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Alternative proofs of some classical metric fixed point theorems by using approximate fixed point sequences

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Abstract The notion of *approximate fixed point sequence*, emphasized in Chidume (Geometric properties of Banach spaces and nonlinear iterations. Lecture Notes in Mathematics, 1965. Springer-Verlag London, Ltd., London, 2009), is a very useful tool in proving convergence theorems for fixed point iterative schemes in the class of nonexpansive-type mappings. In the present paper, our aim is to present simple and unified alternative proofs of some classical fixed point theorems emerging from Banach contraction principle, by using a technique based on the concepts of approximate fixed point sequence and graphic contraction.

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1 Introduction

In the monograph [18], Chidume illustrated the role of approximate fixed point sequences in proving convergence theorems for fixed point iterative schemes in the class of nonexpansive-type mappings.

To exemplify this, let K be a nonempty closed convex subset of a real Banach space X and $T : K \to K$ a nonexpansive map, i.e., a map satisfying

$$\|Tx - Ty\| \le \|x - y\|, \ x, y \in K. \tag{1}$$

For arbitrary $x_0, u \in K$, let $\{x_n\}$ be the Halpern-type iterative sequence defined by

$$x_{n+1} = \lambda_n u + (1 - \lambda_n)Sx_n, n \ge 0,$$

where $\lambda_n \in [0, 1]$ and $S = (1 - \delta)I + \delta T$, for $\delta \in (0, 1)$ (I denotes the identity map).

If X has uniformly Gâteaux differentiable norm and $\{\lambda_n\}$ satisfies some conditions, then (see [19] and [18], page 214) $\{x_n\}$ is *an approximate fixed point sequence* with respect to the averaged map S, that is,

$$\|x_n - Sx_n\| \to 0 \text{ as } n \to \infty.$$

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The property of having an approximate fixed point sequence is very important for the class of nonexpansive-type mappings; see for example the very recent paper [64]. So, there are many convergence results for iterative algorithms in such classes of mappings which are proven by using the properties of some approximate fixed point sequence; see [1–3,6,19,21,22,26,27,49,52,53,59,65,67] and the references therein.

In this paper our aim is to emphasize, by means of several examples, how one can simplify and unify the proofs of some classical fixed point theorems emerging from Banach contraction principle, such as Kannan fixed point theorem, Chatterjea fixed point theorem, Bianchini fixed point theorem, and Zamfirescu fixed point theorem, using a technique based on the concepts of graphic contractions and approximate fixed point sequence.

2 Graphic contractions

An important concept that will be useful in this paper is given in the next definition; see for example [5,48,56–58].

Definition 2.1 Let (X, d) be a metric space. A mapping $T : X \to X$ is called a *graphic contraction (orbital contraction)* if

$$d(Tx, T^2x) \leq \alpha d(x, Tx),\ \forall x \in X, \tag{2}$$

where $\alpha \in (0, 1)$.

In the following examples, (X, d) is supposed to be a metric space.

Example 2.2 Any *Banach contraction*, i.e., any mapping $T : X \to X$ satisfying the inequality

$$d(Tx, Ty) \leq ad(x, y), \forall x, y \in X, \tag{3}$$

for some $a \in [0, 1)$, is a graphic contraction with $\alpha = a$.

Example 2.3 (Kannan [34]) Any *Kannan mapping*, i.e., any mapping $T : X \to X$ satisfying the inequality

$$d(Tx, Ty) \leq b(d(x, Tx) + d(y, Ty)), \forall x, y \in X, \tag{4}$$

for some $b \in [0, 1/2)$, is a graphic contraction with $\alpha = \dfrac{b}{1-b}$.

Example 2.4 (Ćirić [24]; Reich [51]; Rus [55]) Any *Ćirić-Reich-Rus contraction*, i.e., any mapping $T : X \to X$ satisfying

$$d(Tx, Ty) \leq ad(x, y) + b\left(d(x, Tx) + d(y, Ty)\right), \forall x, y \in X, \tag{5}$$

where $a, b \geq 0$ and $a + 2b < 1$, is a graphic contraction with $\alpha = \dfrac{a+b}{1-b}$.

Example 2.5 (Bianchini [62]) Any *Bianchini mapping*, i.e., any mapping $T : X \to X$ satisfying

$$d(Tx, Ty) \leq h \max\{d(x, Tx), d(y, Ty)\}, \forall x, y \in X, \tag{6}$$

for some $h \in [0, 1)$, is a graphic contraction with $\alpha = h$.

Example 2.6 (Chatterjea [17]) Any *Chatterjea mapping* $T : X \to X$, i.e., any mapping satisfying

$$d(Tx, Ty) \leq c(d(x, Ty) + d(y, Tx)), \forall x, y \in X, \tag{7}$$

for some $c \in [0, 1/2)$, is a graphic contraction with $\alpha = \dfrac{c}{1-c}$.

Example 2.7 (Zamfirescu [66]) Any *Zamfirescu mapping*, i.e., any mapping $T : X \to X$ for which there exist $a, b, c \geq 0$ satisfying $a < 1$, $b < 1/2$, $c < 1/2$ such that for each $x, y \in X$ at least one of the following conditions is true:

(i) $d(Tx, Ty) \leq ad(x, y)$;

(ii) $d(Tx, Ty) \leq b\,(d(x, Tx) + d(y, Ty))$;
(iii) $d(Tx, Ty) \leq c\,(d(x, Ty) + d(y, Tx))$,

is a graphic contraction with

$$\alpha = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}.$$

Example 2.8 (Ćirić [24]) Any strong Ćirić quasi contraction, i.e., any mapping $T : X \to X$ satisfying, for all $x, y \in X$,

$$d(Tx, Ty) \leq h \cdot \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\right\} \tag{8}$$

for some $h \in [0, 1)$, is a graphic contraction with $\alpha = h$.

Example 2.9 (Hardy and Rogers [32]) Any Hardy and Rogers contraction, i.e., any mapping $T : X \to X$ satisfying, for all $x, y \in X$,

$$d(Tx, Ty) \leq a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx) \tag{9}$$

for $a_1, a_2, a_3, a_4, a_5 \geq 0$ and $a_1 + a_2 + a_3 + a_4 + a_5 < 1$, is a graphic contraction with

$$\alpha = \frac{2a_1 + a_2 + a_3 + a_4 + a_5}{2 - a_2 - a_3 - a_4 - a_5}.$$

Example 2.10 (Berinde [5]) Any *almost contraction*, that is, any mapping $T : X \to X$ satisfying

$$d(Tx, Ty) \leq ad(x, y) + L \cdot d(y, Tx), \forall x, y \in X, \tag{10}$$

where $a \in [0, 1)$ and $L \geq 0$, is a graphic contraction with $\alpha = a$.

The following notion is related to that of graphic contraction, as it is shown by Lemma 2.12. According to MathScinet, the first papers that consider explicitly this concept are by Lin [41] and Khamsi [37].

Definition 2.11 s Let (X, d) be a metric space and $T : X \to X$ a self-mapping. A sequence $\{x_n\} \subset X$ is called an approximate fixed point sequence with respect to T if

$$d(x_n, Tx_n) \to 0 \text{ as } n \to \infty. \tag{11}$$

The next lemma will be extremely useful in proving some classical fixed results in metric fixed point theory.

Lemma 2.12 *Let (X, d) be a metric space. Any graphic contraction $T : X \to X$ admits an approximate fixed point sequence.*

Proof Denote

$$D = \{d(x, Tx) : x \in X\}, \quad D_1 = \{d(Tx, T^2x) : x \in X\}$$

and

$$\delta = \inf D.$$

Obviously, $\delta \geq 0$.

Asume $\delta > 0$. Then, since $D_1 \subseteq D$, by using (2) we get

$$\delta = \inf D \leq \inf D_1 \leq \alpha \inf D = \alpha\delta < \delta,$$

a contradiction. So, $\delta = 0$, i.e.,

$$\inf\{d(x, Tx) : x \in X\} = 0,$$

which, by the definition of infimum, shows that there exists a sequence $\{x_n\} \subset X$ such that

$$d(x_n, Tx_n) \to 0, \quad \text{as } n \to \infty. \tag{12}$$

□

Remark 2.13 1. Note that the *Picard iteration* associated with a graphic contraction T, i.e., the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$, $n \geq 0$, for some $x_0 \in X$, is an approximate fixed point sequence with respect to T.
However, the approximate fixed point sequence $\{x_n\}$ ensured by Lemma 2.12 is not necessarily the Picard iteration associated with T.
2. The main idea behind Lemma 2.12 is taken from Joseph and Kwack [33].
3. There exist mappings which are not graphic contractions but they admit an approximate fixed point sequence. Indeed, let $X = [0, 1]$ with the usual metric and $T : X \to X$ be given by $Tx = \frac{7}{8}$, if $0 \leq x < 1$ and $T1 = \frac{1}{4}$. Then T has an approximate fixed point sequence $\{x_n\}$ (see [2], Example 2.1), T is asymptotically regular on X but T is not a graphic contraction (just take $x = 1$ in (2) to get $\alpha \geq 5$, a contradiction).

To shorten the statements of the fixed point theorems presented in this paper, we also need the following concepts.

Let $T : X \to X$ be a mapping. Denote by

$$Fix\,(T) = \{x \in X : Tx = x\}$$

the set of all fixed points of T. The map T is called a *weakly Picard operator*, see for example [58], if
$(p1)$ $Fix\,(T) \neq \emptyset$;
$(p2)$ the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n\,, \quad n = 0, 1, 2, \ldots \tag{13}$$

converges to some $p \in Fix\,(T)$, for any $x_0 \in X$.

If T is a weakly Picard operator and $Fix\,(T) = \{p\}$, then T is called a *Picard operator*.

Our first main result in this section is an alternative proof of the well-known Ćirić-Reich-Rus fixed point theorem, from which are then obtained as particular cases the classical fixed point theorems due to Banach [4] and Kannan [34].

The innovation brought by Lemma 2.12 is that the Cauchyness is established for an arbitrary approximate fixed point sequence and not necessarily for the Picard iteration.

Theorem 2.14 (Ćirić [24]; Reich [51]; Rus [55]) *Let (X, d) be a complete metric space and $T : X \to X$ be a Ćirić–Reich–Rus contraction. Then T is a Picard operator.*

Proof By Example 2.4, T is a graphic contraction with $\alpha = \frac{a+b}{1-b} < 1$.

Hence, by Lemma 2.12, there exists an approximate fixed point sequence $\{x_n\}$ with respect to T, that is, a sequence $\{x_n\} \subset X$ with the property

$$d(x_n, Tx_n) \to 0, \text{ as } n \to \infty. \tag{14}$$

Now, for n, m positive integers, by the contraction condition (5) we have

$$d(x_n, x_m) \leq \frac{1+b}{1-a} \cdot (d(x_n, Tx_n) + d(x_m, Tx_m)),$$

which, by virtue of (14), shows that $\{x_n\}$ is a Cauchy sequence. Let

$$\lim_{n\to\infty} x_n = p. \tag{15}$$

By using once again the Ćirić–Reich–Rus condition (5), we obtain

$$d(p, Tp) \leq \frac{1+a}{1-b} \cdot d(x_n, p) + \frac{1+b}{1-b} \cdot d(x_n, Tx_n),$$

which, by (14) and (15), proves that $Tp = p$, i.e., $Fix\,(T) \neq \emptyset$.

Assume that $q \neq p$ is another fixed point of T. Then, by (5)

$$0 < d(p, q) = d(Tp, Tq) \leq a \cdot d(p, q) < d(p, q),$$

a contradiction. This proves that $Fix\,(T) = \{p\}$.

Now, let $\{y_n\} \subset X$ be the Picard iteration defined by $y_0 \in X$ and

$$y_{n+1} = f(y_n), n \geq 0. \tag{16}$$

Then, by (5) one obtains

$$d(y_{n+1}, p) \leq \alpha d(y_n, p), n \geq 0 \tag{17}$$

which, by induction, yields

$$d(y_n, p) \leq \alpha^n d(y_0, p), n \geq 1. \tag{18}$$

This proves that $\{y_n\}$ converges to p as $n \to \infty$. So, T is a Picard operator. □

Corollary 2.15 (Banach [4], [16]) *Let (X, d) be a complete metric space and $T : X \to X$ a Banach contraction. Then T is a Picard operator.*

Proof Any Banach contraction is a Ćirić–Reich–Rus contraction with the constant $b = 0$.

We apply Theorem 2.14 and get the conclusion. □

Corollary 2.16 (Kannan [34]) *Let (X, d) be a complete metric space and $T : X \to X$ a Kannan mapping. Then T is a Picard operator.*

Proof Any Kannan mapping is a Ćirić–Reich–Rus contraction with the coefficient $a = 0$.

The conclusion follows by applying Theorem 2.14. □

Theorem 2.17 (Bianchini [62]) *Let (X, d) be a complete metric space and $T : X \to X$ a Bianchini mapping. Then T is a Picard operator.*

Proof By Example 2.5, T is a graphic contraction with $\alpha = h$. Hence, by Lemma 2.12, there exists an approximate fixed point sequence $\{x_n\}$ with respect to T, i.e., a sequence $\{x_n\} \subset X$ such that

$$d(x_n, Tx_n) \to 0, \text{ as } n \to \infty. \tag{19}$$

Now, for n, m positive integers, by the Bianchini contraction condition (6) we have

$$d(x_n, x_m) \leq h \max\{d(x_n, Tx_n), d(x_m, Tx_m)\} + d(x_n, Tx_n) + d(x_m, Tx_m),$$

which, by (19), shows that $\{x_n\}$ is a Cauchy sequence. Let

$$\lim_{n\to\infty} x_n = p. \tag{20}$$

Again by the Bianchini contraction condition (6), we get

$$d(p, Tp) \leq d(x_n, p) + d(x_n, Tx_n) + h \max\{d(x_n, Tx_n), d(p, Tp)\}.$$

Now, if $\max\{d(x_n, Tx_n), d(p, Tp)\} = d(x_n, Tx_n)$, then it follows that

$$d(p, Tp) \leq d(x_n, p) + (h + 1) \cdot d(x_n, Tx_n)$$

which, by (19) and (20), proves that $Tp = p$.

If $\max\{d(x_n, Tx_n), d(p, Tp)\} = d(p, Tp)$, then

$$d(p, Tp) \leq \frac{1}{1-h}(d(x_n, p) + d(x_n, Tx_n)),$$

which, by (19) and (20), also proves that $Tp = p$, i.e., $Fix\,(T) \neq \emptyset$.

Assume that $q \neq p$ is another fixed point of T. Then, by (6)

$$0 < d(p, q) = d(Tp, Tq) \leq h \max\{d(p, p), d(q, q)\} = 0,$$

a contradiction. This proves that $Fix\,(T) = \{p\}$.

Now, let $\{y_n\} \subset X$ be defined by $y_0 \in X$ and

$$y_{n+1} = f(y_n), n \geq 0. \tag{21}$$

Then, by Example 2.5, one obtains

$$d(y_{n+1}, p) \leq hd(y_n, p), n \geq 0$$

which, by induction, yields

$$d(y_n, p) \leq h^n d(y_0, p), n \geq 0, \tag{22}$$

and this proves that $\{y_n\}$ converges to p, for any starting point $y_0 \in X$.

□

Theorem 2.18 (Chatterjea [17]) *Let (X, d) be a complete metric space and $T : X \to X$ a Chatterjea mapping. Then T is a Picard operator.*

Proof By Example 2.6, T is a graphic contraction with $\alpha = \dfrac{c}{1-c} < 1$. Hence, by Lemma 2.12, there exists an approximate fixed point sequence $\{x_n\}$ with respect to T, i.e., a sequence $\{x_n\} \subset X$ such that

$$d(x_n, Tx_n) \to 0, \text{ as } n \to \infty. \tag{23}$$

By Chatterjea contraction condition (7) and for n, m positive integers, we get

$$d(x_n, x_m) \leq \frac{c}{1-2c} \cdot (d(x_n, Tx_n) + d(x_m, Tx_m)),$$

which, by (23), shows that $\{x_n\}$ is a Cauchy sequence. Let

$$\lim_{n\to\infty} x_n = p. \tag{24}$$

Again, by the Chatterjea contraction condition (7) we get

$$d(p, Tp) \leq \frac{2c+1}{1-c} \cdot d(x_n, p) + \frac{c+1}{1-c} \cdot d(x_n, Tx_n),$$

which, by (23) and (24), proves that $Tp = p$, i.e., $Fix\,(T) \neq \emptyset$.

Assume that $q \neq p$ is another fixed point of T. Then, by (7)

$$0 < d(p, q) = d(Tp, Tq) \leq 2c \cdot d(p, q) < d(p, q),$$

a contradiction. This proves that $Fix\,(T) = \{p\}$.

Now, let $\{y_n\} \subset X$ be defined by $y_0 \in X$ and

$$y_{n+1} = f(y_n), n \geq 0. \tag{25}$$

Then, by Example 2.6 one obtains

$$d(y_{n+1}, p) \leq \alpha \cdot d(y_n, p), n \geq 0$$

which, by induction, yields

$$d(y_n, p) \leq \alpha^n \cdot d(y_0, p), n \geq 0, \tag{26}$$

and this proves that $\{y_n\}$ converges to p, for any starting point $y_0 \in X$.

□

Theorem 2.19 (Zamfirescu [66]) *Let (X, d) be a complete metric space and $T : X \to X$ a Zamfirescu mapping. Then T is a Picard operator.*

Proof By Example 2.7, T is a graphic contraction with

$$\alpha = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\} < 1.$$

Hence, by Lemma 2.12, there exists an approximate fixed point sequence $\{x_n\}$ with respect to T, i.e., a sequence $\{x_n\} \subset X$ such that

$$d(x_n, Tx_n) \to 0, \text{ as } n \to \infty. \tag{27}$$

Now, if for $x_n, x_m \in X$ and T we have condition (i) in Example 2.7 satisfied, then

$$d(x_n, x_m) \le \frac{1}{1-a} \cdot (d(x_n, Tx_n) + d(x_m, Tx_m)). \tag{28}$$

If for $x_n, x_m \in X$ and T we have condition (ii) in Example 2.7 satisfied, then

$$d(x_n, x_m) \le (b+1) \cdot (d(x_n, Tx_n) + d(x_m, Tx_m)), \tag{29}$$

while, if for $x_n, x_m \in X$ and T we have condition (iii) in Example 2.7 satisfied, then by the proof of Theorem 2.18 we have

$$d(x_n, x_m) \le \frac{c}{1-2c} \cdot (d(x_n, Tx_n) + d(x_m, Tx_m)). \tag{30}$$

By (27), (28), (29) and (30), we obtain that $\{x_n\}$ is a Cauchy sequence. Let

$$\lim_{n\to\infty} x_n = p. \tag{31}$$

On the other hand, we have

$$d(p, Tp) \le \begin{cases} \frac{1}{1-a} \cdot (d(x_n, p) + d(x_n, Tx_n)), \text{ if, for } x_n \text{ and } p, \text{ (i) holds} \\ \frac{1+b}{1-b} \cdot d(x_n, p) + \frac{1}{1-b} \cdot d(x_n, Tx_n), \text{ if, for } x_n \text{ and } p, \text{ (ii) holds} \\ \frac{1+c}{1-2c} \cdot (d(x_n, p) + d(x_n, Tx_n)), \text{ if, for } x_n \text{ and } p, \text{ (iii) holds,} \end{cases}$$

which, by (27) and (31), proves that $Tp = p$, i.e., $Fix\,(T) \neq \emptyset$.

Assume that $q \neq p$ is another fixed point of T. Then, by considering separately each of the cases (i), (ii) and (iii), we obtain the contradiction

$$0 < d(p, q) = d(Tp, Tq) < d(p, q),$$

which proves that $Fix\,(T) = \{p\}$.

Now, let $\{y_n\} \subset X$ be defined by $y_0 \in X$ and

$$y_{n+1} = Ty_n, n \ge 0. \tag{32}$$

Then, by Example 2.7 one obtains

$$d(y_{n+1}, y_n) \le \alpha \cdot d(y_n, y_{n-1}), n \ge 1$$

which, by induction, yields

$$d(y_{n+p}, y_n) \le \alpha^n \cdot d(y_0, p), n \ge 0, \tag{33}$$

and this proves that $\{y_n\}$ converges to p, for any starting point $y_0 \in X$.

□

Remark 2.20 Note that to prove the fixed point theorem corresponding to almost contractions (Example 2.10), which are weakly Picard operators, we have to use Picard iteration as approximate fixed point sequence and not an arbitrary approximate fixed point sequence as above; see the complete proof in [5].

3 Maia fixed point theorems

One of the most interesting generalizations of the contraction mapping principle is the so-called Maia fixed point theorem, see [43], which was obtained by splitting the assumptions in the contraction mapping principle among two metrics defined on the same set. We provide an alternate proof to this result by using the concept of approximate fixed point sequence.

Theorem 3.1 (Maia [43]) *Let X be a nonempty set, d and ρ two metrics on X and $T : X \to X$ a mapping. Suppose that*

(i) *$d(x, y) \le \rho(x, y)$, for each $x, y \in X$;*
(ii) *(X, d) is a complete metric space;*
(iii) *$T : X \to X$ is continuous with respect to the metric d;*
(iv) *T is a contraction mapping with respect to the metric ρ, with contraction coefficient $a \in [0, 1)$.*

Then T is a Picard operator.

Proof By assumption (iv), T is a graphic contraction with respect to the metric ρ, with $\alpha = a$. Then, by Lemma 2.12, there exists an approximate fixed point sequence $\{x_n\}$ with respect to T, i.e., a sequence $\{x_n\} \subset X$ such that

$$\rho(x_n, Tx_n) \to 0, \quad \text{as } n \to \infty. \tag{34}$$

For this sequence, by the contraction condition

$$\rho(Tx, Ty) \le a\rho(x, y), \; x, y \in X,$$

we obtain

$$\rho(x_n, x_m) \le (1 - a) \cdot (\rho(x_n, Tx_n) + \rho(x_m, Tx_m)),$$

which, by virtue of (34), shows that $\{x_n\}$ is a Cauchy sequence in the metric space (X, ρ).

By (i), $\{x_n\}$ is a Cauchy sequence in the metric space (X, d), too, and by (ii) it follows that it converges with respect to the metric d. Let

$$p = \lim_{n\to\infty} Tx_n \quad (\Longleftrightarrow \lim_{n\to\infty} d(Tx_n, p) = 0).$$

Now, by (iii) we obtain that $p \in Fix\,(T)$ and by (iv) that $Fix\,(T) = \{p\}$. □

Remark 3.2 Theorem 3.1 remains valid if one replaces assumption (i) by the following more general condition:
(i') There exists $C > 0$ such that $d(x, y) \le C \cdot \rho(x, y)$, for each $x, y \in X$.
If we have $\rho \equiv d$, from Theorem 3.1 one obtains the Picard–Banach fixed point principle (Corollary 2.15).

A more general Maia-type result, which generalizes Theorem 2.14, is given by the following:

Theorem 3.3 *Let X be a nonempty set, d and ρ two metrics on X and $T : X \to X$ a mapping. Suppose that*

(i) *there exists $C > 0$ such that $d(x, y) \le C \cdot \rho(x, y)$, for each $x, y \in X$;*
(ii) *(X, d) is a complete metric space;*
(iii) *$T : X \to X$ is continuous with respect to the metric d;*
(iv) *T is a Ćirić–Reich–Rus contraction with respect to the metric ρ, with contraction coefficients $a, b \in [0, 1)$.*

Then T is a Picard operator.

Proof Based on the same arguments like in the proof of Theorem 2.14 and using assumption (iv), we can easily deduce that T is a graphic contraction with respect to the metric ρ, with $\alpha = \dfrac{a+b}{1-b} < 1$.

Then, by Lemma 2.12, there exists an approximate fixed point sequence $\{x_n\}$ with respect to T, i.e., a sequence $\{x_n\} \subset X$ such that

$$\rho(x_n, Tx_n) \to 0, \quad \text{as } n \to \infty. \tag{35}$$

For this sequence, by the Ćirić–Reich–Rus contraction condition

$$\rho(Tx, Ty) \le a\rho(x, y) + b\left(\rho(x, Tx) + \rho(y, Ty)\right), \tag{36}$$

valid for all $x, y \in X$, we obtain

$$\rho(x_n, x_m) \leq \frac{1+b}{1-a} \cdot (\rho(x_n, Tx_n) + \rho(x_m, Tx_m)),$$

which, by virtue of (35), shows that $\{x_n\}$ is a Cauchy sequence in the metric space (X, ρ).

By (i), it follows that $\{x_n\}$ is a Cauchy sequence in the metric space (X, d), too, and by (ii) we deduce that it converges with respect to the metric d. Let

$$p = \lim_{n\to\infty} Tx_n \quad (\Longleftrightarrow \lim_{n\to\infty} d(Tx_n, p) = 0).$$

Now, by (iii) we obtain that $p \in Fix\,(T)$ and by (iv) that $Fix\,(T) = \{p\}$. □

Remark 3.4 Note that in the case of Ćirić–Reich–Rus contractions, condition (ii) in Theorem 3.3 is not always satisfied, because these mappings are in general not continuous, see the examples in [11].

If in Theorem 3.3 we have $\rho \equiv d$, then one obtains the Ćirić–Reich–Rus fixed point theorem (Theorem 2.14).

4 Conclusions

1. We presented simple and unified alternative proofs, based on the concepts of *graphic contraction* and *approximate fixed point sequence*, for some classic metric fixed point theorems emerging from Picard–Banach contraction mapping principle: Kannan fixed point theorem (Kannan [34]); Ćirić-Reich-Rus fixed point theorem (Ćirić [24], Reich [51], Rus [55]); Bianchini fixed point theorem (Bianchini [62]); Chatterjea fixed point theorem (Chatterjea [17]) and Maia's fixed point theorem (Maia [43]).
2. Similar proofs could be given for other important classes of contractive-type mappings that are related to the Banach contractions: strong Ćirić quasi contractions (see Example 2.8); Hardy and Rogers contractions (see Example 2.9) etc. which are left as exercises for the reader.
3. In connection with Maia-type fixed point theorems, it is an open problem to find weaker conditions than the continuity of the mapping T involved in Theorems 3.1 and 3.3.
4. The technique of proof used in the present paper, essentially based on the concepts of graphic contraction and approximate fixed point sequence, could also be nontrivially applied to other classes of self and nonself single-valued mappings in the literature on metric fixed point theory, see [5,7–12,14,20,23,28,31,36,38, 39,42,44,46,50,60,61,63] etc.
5. There exists another important technique for proving metric fixed point theorems which is based on the property of asymptotic regularity of the mappings, see [14,29–31], and which is naturally closely related but independent to the technique emphasized in the current paper, in view of Theorem 3.1 in [13], which shows that, for a nonempty set X and a mapping $T : X \to X$, the following statements are equivalent:
   (a) there exists a complete metric on X with respect to which T is a continuous graphic contraction;
   (b) $Fix\,(T) \neq \emptyset$ and there exists a metric on X with respect to which T is asymptotically regular.

So, by also having in view Remark 2.13 (3), it would be very important to compare directly the two methods, the one based on graphic contractions (and approximate fixed point sequences) and the other based on asymptotical regularity, for some concrete classes of mappings to establish, if possible, which one is more reliable.

For example, in the case of Kannan mappings, one can compare the proof of Corollary 2.16 to the proof of the corresponding result in [29]–[31] and conclude that the two methods exhibit slightly different facets of the fixed point problem under study.

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Declarations

Data availability Not applicable.

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