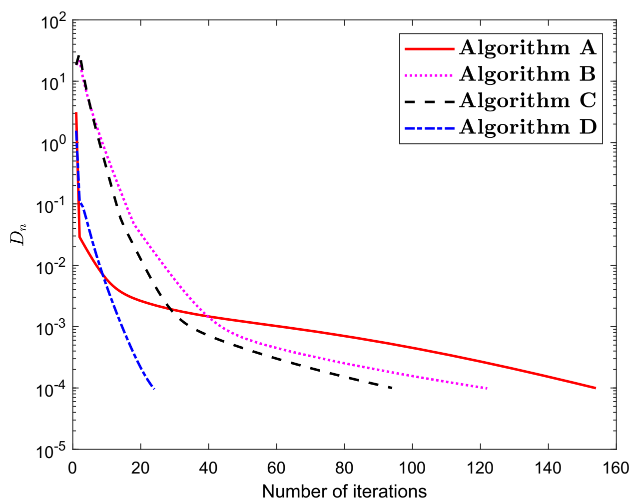

Fig. 2 Numerical illustration of Algorithm 1 with Algorithm 1 in [16] and Algorithm 3.1 in [1] and Algorithm 3.1 in [27] while $m = 10$

with $q \in \mathbb{R}^m$ and

$$M = NN^T + B + D.$$

In above definition, we take $N = \text{rand}(m)$ to be a random matrix and $B = 0.5K - 0.5K^{\mathrm{T}}$ to be a skew-symmetric matrix with $K = \text{rand}(m)$ and $D = \text{diag}(\text{rand}(m, 1))$ is a diagonal matrix. The set $\mathcal{D}$ is taken as follows:

$$\mathcal{D} = \{u \in \mathbb{R}^m : Qu \leq b\},$$

where $Q = \text{rand}(100, m)$ and $b = \text{rand}(100, 1)$. It is obvious that $\mathcal{S}$ is monotone and that Lipschitz is continuous by $L = \|M\|$. The starting point for this experiment are $u_0 = u_1 = (1, 1, \ldots, 1)$. The numerical consequences of these methods are seen in Figs. 1, 2, 3 and 4 and Tables 1 and 8. The control requirement shall be taken as follows:

(1) Algorithm 1 in [16] (shortly, **Algorithm A**): $\tau = \frac{0.7}{L}$, $D_n = \|u_n - y_n\| \leq 10^{-4}$;

(2) Algorithm 3.1 in [1] (shortly, **Algorithm B**): $\alpha = 0.60$, $\tau = \frac{0.7}{L}$, $\epsilon_n = \frac{1}{(n+1)^2}$, $\psi_n = \frac{1}{(n+2)}$, $\theta_n = \frac{5}{10}(1 - \psi_n)$, $D_n = \|t_n - y_n\| \leq 10^{-4}$;

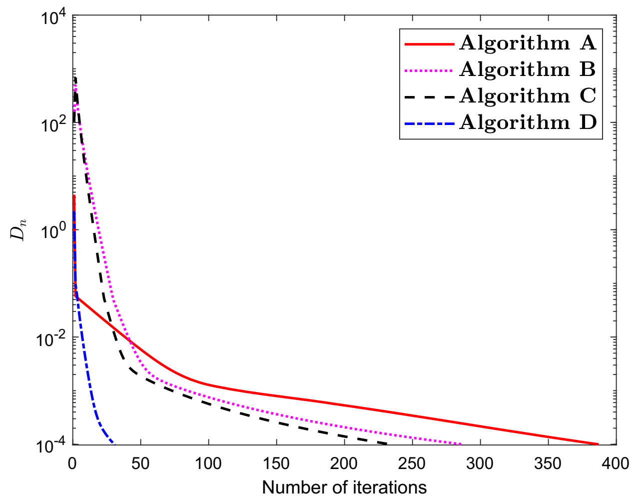

Fig. 3 Numerical illustration of Algorithm 1 with Algorithm 1 in [16] and Algorithm 3.1 in [1] and Algorithm 3.1 in [27] while $m = 20$

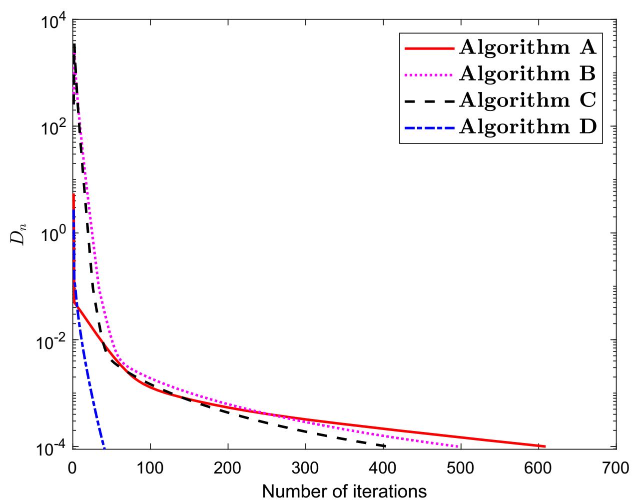

Fig. 4 Numerical illustration of Algorithm 1 with Algorithm 1 in [16] and Algorithm 3.1 in [1] and Algorithm 3.1 in [27] while $m = 30$

(3) Algorithm 3.1 in [27] (shortly, **Algorithm C**): $\alpha = 0.60$, $\tau = \frac{0.7}{L}$, $\psi_n = \frac{1}{(n+2)}$, $\epsilon_n = \frac{1}{(n+1)^2}$, $f(u) = \frac{u}{3}$, $D_n = \|t_n - y_n\| \leq 10^{-4}$;

(4) Algorithm 1 (shortly, **Algorithm D**): $\tau = \frac{0.7}{L}$, $\alpha = 0.60$, $\epsilon_n = \frac{1}{(n+1)^2}$, $\psi_n = \frac{1}{(n+2)}$, $D_n = \|t_n - y_n\| \leq 10^{-4}$ (Table 2).

Example 4.2 Consider the non-linear complementarity problem of Kojima–Shindo where the operator $\mathcal{S} : \mathbb{R}^4 \to \mathbb{R}^4$ is described by

$$\mathcal{S}(u) = \begin{pmatrix} u_1 + u_2 + u_3 + u_4 - 4u_2u_3u_4 \\ u_1 + u_2 + u_3 + u_4 - 4u_1u_3u_4 \\ u_1 + u_2 + u_3 + u_4 - 4u_1u_2u_4 \\ u_1 + u_2 + u_3 + u_4 - 4u_1u_2u_3 \end{pmatrix}.$$

Moreover, the feasible set $\mathcal{D}$ is defined by

$$\mathcal{D} = \{u \in \mathbb{R}^4 : 1 \leq u_i \leq 5, \ i = 1, 2, 3, 4\}.$$

Table 1 Numerical values for Figs. 1, 2, 3 and 4

m	Algorithm A		Algorithm B	
	Number of iterations	Elapsed time	Number of iterations	Elapsed time
5	55	1.271423	44	0.8473277
10	154	3.984347	122	2.4970683
20	387	8.871673	287	5.7369488
30	609	16.065671	496	10.8786385

Table 2 Numerical values for Figs. 1, 2, 3 and 4

m	Algorithm C		Algorithm D	
	Number of iterations	Elapsed time	Number of iterations	Elapsed time
5	33	0.7823692	19	0.436875
10	94	2.1836821	24	0.570326
20	234	4.9718569	31	0.738349
30	406	9.3679587	41	1.039996

Table 3 Example 4.2: Numerical study of Algorithm 1 in [16] and $u_0 = u_1 = (1, 2, 5, 4)^{\mathrm{T}}$

Iter (n)	u_1	u_2	u_3	u_4
1	2.46850686712594	3.43518531832707	4.11170031354167	4.85198081270192
2	3.26306718965650	4.10302677285378	4.68884806816664	5.16263925114349
3	3.68345230081183	4.45301272004093	4.90061928579334	5.15843849896288
4	4.11679011556020	4.77035387197085	5.02276694727452	5.16509362041636
5	4.53653342828402	4.92668529039587	5.07024397379473	5.15044222135542
6	4.99407054939005	4.99407309848611	4.99407400449338	4.99407450402104
7	4.99429687804173	4.99429825094701	4.99429873890904	4.99429900794721
8	4.99450616557211	4.99450690502187	4.99450716783928	4.99450731274381
9	4.99470040849581	4.99470080677679	4.99470094833498	4.99470102638317
10	4.99488125238743	4.99488146691464	4.99488154316252	4.99488158520184
⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮
33	4.99949439152920	4.99949439152920	4.99949439152920	4.99949439152920
34	4.99949608585900	4.99949608585900	4.99949608585900	4.99949608585900
35	4.99949776887094	4.99949776887094	4.99949776887094	4.99949776887094
36	4.99949944067807	4.99949944067807	4.99949944067807	4.99949944067807
37	4.99950110139188	4.99950110139188	4.99950110139188	4.99950110139188
CPU time is seconds	0.5464821			

It is clear to see that $\mathcal{S}$ is not monotone on the set $\mathcal{D}$. By the use of Monte-Carlo technique [9], it can be shown that $\mathcal{S}$ is pseudo-monotone on $\mathcal{D}$. There exits a unique solution $u^* = (5, 5, 5, 5)^T$ for given problem. The starting point for this experiment are $u_0 = u_1 = (1, 1, \ldots, 1)$ and $D_n = \|t_n - y_n\| \leq 10^{-3}$. The numerical consequences of these methods are seen in Tables 3, 4, 5 and 6. The control requirement shall be taken as follows: (1) Algorithm 1 in [16] (shortly, **Algorithm A**): $\tau = \frac{0.7}{L}$, $D_n = \|u_n - y_n\| \leq 10^{-4}$;

(2) Algorithm 3.1 in [1] (shortly, **Algorithm B**): $\alpha = 0.60$, $\tau = \frac{0.7}{L}$, $\epsilon_n = \frac{1}{(n+1)^2}$, $\psi_n = \frac{1}{(n+2)}$, $\theta_n = \frac{5}{10}(1 - \psi_n)$, $D_n = \|t_n - y_n\| \leq 10^{-4}$;

(3) Algorithm 3.1 in [27] (shortly, **Algorithm C**): $\alpha = 0.60$, $\tau = \frac{0.7}{L}$, $\psi_n = \frac{1}{(n+2)}$, $\epsilon_n = \frac{1}{(n+1)^2}$, $f(u) = \frac{u}{3}$, $D_n = \|t_n - y_n\| \leq 10^{-4}$;

(4) Algorithm 1 (shortly, **Algorithm D**): $\tau = \frac{0.7}{L}$, $\alpha = 0.60$, $\epsilon_n = \frac{1}{(n+1)^2}$, $\psi_n = \frac{1}{(n+2)}$, $D_n = \|t_n - y_n\| \leq 10^{-4}$.

Table 4 Example 4.2: Numerical study of Algorithm 1 in [16] and $u_0 = u_1 = (-3, -4, 1, -5)^{\mathrm{T}}$

Iter (n)	u_1	u_2	u_3	u_4
1	3.95798700420275	4.90319832859659	5.24763662592025	4.24662349851835
2	4.88226208073189	5.01271766369566	5.02408134785700	4.91453336394776
3	4.97849884544991	4.98904776095972	4.98511967764930	4.98010334552711
4	4.99054471487201	4.98445005743840	4.98262529908410	4.98878003127405
5	4.99021445709950	4.98791312658462	4.98770097784552	4.98964686129797
6	4.99753023091885	4.99753023091885	4.99753023091885	4.99753023091885
7	4.99761845958073	4.99761845958073	4.99761845958073	4.99761845958073
8	4.99770060194473	4.99770060194473	4.99770060194473	4.99770060194473
9	4.99777726680043	4.99777726680043	4.99777726680043	4.99777726680043
10	4.99784898435845	4.99784898435845	4.99784898435845	4.99784898435845
⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮
38	4.99948317721650	4.99948317721650	4.99948317721650	4.99948317721650
39	4.99948715298345	4.99948715298345	4.99948715298345	4.99948715298345
40	4.99949106804850	4.99949106804850	4.99949106804850	4.99949106804850
41	4.99949492379131	4.99949492379131	4.99949492379131	4.99949492379131
42	4.99949872155006	4.99949872155006	4.99949872155006	4.99949872155006
41	4.99950246262296	4.99950246262296	4.99950246262296	4.99950246262296
CPU time is seconds	0.597347			

Table 5 Example 4.2: Numerical study of Algorithm 1 and $u_0 = u_1 = (1, 2, 5, 4)^{\mathrm{T}}$

Iter (n)	u_1	u_2	u_3	u_4
1	3.91488016372854	4.85252928297821	5.22585944011412	5.72818199081251
2	4.57264127647102	5.00300001557819	5.12274776629554	5.26642638995636
3	4.90746494889927	5.02422967456386	5.04941197855862	5.07649346544389
4	4.97510231126800	4.99574420343289	4.99882318198317	5.00140298171803
5	5.02219257897729	5.01883397835351	5.01760392054566	5.01600966350793
6	5.00045550321418	5.00045550321418	5.00045550321418	5.00045550321418
7	5.00044072203028	5.00044072203028	5.00044072203028	5.00044072203028
8	5.00042473156775	5.00042473156775	5.00042473156775	5.00042473156775
9	5.00041120014611	5.00041120014610	5.00041120014610	5.00041120014610
10	5.00039765754469	5.00039765754474	5.00039765754474	5.00039765754474
⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮
15	5.00017419442177	5.00017419442177	5.00017419442177	5.00017419442177
16	5.00017178052361	5.00017178052361	5.00017178052361	5.00017178052361
17	5.00016943260831	5.00016943260831	5.00016943260831	5.00016943260831
18	5.00016714801455	5.00016714801455	5.00016714801455	5.00016714801455
19	5.00016492420973	5.00016492420973	5.00016492420973	5.00016492420973
20	5.00016275880259	5.00016275880259	5.00016275880259	5.00016275880259
CPU time is seconds	0.142008			

Example 4.3 Let $\mathcal{X} = l_2$ be a real Hilbert space with the sequences of real numbers satisfying the following condition

$$\|u_1\|^2 + \|u_2\|^2 + \cdots + \|u_n\|^2 + \cdots < +\infty. \tag{57}$$

Assume that a mapping $\mathcal{S} : \mathcal{D} \to \mathcal{D}$ is defined by

$$G(u) = (5 - \|u\|)u, \quad \forall\, u \in \mathcal{X},$$

where $\mathcal{D} = \{u \in \mathcal{X} : \|u\| \le 3\}$. We can easily see that $\mathcal{S}$ is weakly sequentially continuous on $\mathcal{X}$ and the solution set is $VI(\mathcal{D}, \mathcal{S}) = \{0\}$. Moreover, $\mathcal{S}$ is L-Lipschitz continuous with $L = 11$. The mapping $\mathcal{S}$ is pseudomonotone on $\mathcal{D}$ but not monotone (see for more details [33]). Let considered the following projection

Table 6 Example 4.2: Numerical study of Algorithm 1 and $u_0 = u_1 = (-3, -4, 1, -5)^T$

Iter (n)	u_1	u_2	u_3	u_4
1	3.91488016372854	4.85252928297821	5.22585944011412	5.72818199081251
2	4.57121385239430	5.00184252170840	5.12166538077650	5.26543412274350
3	4.90757824004344	5.02439246837912	5.04958273583595	5.07667166195147
4	4.97308781606032	4.99430374686646	4.99747028860293	5.00012514266696
5	5.02171157115764	5.01876924007345	5.01764240206392	5.01616645155788
6	5.00038452174424	5.00038452174425	5.00038452174425	5.00038452174425
7	5.00037212854981	5.00037212854981	5.00037212854981	5.00037212854981
8	5.00035831848008	5.00035831848008	5.00035831848008	5.00035831848008
9	5.00034686831722	5.00034686831722	5.00034686831722	5.00034686831722
10	5.00033525865097	5.00033525865097	5.00033525865097	5.00033525865097
⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮
13	5.00017878677236	5.00017878677236	5.00017878677236	5.00017878677236
14	5.00017574742264	5.00017574742264	5.00017574742264	5.00017574742264
15	5.00017280968570	5.00017280968570	5.00017280968570	5.00017280968570
16	5.00016996854741	5.00016996854741	5.00016996854741	5.00016996854741
17	5.00016721932153	5.00016721932153	5.00016721932153	5.00016721932153
18	5.00016455761801	5.00016455761801	5.00016455761801	5.00016455761801
CPU time is seconds	0.129242			

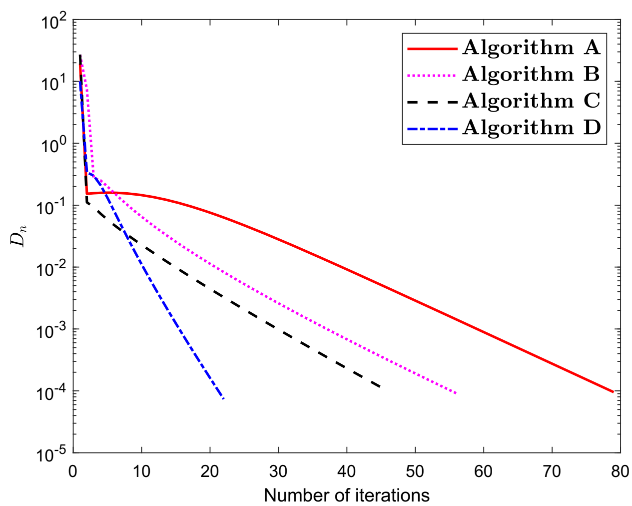

Fig. 5 Numerical illustration of Algorithm 1 with Algorithm 1 in [16] and Algorithm 3.1 in [1] and Algorithm 3.1 in [27] while $u_0 = u_1 = (1, 1, \ldots, 1_{5000}, 0, 0, \ldots)$

formula:

$$P_{\mathcal{D}}(u) = \begin{cases} u & \text{if } \|u\| \leq 3, \\ \frac{3u}{\|u\|}, & \text{otherwise.} \end{cases}$$

The numerical consequences of these methods are seen in Figs. 5, 6, 7 and 8 and Table 7. The control requirement shall be taken as follows:

(1) Algorithm 1 in [16] (shortly, **Algorithm A**): $\tau = \frac{0.8}{L}$, $D_n = \|u_n - y_n\| \leq 10^{-4}$;

(2) Algorithm 3.1 in [1] (shortly, **Algorithm B**): $\alpha = 0.70$, $\tau = \frac{0.8}{L}$, $\epsilon_n = \frac{1}{(n+1)^2}$, $\psi_n = \frac{1}{(n+2)}$, $\theta_n = \frac{5}{10}(1 - \psi_n)$, $D_n = \|t_n - y_n\| \leq 10^{-4}$;

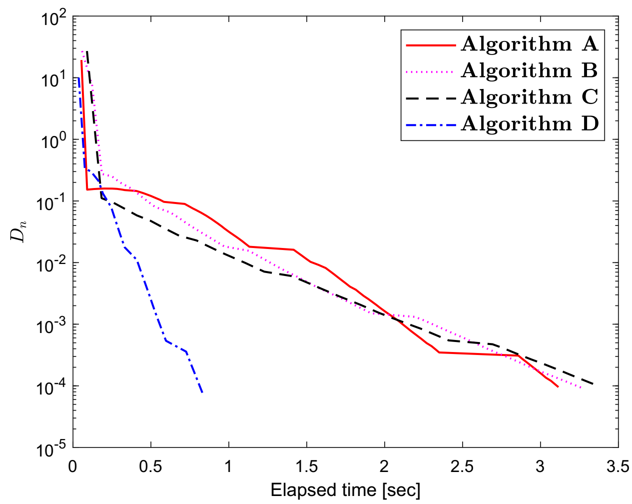

Fig. 6 Numerical illustration of Algorithm 1 with Algorithm 1 in [16] and Algorithm 3.1 in [1] and Algorithm 3.1 in [27] while $u_0 = u_1 = (1, 1, \ldots, 1_{5000}, 0, 0, \ldots)$

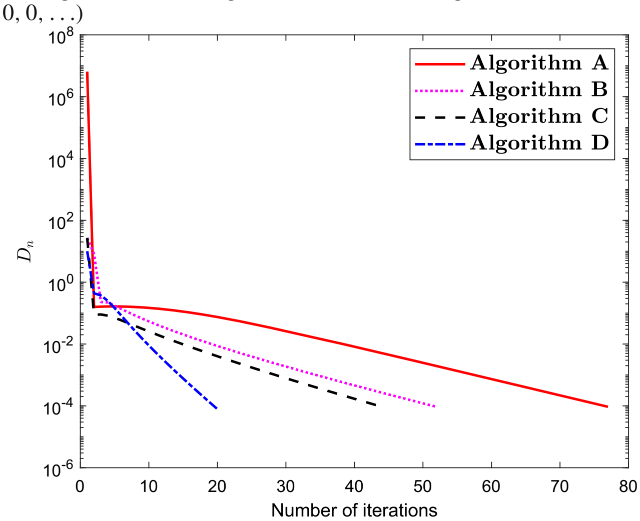

Fig. 7 Numerical illustration of Algorithm 1 with Algorithm 1 in [16] and Algorithm 3.1 in [1] and Algorithm 3.1 in [27] while $u_0 = u_1 = (1, 2, \ldots, 5000, 0, 0, \ldots)$

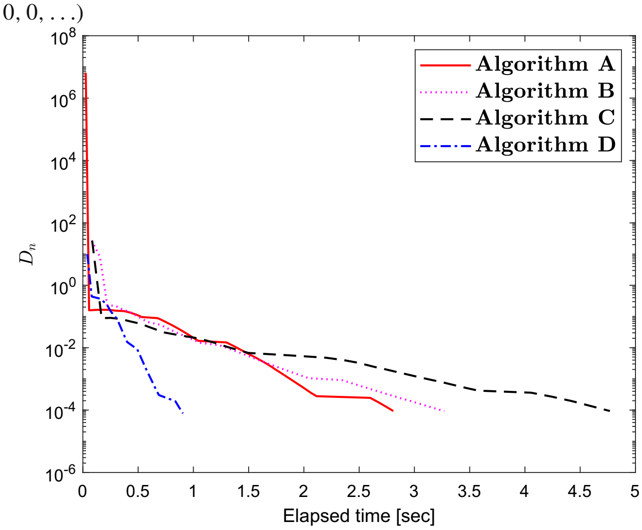

Fig. 8 Numerical illustration of Algorithm 1 with Algorithm 1 in [16] and Algorithm 3.1 in [1] and Algorithm 3.1 in [27] while $u_0 = u_1 = (1, 2, \ldots, 5000, 0, 0, \ldots)$

Table 7 Numerical values for Figs. 5, 6, 7 and 8

m	Algorithm A		Algorithm B	
	Number of iterations	Elapsed time	Number of iterations	Elapsed time
$(1, 1, \ldots, 1_{5000}, 0, 0, \ldots)$	79	3.1150363	56	3.2711516
$(1, 2, \ldots, 5000, 0, 0, \ldots)$	77	2.8090430	52	3.27137490

Table 8 Numerical values for Figs. 5, 6, 7 and 8

m	Algorithm C		Algorithm D	
	Number of iterations	Elapsed time	Number of iterations	Elapsed time
$(1, 1, \ldots, 1_{5000}, 0, 0, \ldots)$	46	3.3728652	22	0.8332482
$(1, 2, \ldots, 5000, 0, 0, \ldots)$	44	4.76874250	20	0.9056598

(3) Algorithm 3.1 in [27] (shortly, **Algorithm C**): $\alpha = 0.70$, $\tau = \frac{0.8}{L}$, $\psi_n = \frac{1}{(n+2)}$, $\epsilon_n = \frac{1}{(n+1)^2}$, $f(u) = \frac{u}{3}$, $D_n = \|t_n - y_n\| \leq 10^{-4}$;

(4) Algorithm 1 (shortly, **Algorithm D**): $\tau = \frac{0.8}{L}$, $\alpha = 0.70$, $\epsilon_n = \frac{1}{(n+1)^2}$, $\psi_n = \frac{1}{(n+2)}$, $D_n = \|t_n - y_n\| \leq 10^{-4}$ (Table 8).

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Declarations

Conflict of interest The authors have not disclosed any competing interests.

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