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Some homotypical closed varieties of semigroups

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Abstract We show that for each $n \geq 2 \in \mathbb{N}$, the varieties $\mathbb{V}_n = [x_1x_2x_3 = x_1^n x_{i_1}x_{i_2}x_{i_3}]$ where i is any non-trivial permutation of $\{1, 2, 3\}$ are closed. Further, we show that for each $n \in \mathbb{N}$, the varieties $\mathcal{V}_n = [x_1x_2x_3 = x_1^n x_{i_1}x_{i_2}x_{i_3}]$ where i is any non-trivial permutation of $\{1, 2, 3\}$ other than the permutation (231) are closed.

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1 Introduction and preliminaries

Let U be a subsemigroup of a semigroup S. Following Isbell [10], we say that U *dominates* an element d of S, if for every semigroup T and all morphisms $\beta, \gamma : S \to T$, $u\beta = u\gamma$, for all $u \in U$ implies that $\mathrm{d}\beta = \mathrm{d}\gamma$. The set of all elements of S dominated by U is called the *dominion* of U in S and we denote it by $Dom(U, S)$. It can be easily verified that $Dom(U, S)$ is a subsemigroup of S containing U. A subsemigroup U of a semigroup S is said to be *closed* in S if $Dom(U, S) = U$ and it is said to be *absolutely closed* if it is closed in every containing semigroup S.

A class $\mathcal{V}$ of semigroups is a *variety of semigroups* if it is closed under subsemigroups, morphic images and arbitrary direct products. Also it is well known that each variety of semigroups is an equational class and vice versa Howie [9]. Therefore $\mathcal{V}$ is a variety if and only if each member of $\mathcal{V}$ satisfies certain finite or infinite identities of fixed type. If the variety $\mathcal{V}$ of semigroups satisfies identities $u_i = v_i$ $(i \geq 1)$, then we write it as $\mathcal{V} = [u_i = v_i : i \geq 1]$. For example some well known varieties of semigroups are the variety $\mathcal{V}_1 = [xy = yx]$ of all commutative semigroups, the variety $\mathcal{V}_2 = [x^2 = x]$ of all bands and the variety $\mathcal{V}_3 = [x_1x_2 \ldots x_n = x_{i_1}x_{i_2} \ldots x_{i_n}]$ of all permutative semigroups, where i is a non-trivial permutation on $\{1, 2, \ldots, n\}$. Let $u(x_1, x_2, \ldots, x_n) = v(x_1, x_2, \ldots, x_n)$ be an identity with variables $x_1, x_2, \ldots, x_n$. Then the identity $u(x_1, x_2, \ldots, x_n) = v(x_1, x_2, \ldots, x_n)$ is called *homotypical* if $c(u) = c(v)$, where $c(w)$, for any word w, is the set of variables appearing in w. A semigroup variety admitting a homotypical identity is called a *homotypical* variety and *heterotypical* otherwise.

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We say that a variety $\mathcal{V}$ is closed if for all $U, S \in \mathcal{V}$ with U as a subsemigroup of S, $Dom(U, S) = U$. Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two varieties of semigroups such that $\mathcal{V}_1$ is a subvariety of $\mathcal{V}_2$. Then we say that $\mathcal{V}_1$ is $\mathcal{V}_2$-closed if for every $U \in \mathcal{V}_1$ and $S \in \mathcal{V}_2$ with U as subsemigroup of S, $Dom(U, S) = U$.

Isbell [10], provided the most useful characterization of semigroup dominions which is called as Isbell's Zigzag Theorem and is as follows:

Theorem 1.1 ([9] Theorem 8.3.5) *Let U be a subsemigroup of a semigroup S and $d \in S$. Then $d \in Dom(U, S)$ if and only if $d \in U$ or there exists a system of equalities for d as follows.*

$$\begin{aligned} d &= a_0 y_1 & a_0 &= x_1 a_1 \\ a_{2i-1} y_i &= a_{2i} y_{i+1} & x_i a_{2i} &= x_{i+1} a_{2i+1} \quad (i = 1, 2, \ldots, m-1) \\ a_{2m-1} y_m &= a_{2m} & x_m a_{2m} &= d, \end{aligned} \tag{1.1}$$

where $a_i \in U$ $(0 \le i \le 2m)$ and $x_i, y_i \in S$ $(1 \le i \le m)$.

The above system of equalities (1.1) is called as the *zigzag of length m* in S over U with value d. In whatever follows, by zigzag equations, we shall mean equations of the type (1.1). We further mention that the bracketed statements shall mean the statements dual to other.

2 Homotypical closed varieties of semigroups

Scheiblich [11], had shown that the variety of normal bands was closed. However, the variety of normal bands was not absolutely closed as Higgins [7] had shown that the variety of left [right] normal bands was not absolutely closed. Howie [8] had shown that the variety of rectangular bands was not absolutely closed. Therefore finding closed varieties of bands remains an open problem for quite long. Recently this problem gained attention as Alam and Khan [6], Ahanger and Shah [2], and Ahanger et al. [3], have extended the above results. However, the problem of finding closed varieties of bands has not been completely settled. Since any band variety is a homotypical variety, so it becomes interesting to find closed homotypical varieties of semigroups. In this paper we find some closed homotypical varieties of semigroups.

In [1], Shabnam et al. have shown that the variety of semigroups satisfying the identity $x_1x_2x_3 = x_1x_1x_3x_2$ is closed. In the next proposition, we show that for each $n \in \mathbb{N}$, the variety of semigroups satisfying an identity of the type $x_1x_2x_3 = x_1^n x_1x_3x_2$ is closed and have generalized the above result.

Lemma 2.1 *For each $n \in \mathbb{N}$, let $\mathcal{V}_n = [x_1x_2x_3 = x_1^n x_1x_3x_2]$ and let $S \in \mathcal{V}_n$. Then S also satisfies the identity $xyz = xy^{n+1}z$.*

Proof For any $x, y, z \in S$, we have

$$\begin{aligned} xyz &= x^n(x)(z)(y) \quad (\text{since } S \text{ is in } \mathcal{V}_n) \\ &= x^n x^n xyz \quad (\text{since } S \text{ is in } \mathcal{V}_n) \\ &= x^n x^{n-1}(x)(x)(y)z \\ &= x^{2n-1}x^{n+1}yxz \quad (\text{since } S \text{ is in } \mathcal{V}_n) \\ &= x^{3n}(y)(x)(z) \\ &= x^{3n}y^{n+1}zx \quad (\text{since } S \text{ is in } \mathcal{V}_n) \\ &= x^{2n-1}(x^n)(x)\left(y^{n+1}\right)(zx) \\ &= x^{2n-1}xzxy^{n+1} \quad (\text{since } S \text{ is in } \mathcal{V}_n) \\ &= x^{n-1}(x^n)(x)(z)(x)y^{n+1} \\ &= (x^n)(x)(z)\left(y^{n+1}\right) \quad (\text{since } S \text{is in } \mathcal{V}_n) \\ &= xy^{n+1}z \quad (\text{since } S \text{ is in } \mathcal{V}_n)\,. \end{aligned}$$

□

Proposition 2.2 *For each $n \in \mathbb{N}$, the variety $\mathcal{V}_n = [x_1x_2x_3 = x_1^n x_1x_3x_2]$ of semigroups is closed.*

Proof Let S, U be in $\mathcal{V}_n$ such that U is a subsemigroup of S. Take any $d \in Dom(U, S) \setminus U$ and let (1.1) be zigzag equations in S over U with value d of length m.

To prove the proposition we first prove the following lemma.

Lemma 2.3 *For all* $k = 1, 2, \ldots, m-1$,

$$d = \left(\prod_{i=1}^{k} x_i^n\right) x_{k+1} \left(\prod_{i=1}^{k+1} a_{2k-(2i-3)}\right) \left(\prod_{i=1}^{k} a_{2i-1}^{n-1}\right) y_{k+1}.$$

Proof We prove the lemma by applying induction on k. For $k = 1$, we have

$$\begin{aligned}
d &= x_1 a_1 y_1 \quad \text{(by zigzag equations)}\\
&= x_1 a_1^{n+1} y_1 \quad \text{(by Lemma 2.1)}\\
&= x_1 a_1^n a_2 y_2 \quad \text{(by zigzag equations)}\\
&= x_1^n x_1 a_2 a_1^n y_2 \quad (\text{since } S \text{ is in} \mathcal{V}_n)\\
&= x_1^n x_2 a_3 a_1^n y_2 \quad \text{(by zigzag equations)}\\
&= x_1^n x_2 a_3 a_1 a_1^{n-1} y_2.
\end{aligned}$$

Hence the result holds for $k = 1$. Assume inductively that the result holds for $k = l < m-1$. We prove that the result also holds for $k = l + 1$. Now

$$\begin{aligned}
d &= \left(\prod_{i=1}^{l} x_i^n\right) x_{l+1} \left(\prod_{i=1}^{l+1} a_{2l-(2i-3)}\right) \left(\prod_{i=1}^{l} a_{2i-1}^{n-1}\right) y_{l+1} \quad \text{(by inductive hypothesis)}\\
&= \left(\prod_{i=1}^{l} x_i^n\right) x_{l+1} a_{2l+1} \left(\prod_{i=2}^{l+1} a_{2l-(2i-3)}\right) \left(\prod_{i=1}^{l} a_{2i-1}^{n-1}\right) y_{l+1}\\
&= \left(\prod_{i=1}^{l} x_i^n\right) x_{l+1} a_{2l+1}^{n+1} \left(\prod_{i=2}^{l+1} a_{2l-(2i-3)}\right) \left(\prod_{i=1}^{l} a_{2i-1}^{n-1}\right) y_{l+1} \quad \text{(by Lemma 2.1)}\\
&= \left(\prod_{i=1}^{l} x_i^n\right) x_l a_{2l} a_{2l+1}^n \left(\prod_{i=2}^{l+1} a_{2l-(2i-3)}\right) \left(\prod_{i=1}^{l} a_{2i-1}^{n-1}\right) y_{l+1} \quad \text{(by zigzag equations)}\\
&= \left(\prod_{i=1}^{l-1} x_i^n\right) x_l^n x_l a_{2l} a_{2l+1}^n \left(\prod_{i=2}^{l+1} a_{2l-(2i-3)}\right) \left(\prod_{i=1}^{l} a_{2i-1}^{n-1}\right) y_{l+1}\\
&= \left(\prod_{i=1}^{l-1} x_i^n\right) x_l \left(\prod_{i=2}^{l+1} a_{2l-(2i-3)}\right) \left(\prod_{i=1}^{l} a_{2i-1}^{n-1}\right) a_{2l} a_{2l+1}^n y_{l+1} \quad (\text{since } S \text{ is in } \mathcal{V}_n)\\
&= \left(\prod_{i=1}^{l-1} x_i^n\right) x_l^n x_l a_{2l} \left(\prod_{i=2}^{l+1} a_{2l-(2i-3)}\right) \left(\prod_{i=1}^{l} a_{2i-1}^{n-1}\right) a_{2l+1}^n y_{l+1} \quad (\text{since } S \text{ is in } \mathcal{V}_n)\\
&= \left(\prod_{i=1}^{l} x_i^n\right) x_{l+1} \left(\prod_{i=1}^{l+1} a_{2l-(2i-3)}\right) \left(\prod_{i=1}^{l} a_{2i-1}^{n-1}\right) a_{2l+1}^{n-1} a_{2l+2} y_{l+2} \quad \text{(by zigzag equations)}\\
&= \left(\prod_{i=1}^{l} x_i^n\right) x_{l+1}^n x_{l+1} a_{2l+2} \left(\prod_{i=1}^{l+1} a_{2l-(2i-3)}\right) \left(\prod_{i=1}^{l} a_{2i-1}^{n-1}\right) a_{2l+1}^{n-1} y_{l+2} \quad (\text{since } S \text{ is in } \mathcal{V}_n)\\
&= \left(\prod_{i=1}^{l+1} x_i^n\right) x_{l+2} a_{2l+3} \left(\prod_{i=1}^{l+1} a_{2l-(2i-3)}\right) \left(\prod_{i=1}^{l} a_{2i-1}^{n-1}\right) a_{2l+1}^{n-1} y_{l+2} \quad \text{(by zigzag equations)}\\
&= \left(\prod_{i=1}^{l+1} x_i^n\right) x_{l+2} \left(\prod_{i=1}^{l+2} a_{2(l+1)-(2i-3)}\right) \left(\prod_{i=1}^{l+1} a_{2i-1}^{n-1}\right) y_{l+2},
\end{aligned}$$

as required. □

Now to complete the proof of proposition, we have

$$
\begin{aligned}
d &= \left(\prod_{i=1}^{m-1} x_i^n\right) x_m \left(\prod_{i=1}^{m} a_{2m-(2i-1)}\right)\left(\prod_{i=1}^{m-1} a_{2i-1}^{n-1}\right) y_m \quad \text{(by Lemma 2.3)}\\
&= \left(\prod_{i=1}^{m-1} x_i^n\right) x_m a_{2m-1}^{n+1} \left(\prod_{i=2}^{m} a_{2m-(2i-1)}\right)\left(\prod_{i=1}^{m-1} a_{2i-1}^{n-1}\right) y_m \quad \text{(by Lemma 2.1)}\\
&= \left(\prod_{i=1}^{m-1} x_i^n\right) x_{m-1} a_{2m-2} a_{2m-1}^{n} \left(\prod_{i=2}^{m} a_{2m-(2i-1)}\right)\left(\prod_{i=1}^{m-1} a_{2i-1}^{n-1}\right) y_m \quad \text{(by zigzag equations)}\\
&= \left(\prod_{i=1}^{m-2} x_i^n\right) x_{m-1}^n x_{m-1} a_{2m-2} a_{2m-1}^{n} \left(\prod_{i=2}^{m} a_{2m-(2i-1)}\right)\left(\prod_{i=1}^{m-1} a_{2i-1}^{n-1}\right) y_m\\
&= \left(\prod_{i=1}^{m-2} x_i^n\right) x_{m-1} \left(\prod_{i=2}^{m} a_{2m-(2i-1)}\right)\left(\prod_{i=1}^{m-1} a_{2i-1}^{n-1}\right) a_{2m-2} a_{2m-1}^{n} y_m \quad \text{(since } S \text{ is in } \mathcal{V}_n)\\
&= \left(\prod_{i=1}^{m-2} x_i^n\right) x_{m-2} a_{2m-4} \left(\prod_{i=3}^{m} a_{2m-(2i-1)}\right)\left(\prod_{i=1}^{m-1} a_{2i-1}^{n-1}\right) a_{2m-2} a_{2m-1}^{n-1} a_{2m} \quad \text{(by zigzag equations)}\\
&= \left(\prod_{i=1}^{m-3} x_i^n\right) x_{m-2}^n x_{m-2} a_{2m-4} \left(\prod_{i=3}^{m} a_{2m-(2i-1)}\right)\left(\prod_{i=1}^{m-2} a_{2i-1}^{n-1}\right) a_{2m-3}^{n-1} a_{2m-2} a_{2m-1}^{n-1} a_{2m}\\
&= \left(\prod_{i=1}^{m-3} x_i^n\right) x_{m-2} \left(\prod_{i=3}^{m} a_{2m-(2i-1)}\right)\left(\prod_{i=1}^{m-2} a_{2i-1}^{n-1}\right) a_{2m-4} a_{2m-3}^{n-1} a_{2m-2} a_{2m-1}^{n-1} a_{2m} \quad \text{(since } S \text{ is in } \mathcal{V}_n)\\
&= \left(\prod_{i=1}^{m-3} x_i^n\right) x_{m-3} a_{2m-6} \left(\prod_{i=4}^{m} a_{2m-(2i-1)}\right)\left(\prod_{i=1}^{m-2} a_{2i-1}^{n-1}\right) a_{2m-4} a_{2m-3}^{n-1} a_{2m-2} a_{2m-1}^{n-1} a_{2m}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by zigzag equations)}\\
&\ \vdots\\
&= x_1^n x_1 a_2 a_1 a_1^{n-1} \left(\prod_{i=2}^{m} a_{2i-1}^{n-1} a_{2i}\right)\\
&= \left(x_1^n\right)(x_1)(a_2)\left(a_1^n\right)\left(\prod_{i=2}^{m} a_{2i-1}^{n-1} a_{2i}\right)\\
&= x_1 a_1^n a_2 \left(\prod_{i=2}^{m} a_{2i-1}^{n-1} a_{2i}\right) \quad \text{(since } S \text{ is in } \mathcal{V}_n)\\
&= a_0 a_1^{n-1} a_2 \left(\prod_{i=2}^{m} a_{2i-1}^{n-1} a_{2i}\right) \quad \text{(by zigzag equations)}\\
&= a_0 \left(\prod_{i=1}^{m} a_{2i-1}^{n-1} a_{2i}\right).
\end{aligned}
$$

Thus $d \in U$ as required. □

In the next proposition, we show that for each $n \in \mathbb{N}$, the variety of semigroups satisfying an identity of the type $x_1x_2x_3 = x_1^n x_3 x_2 x_1$ is closed.

Lemma 2.4 *For each $n \in \mathbb{N}$, let $\mathcal{V}_n = [x_1x_2x_3 = x_1^n x_3x_2x_1]$ and $S \in \mathcal{V}_n$. Then S also satisfies the identity $xyz = xyzy^n$.*

Proof For any $x, y, z \in S$, we have

$$\begin{aligned} xyz &= x^n zyx \quad (\text{since } S \text{ is in } \mathcal{V}_n) \\ &= x^{n-1}(x)(z)(yx) \\ &= x^{n-1}x^n yxzx \quad (\text{since } S \text{ is in } \mathcal{V}_n) \\ &= x^{n-1}x^n (y)(x)(z)x \\ &= x^{n-1}(x^n)\left(y^n\right)(z)(x)yx \quad (\text{since } S \text{ is in } \mathcal{V}_n) \\ &= \left(x^n\right)\left(zy^n\right)(y)(x) \quad (\text{since } S \text{ is in } \mathcal{V}_n) \\ &= xyzy^n \quad (\text{since } S \text{ is in } \mathcal{V}_n). \end{aligned}$$

□

Proposition 2.5 *For each $n \in \mathbb{N}$, the variety $\mathcal{V}_n = [x_1x_2x_3 = x_1^n x_3x_2x_1]$ of semigroups is closed.*

Proof Let S, U be in $\mathcal{V}_n$ such that U is a subsemigroup of S. Take any $d \in Dom(U, S) \setminus U$ and let (1.1) be zigzag equations in S over U with value d of length m.

To prove the proposition we first prove the following lemma.

Lemma 2.6 *For all $k = 1, 2, \ldots, m-1$,*

$$d = d\left(\prod_{i=1}^{k} a_{2i-1}^n\right).$$

Proof We prove the lemma by applying induction on k. For $k = 1$, we have

$$\begin{aligned} d &= x_1a_1y_1 \quad (\text{by zigzag equations}) \\ &= x_1a_1y_1a_1^n \quad (\text{by Lemma 2.4}) \\ &= da_1^n \quad (\text{by zigzag equations}). \end{aligned}$$

Hence the result holds for $k = 1$. Assume inductively that the result holds for $k = l < m - 1$. We will show that the result also holds for $k = l + 1$. Now

$$\begin{aligned} d &= d\left(\prod_{i=1}^{l} a_{2i-1}^n\right) \quad (\text{by inductive hypothesis}) \\ &= x_{l+1}a_{2l+1}y_{l+1}\left(\prod_{i=1}^{l} a_{2i-1}^n\right) \quad (\text{by zigzag equations}) \\ &= x_{l+1}a_{2l+1}y_{l+1}\left(\prod_{i=1}^{l} a_{2i-1}^n\right)a_{2l+1}^n \quad (\text{by Lemma 2.4}) \\ &= d\left(\prod_{i=1}^{l+1} a_{2i-1}^n\right) \quad (\text{by zigzag equations}), \end{aligned}$$

as required. □

Now to complete the proof of proposition, we have

$$\begin{aligned} d &= d\left(\prod_{i=1}^{m} a_{2i-1}^n\right) \quad (\text{by Lemma 2.6}) \\ &= a_0y_1a_1^n\left(\prod_{i=2}^{m} a_{2i-1}^n\right) \quad (\text{by zigzag equations}) \end{aligned}$$

$$= a_0 y_1 a_1 a_1^{n-1} \left(\prod_{i=2}^{m} a_{2i-1}^n \right)$$

$$= a_0^n a_1 y_1 a_0 a_1^{n-1} \left(\prod_{i=2}^{m} a_{2i-1}^n \right) \quad \text{(since } S \text{ is in } \mathcal{V}_n\text{)}$$

$$= a_0^n a_2 y_2 a_0 a_1^{n-1} a_3^n \left(\prod_{i=3}^{m} a_{2i-1}^n \right) \quad \text{(by zigzag equations)}$$

$$= a_0^n \, (a_2) \left(y_2 a_0 a_1^{n-1} \right) (a_3) a_3^{n-1} \left(\prod_{i=3}^{m} a_{2i-1}^n \right)$$

$$= a_0^n a_2^n a_3 y_2 a_0 a_1^{n-1} a_2 a_3^{n-1} \left(\prod_{i=3}^{m} a_{2i-1}^n \right) \quad \text{(since } S \text{ is in } \mathcal{V}_n\text{)}$$

$$= a_0^n a_2^n a_4 y_3 a_0 a_1^{n-1} a_2 a_3^{n-1} \left(\prod_{i=3}^{m} a_{2i-1}^n \right) \quad \text{(by zigzag equations)}$$

$$\vdots$$

$$= \left(a_0^n a_2^n \ldots a_{2m-4}^n \right) a_{2m-2} y_m \left(a_0 a_1^{n-1} a_2 a_3^{n-1} \ldots a_{2m-4} a_{2m-3}^{n-1} \right) a_{2m-1}^n$$

$$= \left(\prod_{i=1}^{m-1} a_{2i-2}^n \right) (a_{2m-2}) \left(y_m (\prod_{i=1}^{m-1} a_{2i-2} a_{2i-1}^{n-1} \right) (a_{2m-1}) \, a_{2m-1}^{n-1}$$

$$= \left(\prod_{i=1}^{m-1} a_{2i-2}^n \right) a_{2m-2}^n a_{2m-1} y_m \left(\prod_{i=1}^{m-1} a_{2i-2} a_{2i-1}^{n-1} \right) a_{2m-2} a_{2m-1}^{n-1} \quad \text{(since } S \text{ is in } \mathcal{V}_n\text{)}$$

$$= \left(\prod_{i=1}^{m} a_{2i-2}^n \right) a_{2m} \left(\prod_{i=1}^{m} a_{2i-2} a_{2i-1}^{n-1} \right) \quad \text{(by zigzag equations),}$$

which is in U as required. □

In [5], Alam and Khan have shown that the variety of semigroups satisfying an identity $x_1x_2x_3 = x_1x_2x_1x_3$ is closed. In the next proposition, we extend this result by proving that for each $n \in \mathbb{N}$, the variety of semigroups determined by identities of the type $x_1x_2x_3 = x_1^n x_2 x_1 x_3$ is closed.

Proposition 2.7 *For each $n \in \mathbb{N}$, the variety $\mathcal{V}_n = [x_1x_2x_3 = x_1^n x_2 x_1 x_3]$ of semigroups is closed.*

Proof let S, U be in $\mathcal{V}_n$ such that U is a subsemigroup of S. Take any $d \in Dom(U, S) \setminus U$ and let (1.1) be zigzag equations in S over U with value d of length m.

To prove the proposition, we first prove the following lemma.

Lemma 2.8 *For all $k = 1, 2, \ldots, m$, $d = (\prod_{i=1}^{k} x_i^n a_{2i-1}^n)d$.*

Proof We prove the result by applying induction on k. For $k = 1$,

$$\begin{aligned} d &= x_1 a_1 y_1 \quad \text{(by zigzag equations)} \\ &= x_1^n a_1 x_1 y_1 \quad \text{(since } S \text{ is in } \mathcal{V}_n\text{)} \\ &= x_1^n a_1^n x_1 a_1 y_1 \quad \text{(since } S \text{ is in } \mathcal{V}_n\text{)} \\ &= x_1^n a_1^n d \quad \text{(by zigzag equations),} \end{aligned}$$

as required. Assume that the result holds for $k = l < m$. We will show the result also holds for $k = l + 1$. Now

$$d = \left(\prod_{i=1}^{l} x_i^n a_{2i-1}^n \right) d \quad \text{(by inductive hypothesis)}$$

$$= \left(\prod_{i=1}^{l} x_i^n a_{2i-1}^n\right) x_{l+1} a_{2l+1} y_{l+1} \text{ (by zigzag equations)}$$
$$= \left(\prod_{i=1}^{l} x_i^n a_{2i-1}^n\right) x_{l+1}^n a_{2l+1} x_{l+1} y_{l+1} \quad \text{(since } S \text{ is in } \mathcal{V}_n\text{)}$$
$$= \left(\prod_{i=1}^{l} x_i^n a_{2i-1}^n\right) x_{l+1}^n a_{2l+1}^n x_{l+1} a_{2l+1} y_{l+1} \quad \text{(since } S \text{ is in } \mathcal{V}_n\text{)}$$
$$= \left(\prod_{i=1}^{l+1} x_i^n a_{2i-1}^n\right) d \text{ (by zigzag equations),}$$

as required. □

Now to complete the proof of proposition, we have

$$d = \left(\prod_{i=1}^{m} x_i^n a_{2i-1}^n\right) d \text{ (by Lemma 2.8)}$$
$$= \left(\prod_{i=1}^{m-1} x_i^n a_{2i-1}^n\right) x_m^n a_{2m-1}^n x_m a_{2m} \text{ (by zigzag equations)}$$
$$= \left(\prod_{i=1}^{m-1} x_i^n a_{2i-1}^n\right) x_m a_{2m-1}^n a_{2m} \quad \text{(since } S \text{ is in } \mathcal{V}_n\text{)}$$
$$= \left(\prod_{i=1}^{m-1} x_i^n a_{2i-1}^n\right) x_{m-1} a_{2m-2} a_{2m-1}^{n-1} a_{2m} \text{ (by zigzag equations)}$$
$$= \left(\prod_{i=1}^{m-2} x_i^n a_{2i-1}^n\right) x_{m-1}^n a_{2m-3}^n x_{m-1} a_{2m-2} a_{2m-1}^{n-1} a_{2m}$$
$$= \left(\prod_{i=1}^{m-2} x_i^n a_{2i-1}^n\right) x_{m-1} a_{2m-3}^n a_{2m-2} a_{2m-1}^{n-1} a_{2m} \quad \text{(since } S \text{ is in } \mathcal{V}_n\text{)}$$
$$= \left(\prod_{i=1}^{m-2} x_i^n a_{2i-1}^n\right) x_{m-2} a_{2m-4} a_{2m-3}^{n-1} a_{2m-2} a_{2m-1}^{n-1} a_{2m} \text{ (by zigzag equations)}$$
$$\vdots$$
$$= \left(x_1^n\right)\left(a_1^n\right)(x_1)(a_2)\left(\prod_{i=2}^{m} a_{2i-1}^{n-1} a_{2i}\right)$$
$$= x_1 a_1^n a_2 \left(\prod_{i=2}^{m} a_{2i-1}^{n-1} a_{2i}\right) \quad \text{(since } S \text{ is in } \mathcal{V}_n\text{)}$$
$$= a_0 a_1^{n-1} a_2 \left(\prod_{i=2}^{m} a_{2i-1}^{n-1} a_{2i}\right) \text{ (by zigzag equations)}$$
$$= a_0 \left(\prod_{i=1}^{m} a_{2i-1}^{n-1} a_{2i}\right).$$

Thus $d \in U$ as required. □

In the next proposition, we show that the variety of semigroups satisfying an identity of the type $x_1x_2x_3 = x_1^n x_3 x_1 x_2\ \forall n \in \mathbb{N}$) is closed.

Lemma 2.9 *For each $n \in \mathbb{N}$, let $\mathcal{V}_n = [x_1x_2x_3 = x_1^n x_3x_1x_2]$ and $S \in \mathcal{V}_n$. Then S also satisfies the identity $xyz = xyzy^{2n}$.*

Proof For any $x, y, z \in S$, we have

$$\begin{aligned} xyz &= x^n(z)(x)(y) \quad (\text{since } S \text{is in } \mathcal{V}_n)\\ &= x^n z^n(y)(z)(x) \quad (\text{since } S \text{is in } \mathcal{V}_n)\\ &= x^n z^n y^n xyz \quad (\text{since } S \text{is in } \mathcal{V}_n)\\ &= x^n z^n y^{n-1}(y)(xy)(z)\\ &= \left(x^n\right)\left(z^n y^{2n-1} zy\right)(x)(y) \quad (\text{since } S \text{ is in} \mathcal{V}_n)\\ &= xy\left(z^n\right)\left(y^{2n-1}\right)(z)(y) \quad (\text{since } S \text{ is in} \mathcal{V}_n)\\ &= xyzyy^{2n-1} \quad (\text{since } S \text{ is in } \mathcal{V}_n)\\ &= xyzy^{2n}. \end{aligned}$$

□

Proposition 2.10 *For each $n \in \mathbb{N}$, the variety $\mathcal{V}_n = [x_1x_2x_3 = x_1^n x_3x_1x_2]$ of semigroups is closed.*

Proof Let S, U be in $\mathcal{V}_n$ such that U is a subsemigroup of S. Take any $d \in Dom(U, S) \setminus U$ and let (1.1) be zigzag equations in S over U with value d of length m.

To prove the proposition we first prove the following lemma.

Lemma 2.11 *For all $k = 1, 2, \ldots, m$, $d = d(\prod_{i=1}^{k} a_{2k-(2i-1)}^{2n})$.*

Proof We prove the lemma by applying induction on k. For $k = 1$, we have

$$\begin{aligned} d &= x_1a_1y_1 \quad (\text{by zigzag equations})\\ &= x_1a_1y_1a_1^{2n} \quad (\text{by Lemma 2.9})\\ &= da_1^{2n} \quad (\text{by zigzag equations}). \end{aligned}$$

Therefore, the result holds for $k = 1$. Assume inductively that the result holds for $k = l < m$. We will show that the result also holds for $k = l + 1$. Now

$$\begin{aligned} d &= d\left(\prod_{i=1}^{l} a_{2l-(2i-1)}^{2n}\right) \quad (\text{by inductive hypothesis})\\ &= x_{l+1}a_{2l+1}y_{l+1}\left(\prod_{i=1}^{l} a_{2l-(2i-1)}^{2n}\right) \quad (\text{by zigzag equations})\\ &= x_{l+1}a_{2l+1}y_{l+1}a_{2l+1}^{2n}\left(\prod_{i=1}^{l} a_{2l-(2i-1)}^{2n}\right) \quad (\text{by Lemma 2.9})\\ &= d\left(\prod_{i=1}^{l+1} a_{2(l+1)-(2i-1)}^{2n}\right), \end{aligned}$$

as required. □

Now to complete the proof of proposition, we have

$$\begin{aligned} d &= d\left(\prod_{i=1}^{m} a_{2m-(2i-1)}^{2n}\right) \quad (\text{by Lemma 2.11})\\ &= a_0y_1\left(\prod_{i=1}^{m-1} a_{2m-(2i-1)}^{2n}\right)a_1^{2n} \quad (\text{by zigzag equations}) \end{aligned}$$

$$
\begin{aligned}
&= a_0^n \left(\prod_{i=1}^{m-1} a_{2m-(2i-1)}^{2n}\right) a_1^{2n} a_0 y_1 \quad (\text{since } S \text{ is in } \mathcal{V}_n)\\
&= a_0^n \left(\prod_{i=1}^{m-1} a_{2m-(2i-1)}^{2n}\right) (a_1 a_1^{2n-1} a_0) y_1\\
&= a_0^n \left(\prod_{i=1}^{m-1} a_{2m-(2i-1)}^{2n}\right) a_1^n a_0 a_1 a_1^{2n-1} y_1 \quad (\text{since } S \text{ is in } \mathcal{V}_n)\\
&= a_0^n \left(\prod_{i=1}^{m-1} a_{2m-(2i-1)}^{2n}\right) a_1^n a_0 a_1^{2n-1} a_2 y_2 \quad (\text{by zigzag equations})\\
&= a_0^n \left(\prod_{i=1}^{m-2} a_{2m-(2i-1)}^{2n}\right) a_3^{2n} a_1^n a_0 a_1^{2n-1} a_2 y_2\\
&= a_0^n \left(\prod_{i=1}^{m-2} a_{2m-(2i-1)}^{2n}\right) (a_3) \left(a_3^{2n-1}\right) \left(a_1^n a_0 a_1^{2n-1} a_2\right) y_2\\
&= a_0^n \left(\prod_{i=1}^{m-2} a_{2m-(2i-1)}^{2n}\right) a_3^n \left(a_1^n a_0 a_1^{2n-1} a_2\right) a_3 \left(a_3^{2n-1}\right) y_2 \quad (\text{since } S \text{ is in } \mathcal{V}_n)\\
&= a_0^n \left(\prod_{i=1}^{m-2} a_{2m-(2i-1)}^{2n}\right) a_3^n a_1^n a_0 a_1^{2n-1} a_2 a_3^{2n-1} a_3 y_2\\
&\vdots\\
&= a_0^n a_{2m-1}^n a_{2m-3}^n \ldots a_3^n a_1^n a_0 a_1^{2n-1} a_2 a_3^{2n-1} a_4 \ldots a_{2m-1}^{2n-1} a_{2m-1} y_m\\
&= a_0^n a_{2m-1}^n a_{2m-3}^n \ldots a_3^n a_1^n a_0 a_1^{2n-1} a_2 a_3^{2n-1} a_4 \ldots a_{2m-1}^{2n-1} a_{2m} \quad (\text{by zigzag equations})\\
&= a_0^n \left(\prod_{i=1}^{m} a_{2m-(2i-1)}\right) a_0 \left(\prod_{i=1}^{m} a_{2i-1}^{2n-1} a_{2i}\right)\\
&= a_0 \left(\prod_{i=1}^{m} a_{2i-1}^{2n-1} a_{2i}\right) \left(\prod_{i=1}^{m} a_{2m-(2i-1)}\right) \quad (\text{since } S \text{ is in } \mathcal{V}_n),
\end{aligned}
$$

which is in U as required. □

In [4], Alam and Khan have shown that the variety of semigroups satisfying the identity $xy = xyx$ $[yx = xyx]$ is closed. Next we generalize this result by showing that for each $n \geq 2 \in \mathbb{N}$, the variety of semigroups satisfying the identity $x_1x_2x_3 = x_1^n x_2x_3x_1$ is closed.

Lemma 2.12 *For each $n \geq 2 \in \mathbb{N}$, let $\mathbb{V}_n = [x_1x_2x_3 = x_1^n x_2x_3x_1]$ be a variety of semigroups and S be any member of $\mathbb{V}_n$. Then S satisfies the identity $xyz = x^{2n-1}y^{2n-2}xyzxy^2$.*

Proof For any $x, y, z \in S$, we have

$$
\begin{aligned}
xyz &= x^n yzx \quad (\text{since } S \text{ is in } \mathbb{V}_n)\\
&= x^n y^n zxy \quad (\text{since } S \text{ is in } \mathbb{V}_n)\\
&= x^n y^{n-1} y^n zxy^2 \quad (\text{since } S \text{ is in } \mathbb{V}_n)\\
&= x^n y^{2n-1} zxy^2\\
&= (x)(x^{n-1}) \left(y^{2n-2}\right) yzxy^2\\
&= x^n x^{n-1} y^{2n-2} xyzxy^2 \quad (\text{since } n \geq 2 \text{ and } S \in \mathbb{V}_n)\\
&= x^{2n-1} y^{2n-2} xyzxy^2.
\end{aligned}
$$
□

Proposition 2.13 *For each $n \geq 2$ in $\mathbb{N}$, the variety $\mathbb{V}_n = [x_1x_2x_3 = x_1^n x_2x_3x_1]$ is closed.*

Proof Let S, U be in $\mathbb{V}_n$ such that U is a subsemigroup of S. Take any $d \in Dom(U, S) \setminus U$ and let (1.1) be zigzag equations in S over U with value d of length m.

To prove the proposition we first need to prove the following lemma.

Lemma 2.14 *For all $k = 1, 2, \ldots, m$,*

$$d = \left(\prod_{i=1}^{k} x_i^{2n-1} a_{2i-1}^{2n-2}\right) d \left(\prod_{i=1}^{k} x_i\right) \left(\prod_{i=1}^{k} a_{2i-1}^2\right).$$

Proof We prove the lemma by applying induction on k. For $k = 1$,

$$\begin{aligned} d &= x_1a_1y_1 \text{ (by zigzag equations)} \\ &= x_1^{2n-1}a_1^{2n-2}x_1a_1y_1x_1a_1^2 \text{ (by Lemma 2.12)} \\ &= x_1^{2n-1}a_1^{2n-2}dx_1a_1^2 \text{ (by zigzag equations).} \end{aligned}$$

Assume that the result holds for $k = l < m$. We show that the result also holds for $k = l + 1$.

$$\begin{aligned} d &= \left(\prod_{i=1}^{l} x_i^{2n-1} a_{2i-1}^{2n-2}\right) d \left(\prod_{i=1}^{l} x_i\right) \left(\prod_{i=1}^{l} a_{2i-1}^2\right) \text{ (by inductive hypothesis)} \\ &= \left(\prod_{i=1}^{l} x_i^{2n-1} a_{2i-1}^{2n-2}\right) x_{l+1}a_{2l+1}y_{l+1} \left(\prod_{i=1}^{l} x_i\right) \left(\prod_{i=1}^{l} a_{2i-1}^2\right) \text{ (by zigzag equations)} \\ &= \left(\prod_{i=1}^{l} x_i^{2n-1} a_{2i-1}^{2n-2}\right) x_{l+1}^{2n-1}a_{2l+1}^{2n-2}x_{l+1}a_{2l}y_{l+1} \left(\prod_{i=1}^{l+1} x_i\right) \left(\prod_{i=1}^{l+1} a_{2i-1}^2\right) \text{ (by Lemma 2.12)} \\ &= \left(\prod_{i=1}^{l+1} x_i^{2n-1} a_{2i-1}^{2n-2}\right) d \left(\prod_{i=1}^{l+1} x_i\right) \left(\prod_{i=1}^{l+1} a_{2i-1}^2\right) \text{ (by zigzag equations),} \end{aligned}$$

as required. □

Now to complete the proof of proposition, we have

$$\begin{aligned} d &= \left(\prod_{i=1}^{m-1} x_i^{2n-1} a_{2i-1}^{2n-2}\right) x_m a_{2m-1} y_m \left(\prod_{i=1}^{m-1} x_i\right) \left(\prod_{i=1}^{m-1} a_{2i-1}^2\right) \text{ (by Lemma 2.14)} \\ &= \left(\left(\prod_{i=1}^{m-1} x_i^{2n-1} a_{2i-1}^{2n-2}\right) x_m a_{2m-1}^n y_m \left(\prod_{i=1}^{m-1} x_i\right) \left(\prod_{i=1}^{m-1} a_{2i-1}^2\right) a_{2m-1}\right. \text{ (since } S \text{ is in } \mathbb{V}_n) \\ &= \left(\left(\prod_{i=1}^{m-1} x_i^{2n-1} a_{2i-1}^{2n-2}\right) x_m a_{2m-1}^{2n-1} y_m \left(\prod_{i=1}^{m-1} x_i\right) \left(\prod_{i=1}^{m-1} a_{2i-1}^2\right) a_{2m-1}^2\right. \text{ (since} S \text{ is in } \mathbb{V}_n) \\ &= \left(\prod_{i=1}^{m-1} x_i^{2n-1} a_{2i-1}^{2n-2}\right) x_{m-1} a_{2m-2} a_{2m-1}^{2n-3} a_{2m} \left(\prod_{i=1}^{m-1} x_i\right) \left(\prod_{i=1}^{m} a_{2i-1}^2\right) \text{ (by zigzag equations)} \\ &= \left(\prod_{i=1}^{m-2} x_i^{2n-1} a_{2i-1}^{2n-2}\right) x_{m-1}^{2n-1} a_{2m-3}^{2n-2} x_{m-1} a_{2m-2} a_{2m-1}^{2n-3} a_{2m} \left(\prod_{i=1}^{m-1} x_i\right) \left(\prod_{i=1}^{m} a_{2i-1}^2\right) \\ &= \left(\prod_{i=1}^{m-2} x_i^{2n-1} a_{2i-1}^{2n-2}\right) \left(x_{m-1}^n x_{m-1}^{n-1} a_{2m-3}^{2n-2} x_{m-1}\right) a_{2m-2} a_{2m-1}^{2n-3} a_{2m} \left(\prod_{i=1}^{m-1} x_i\right) \left(\prod_{i=1}^{m} a_{2i-1}^2\right) \end{aligned}$$

$$= \left(\prod_{i=1}^{m-2} x_i^{2n-1} a_{2i-1}^{2n-2}\right) x_{m-1}^{n} a_{2m-3}^{2n-2} a_{2m-2} a_{2m-1}^{2n-3} a_{2m} \left(\prod_{i=1}^{m-2} x_i\right) x_{m-1} \left(\prod_{i=1}^{m} a_{2i-1}^2\right)$$
(since $n \geq 2$ and $S \in \mathbb{V}_n$)
$$= \left(\prod_{i=1}^{m-2} x_i^{2n-1} a_{2i-1}^{2n-2}\right) x_{m-1} a_{2m-3}^{2n-2} a_{2m-2} a_{2m-1}^{2n-3} a_{2m} \left(\prod_{i=1}^{m-2} x_i\right) \left(\prod_{i=1}^{m} a_{2i-1}^2\right) \quad \text{(since } S \text{ is in } \mathbb{V}_n\text{)}$$
$$= \left(\prod_{i=1}^{m-2} x_i^{2n-1} a_{2i-1}^{2n-2}\right) x_{m-2} a_{2m-4} a_{2m-3}^{2n-3} a_{2m-2} a_{2m-1}^{2n-3} a_{2m} \left(\prod_{i=1}^{m-2} x_i\right) \left(\prod_{i=1}^{m} a_{2i-1}^2\right)$$
(by zigzag equations)
$$\vdots$$
$$= x_1^{2n-1} a_1^{2n-2} x_1 a_2 a_3^{2n-3} a_4 \ldots a_{2m-3}^{2n-3} a_{2m-2} a_{2m-1}^{2n-3} a_{2m} x_1 \left(\prod_{i=1}^{m} a_{2i-1}^2\right)$$
$$= \left(x_1^n x_1^{n-1} a_1^{2n-2} x_1\right) a_2 a_3^{2n-3} a_4 \ldots a_{2m-3}^{2n-3} a_{2m-2} a_{2m-1}^{2n-3} a_{2m} x_1 \left(\prod_{i=1}^{m} a_{2i-1}^2\right)$$
$$= x_1^n a_1^{2n-2} a_2 a_3^{2n-3} a_4 \ldots a_{2m-3}^{2n-3} a_{2m-2} a_{2m-1}^{2n-3} a_{2m} x_1 \left(\prod_{i=1}^{m} a_{2i-1}^2\right) \quad \text{(since } n \geq 2 \text{ and } S \in \mathbb{V}_n\text{)}$$
$$= x_1 a_1^{2n-2} a_2 a_3^{2n-3} a_4 \ldots a_{2m-3}^{2n-3} a_{2m-2} a_{2m-1}^{2n-3} a_{2m} \left(\prod_{i=1}^{m} a_{2i-1}^2\right) \quad \text{(since } S \text{ is in } \mathbb{V}_n\text{)}$$
$$= a_0 a_1^{2n-3} a_2 a_3^{2n-3} a_4 \ldots a_{2m-3}^{2n-3} a_{2m-2} a_{2m-1}^{2n-3} a_{2m} \left(\prod_{i=1}^{m} a_{2i-1}^2\right) \quad \text{(by zigzag equations)}$$
$$= \left(\prod_{i=1}^{m} a_{2i-2} a_{2i-1}^{2n-3}\right) \left(\prod_{i=1}^{m} a_{2i-1}^2\right),$$

which is in U as required. □

Theorem 2.15 *All homotypical varieties of semigroups of the following type:*

(i) *for each* $n \geq 2 \in \mathbb{N}$, $\mathbb{V}_n = [x_1x_2x_3 = x_1^n x_{i_1} x_{i_2} x_{i_3}]$ *where* i *is any non-trivial permutation of* $\{1, 2, 3\}$*;*
(ii) *for each* $n \in \mathbb{N}$, $\mathcal{V}_n = [x_1x_2x_3 = x_1^n x_{i_1} x_{i_2} x_{i_3}]$ *where* i *is any non-trivial permutation on* $\{1, 2, 3\}$ *other than the permutation* (231)

are closed.

Proof The proof follows from Propositions 2.2, 2.5, 2.7, 2.10 and 2.13. □

The next result is dual of the Theorem 2.15.

Theorem 2.16 *All homotypical varieties of semigroups of the following type:*

(i) *for each* $n \geq 2 \in \mathbb{N}$, $\mathbb{V}_n = [x_1x_2x_3 = x_{i_1} x_{i_2} x_{i_3} x_3^n]$ *where* i *is any non-trivial permutation of* $\{1, 2, 3\}$*;*
(ii) *for each* $n \in \mathbb{N}$, $\mathcal{V}_n = [x_1x_2x_3 = x_{i_1} x_{i_2} x_{i_3} x_3^n]$ *where* i *is any non-trivial permutation on* $\{1, 2, 3\}$ *other than the permutation* (312)

are closed.

3 Open problems

Problem 3.1 Can the condition of $n \geq 2$ in Proposition 2.13 be relaxed?
Problem 3.2 Is the variety of semigroups satisfying an identity of type $x_1x_2 \ldots x_n = x_1^n x_{i_1} x_{i_2} \ldots x_{i_n}$ $[x_1x_2 \ldots x_n = x_{i_1} x_{i_2} \ldots x_{i_n} x_n^n]$ closed?

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Declarations

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