Arab. J. Math. (2022) 11:539–547
https://doi.org/10.1007/s40065-022-00390-0

Arabian Journal of Mathematics

Tebbani Hossni · Achache Mahdi

Maximal regularity for semilinear non-autonomous evolution equations in temporally weighted spaces

Received: 10 September 2021 / Accepted: 22 August 2022 / Published online: 7 September 2022

Abstract We consider the problem of maximal regularity for the semilinear non-autonomous evolution equations

$$u'(t) + A(t)u(t) = F(t, u),\ t\text{-a.e.},\ u(0) = u_0.$$

Here, the time-dependent operators $A(t)$ are associated with (time dependent) sesquilinear forms on a Hilbert space $\mathcal{H}$. We prove the maximal regularity result in temporally weighted L^2-spaces and other regularity properties for the solution of the previous problem under minimal regularity assumptions on the forms, the initial value u_0 and the inhomogeneous term F. Our results are motivated by boundary value problems.

Mathematics Subject Classification 35K90 · 35K45 · 47D06

1 Introduction

The present paper deals with maximal L^2-regularity for non-autonomous evolution equations in the setting of Hilbert spaces. Before explaining our results, we introduce some notations and assumptions.

Let $(\mathcal{H}, (\cdot,\cdot), \|\cdot\|)$ be a Hilbert space over $\mathbb{R}$ or $\mathbb{C}$. We consider another Hilbert space $\mathcal{V}$ which is densely and continuously embedded into $\mathcal{H}$. We denote by $\mathcal{V}'$ the (anti-) dual space of $\mathcal{V}$ so that

$$\mathcal{V} \hookrightarrow_d \mathcal{H} \hookrightarrow_d \mathcal{V}'.$$

We denote by $\langle, \rangle$ the duality $\mathcal{V}$-$\mathcal{V}'$ and note that $\langle \psi, v\rangle = (\psi, v)$ if $\psi, v \in \mathcal{H}$. Given $\tau \in (0,\infty)$ and consider a family of sesquilinear forms

$$\mathfrak{a} : [0,\tau] \times \mathcal{V} \times \mathcal{V} \to \mathbb{C},$$

such that

- [H1]: $D(\mathfrak{a}(t)) = \mathcal{V}$ (constant form domain),
- [H2]: $|\mathfrak{a}(t, u, v)| \leq M\|u\|_{\mathcal{V}}\|v\|_{\mathcal{V}}$ (uniform boundedness),
- [H3]: Re $\mathfrak{a}(t, u, u) + \nu\|u\|^2 \geq \delta\|u\|^2_{\mathcal{V}}$ $(\forall u \in \mathcal{V})$ for some $\delta > 0$ and some $\nu \in \mathbb{R}$ (uniform quasi-coercivity).

T. Hossni (✉)
Univ. Sétif -1-, Sétif, Algeria
E-mail: maths47@ymail.com

A. Mahdi
Univ. Aix Marseille , CNRS, CPT, Marseille, France
E-mail: achachemahdi@hotmail.fr

Here and throughout this paper, $\|\cdot\|_{\mathcal{V}}$ denotes the norm of $\mathcal{V}$.

To each form $\mathfrak{a}(t)$, we can associate two operators $A(t)$ and $\mathcal{A}(t)$ on $\mathcal{H}$ and $\mathcal{V}'$, respectively. Recall that $u \in \mathcal{H}$ is in the domain $D(A(t))$ if there exists $h \in \mathcal{H}$ such that for all $v \in \mathcal{V}$: $\mathfrak{a}(t, u, v) = (h, v)$. We then set $A(t)u := h$. The operator $\mathcal{A}(t)$ is a bounded operator from $\mathcal{V}$ into $\mathcal{V}'$ such that $\mathcal{A}(t)u = \mathfrak{a}(t, u, \cdot)$. The operator $A(t)$ is the part of $\mathcal{A}(t)$ on $\mathcal{H}$. It is a classical fact that $-A(t)$ and $-\mathcal{A}(t)$ are both generators of holomorphic semigroups $(e^{-rA(t)})_{r\geq 0}$ and $(e^{-r\mathcal{A}(t)})_{r\geq 0}$ on $\mathcal{H}$ and $\mathcal{V}'$, respectively. The semigroup $e^{-rA(t)}$ is the restriction of $e^{-r\mathcal{A}(t)}$ to $\mathcal{H}$. In addition, $e^{-rA(t)}$ induces a holomorphic semigroup on $\mathcal{V}$ (see, e.g., Ouhabaz [15, Chapter 1]). We consider the non-homogeneous Cauchy problem

$$\begin{cases} u'(t) + A(t)\,u(t) = f(t), \ t \in (0, \tau] \\ \qquad\qquad u(0) = u_0. \end{cases} \tag{1.1}$$

By a well-known result of J.L. Lions, the maximal regularity always holds in the space $\mathcal{V}'$. That is for every $f \in L^2(0, \tau; \mathcal{V}')$ and $u_0 \in \mathcal{H}$ there exists a unique $u \in H^1(0, \tau; \mathcal{V}') \cap L^2(0, \tau; \mathcal{V})$ which solves the problem (1.1). The maximal regularity in $\mathcal{H}$ is however more interesting since when dealing with boundary value problems one cannot identify the boundary conditions if the Cauchy problem is considered in $\mathcal{V}'$. The maximal regularity in $\mathcal{H}$ is more difficult to prove.

It has been shown in [12] that the maximal regularity in $\mathcal{H}$ may fail for forms $C^{\frac{1}{2}}$ in time. For $\mathcal{A}(.) \in W^{s,p}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$, with $s < \frac{1}{2}$ the maximal regularity does not hold and this comes from the inclusion $C^{\frac{1}{2}}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}')) \subset W^{s,p}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$.

For $p > 2$ and $\mathcal{A}(.) \in W^{\frac{1}{2},p}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$, the maximal regularity can fail also and this follows from the counterexample in [6]. It is proved in [3] that the maximal regularity holds if $t \mapsto \mathcal{A}(t) \in W^{\frac{1}{2},2}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$ (with some integrability conditions). This result is optimal. For the case of weighted spaces, we refer the reader to the recent paper [4]. The choice of weighted spaces has a big advantages. Among them is to reduce the necessary regularity for initial conditions of evolution equations. Time-weights can be used also to exploit parabolic regularization which is typical for quasilinear parabolic problems.

The main focus of this paper is to consider the semilinear equation

$$u'(t) + \mathcal{A}(t)u(t) = F(t, u), \ t\text{-a.e.}, \ u(0) = u_0. \tag{1.2}$$

Here, the inhomogeneous term F satisfies some continuity condition. Our main result shows that for forms satisfying the uniform Kato square root property (see the next section for the definition) then we have the maximal regularity result in temporally weighted L^2-spaces if $u_0 \in [\mathcal{H}; D(A(0))]_{\frac{1-\beta}{2}}$ and $\mathcal{A} \in W^{\frac{1}{2},2}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}')) \cap C^{\varepsilon}([0, \tau], \mathcal{L}(\mathcal{V}, \mathcal{V}'))$. The Kato square root property plays an important role in the questions of (non-autonomous) maximal regularity and optimal control. We remark that (1.2) was studied recently in [11] in non weighted spaces, such that the non-linearity term F is a bounded valued function on $\mathcal{H}$, satisfies other more regularity assumptions and some Dini-condition holds for $\mathcal{A}(.)$ (see [11, Theorem 5.1] for more details). In the present paper, the regularity assumptions on $\mathcal{A}(.)$ and F are significantly weaker than those from previous results.

To prove our results, we appeal to classical tools from harmonic analysis such as square function estimate and from functional analysis such as interpolation theory or operator theory.

Notation. We denote by $\mathcal{L}(E, F)$ (or $\mathcal{L}(E)$) the space of bounded linear operators from E to F (from E to E).The spaces $L^p(a, b; E)$ and $W^{1,p}(a, b; E)$ denote respectively the Lebesgue and Sobolev spaces of function on (a, b) with values in E. $C^{\alpha}(a, b; E)$ denote the space of Hölder continuous functions of order α, recall that the norms of $\mathcal{H}$ and $\mathcal{V}$ are denoted by $\|\cdot\|$ and $\|\cdot\|_{\mathcal{V}}$. The scalar product of $\mathcal{H}$ is $(\cdot, \cdot)$.
We denote by C, C' or c... all inessential positive constants, their values may change from line to line.
Finally, by $(E, F)_{\theta,p}$ and $[E, F]_{\theta}$ ($\theta \in (0, 1)$ and $p \in (1, \infty)$) we denote the real interpolation space defined by the $K-$method and complex interpolation space, respectively, between E and F. We refer the reader to [14, Definition 1.1.2, Definition 2.1.3] for more details.

2 Preliminaries

In this section, we state several definitions and properties which will play an important role in the proof of our results.

We briefly recall the definitions and we give the basic properties of vector-valued function spaces with temporal weights. For more details, we refer to [4].

For $-1 < \beta < 1$ we set $L^2_\beta(0,\tau;\mathcal{H}) = L^2(0,\tau,t^\beta \mathrm{d}t;\mathcal{H})$, endowed with the norm

$$\|u\|^2_{L^2_\beta(0,\tau,\mathcal{H})} := \int_0^\tau \|u(t)\|^2\, t^\beta\, \mathrm{d}t.$$

It is known that $L^2_\beta(0,\tau;\mathcal{H}) \hookrightarrow L^1_{loc}(0,\tau;\mathcal{H})$. We define the corresponding weighted Sobolev spaces

$$W^{1,2}_\beta(0,\tau;\mathcal{H}) := \left\{u \in W^{1,1}(0,\tau;\mathcal{H}) : u, u' \in L^2_\beta(0,\tau;\mathcal{H})\right\},$$
$$W^{1,2}_{\beta,0}(0,\tau;X) := \left\{u \in W^{1,2}_\beta(0,\tau;\mathcal{H}) : u(0) = 0\right\},$$

which are Banach spaces for the norms, respectively

$$\|u\|^2_{W^{1,2}_\beta(0,\tau;\mathcal{H})} := \|u\|^2_{L^2_\beta(0,\tau;\mathcal{H})} + \|u'\|^2_{L^2_\beta(0,\tau;X)},$$
$$\|u\|^2_{W^{1,2}_{\beta,0}(0,\tau;\mathcal{H})} := \|u'\|^2_{L^2_\beta(0,\tau;\mathcal{H})}.$$

Remark 2.1 The restriction on β comes from several facts. The first one is the embedding $L^2_\beta(0,\tau;\mathcal{H}) \hookrightarrow L^1(0,\tau;\mathcal{H})$. The second one is due to Hardy' inequality and the third reason comes from the fact that functions in $W^{1,2}_\beta(0,\tau;\mathcal{H})$ have a well-defined trace in case that $-1 < \beta < 1$.

Lemma 2.2 *Let $u \in W^{1,2}_{\beta,0}(0,\tau;\mathcal{H})$. We have*

$$\|u\|^2_{L^2_\beta(0,\tau;\mathcal{H})} \le \int_0^\tau s \int_0^s \left\|u'(r)\right\|^2 r^\beta\, \mathrm{d}r\, \mathrm{d}s.. \tag{2.1}$$

Proof Let $u \in W^{1,2}_{\beta,0}(0,\tau;\mathcal{H})$, $s \in (0,\tau)$. Due to Holder's inequality, we get

$$\begin{aligned}\|u(s)\|_X &= \left\|\int_0^s u'(r)\,\mathrm{d}r\right\| \le \int_0^s \left\|u'(r)\right\|\,\mathrm{d}r\\ &\le \left(\int_0^s r^{-\beta}\,\mathrm{d}r\right)^{\frac12}\left(\int_0^s \left\|u'(r)\right\|^2 r^\beta\,\mathrm{d}r\right)^{\frac12}\\ &= s^{\frac{1-\beta}{2}}\left(\int_0^s \left\|u'(r)\right\|^2 r^\beta\,\mathrm{d}r\right)^{\frac12}.\end{aligned}$$

Therefore, (2.1) follows immediately. □

Let us define the space

$$W_\beta\left(D(A(.)),\mathcal{H}\right) := \left\{u \in W^{1,1}(0,\tau;\mathcal{H}),\ \text{s.t } A(.)u \in L^2_\beta(0,\tau;\mathcal{H}),\ u' \in L^2_\beta(0,\tau;\mathcal{H})\right\},$$

with norm

$$\|u\|_{W_\beta(D(A(.),\mathcal{H})} = \|A(.)u\|_{L^2_\beta(0,\tau;\mathcal{H})} + \left\|u'\right\|_{L^2_\beta(0,\tau;\mathcal{H})}.$$

It is easy to see that $W_\beta(D(A(.),\mathcal{H}) \hookrightarrow W^{1,2}_\beta(0,\tau;\mathcal{H})$. For $s \in (0,\tau)$, we define the associated trace space to $W_\beta(D(A(.)),\mathcal{H})$ by

$$TR(s,\beta) := \left\{u(s) : u \in W_\beta(D(A(.)),\mathcal{H})\right\},$$

endowed with norm

$$\|u(s)\|_{TR(s,\beta)} := \inf\left\{\|v\|_{W_\beta(D(A(.)),\mathcal{H})} : v(s) = u(s)\right\}.$$

Remark that $(TR(s,\beta), \|\cdot\|_{TR(s,\beta)})$ is a Banach space and for $u \in W_\beta(D(A(\cdot)), \mathcal{H})$, it follows that $u(s) \in TR(s,\beta)$. Conversely, if $x \in TR(s,\beta)$, then there exists $u \in W_\beta(D(A(\cdot)), \mathcal{H})$ such that $u(s) = x$.

From now, we assume without loss of generality that the forms are coercive, that is $[H3]$ holds with $\nu = 0$. The reason is that by replacing $A(t)$ by $A(t) + \nu$, the solution v of (1.1) is $v(t) = e^{-\nu t}u(t)$ and it is clear that $u \in W^{1,2}_\beta(0,\tau;\mathcal{H}) \cap L^2_\beta(0,\tau;\mathcal{V})$ if and only if $v \in W^{1,2}_\beta(0,\tau;\mathcal{H}) \cap L^2_\beta(0,\tau;\mathcal{V})$.

In the statements below, we shall need the following square root property (called Kato's square root property)

$$D\left(A(t)^{1/2}\right) = \mathcal{V} \text{ and } c_1 \left\|A(t)^{1/2}v\right\| \le \|v\|_{\mathcal{V}} \le c_2 \left\|A(t)^{1/2}v\right\|, \tag{2.2}$$

for all $v \in \mathcal{V}$ and $t \in [0,\tau]$, where the positive constants c_1 and c_2 are independent of t. This assumption is always true for symmetric forms when $\nu = 0$ in [H3]. It is also valid for uniformly elliptic operator on $\mathbb{R}^n$, see [8].

3 Main results

In this section, we state explicitly our main results.

Let $F(t,x) : (0,\tau)\times\mathcal{H} \to \mathcal{H}$ and $F_0(t) = F(t,0)$. Assume that $F_0(.) \in L^2_\beta(0,\tau;\mathcal{H})$ and $(t,x) \mapsto F(t,x)$ satisfies the following continuity property: for any $\varepsilon > 0$ there exists a constant $N_\varepsilon > 0$ such that

$$\|F(.,u) - F(.,v)\|^2_{L^2_\beta(0,\tau;\mathcal{H})} \le \varepsilon\|u - v\|^2_{W_\beta(D(A(.),\mathcal{H})} + N_\varepsilon\|u - v\|^2_{L^2_\beta(0,\tau;\mathcal{H})}, \tag{3.1}$$

for any $u, v \in W_\beta(D(A(.),\mathcal{H})$.

If we assume that $\|F(t,x) - F(t,y)\| \le K\|x - y\|_{\mathcal{V}}$, $K > 0$, $x, y \in \mathcal{V}$, $t \in (0,\tau)$ then the condition (3.1) is satisfied. Indeed, let $u, v \in W_\beta(D(A(.),\mathcal{H})$ one has

$$\begin{aligned}
\|F(.,u) - F(.,v)\|^2_{L^2_\beta(0,\tau;\mathcal{H})} &\le K^2\|u - v\|^2_{L^2_\beta(0,\tau;\mathcal{V})} \\
&= \frac{K^2}{\delta}\int_0^\tau \left(\delta\|u(t) - v(t)\|^2_{\mathcal{V}}\right) t^\beta \,\mathrm{d}t \\
&\le \frac{K^2}{\delta}\int_0^\tau (\mathrm{Re}\ (A(t)\,(u(t) - v(t))\,, u(t) - v(t)))\, t^\beta \,\mathrm{d}t \\
&\le \frac{K^2}{\delta}\int_0^\tau \|A(t)(u(t) - v(t))\|\,\|u(t) - v(t)\|\, t^\beta \,\mathrm{d}t \\
&\le \varepsilon\|A(.)(u - v)\|^2_{L^2_\beta(0,\tau;\mathcal{H})} + N_\varepsilon\|u - v\|^2_{L^2_\beta(0,\tau;\mathcal{H})},
\end{aligned}$$

where $N_\varepsilon = \frac{K^4}{\delta^2\varepsilon}$.

The following theorem is proved in [4, Theorem 5.3].

Theorem 3.1 *Assume that $\mathcal{A} \in W^{\frac{1}{2},2}(0,\tau;\mathcal{L}(\mathcal{V},\mathcal{V}')) \cap C^\varepsilon([0,\tau],\mathcal{L}(\mathcal{V},\mathcal{V}'))$, $\varepsilon > 0$ and (2.2) holds. Then for all $f \in L^2_\beta(0,\tau;\mathcal{H})$ and $u_0 \in [\mathcal{H}; D(A(0))]_{\frac{1-\beta}{2}}$, there exists a unique $u \in W_\beta(D(A(.),\mathcal{H})$ solves (1.1). Moreover, there exists a positive constant $N > 0$ such that*

$$\|u\|_{W_\beta(D(A(.),\mathcal{H})} \le N\Big[\|u_0\|_{[\mathcal{H};D(A(0))]_{\frac{1-\beta}{2}}} + \|f\|_{L^2_\beta(0,\tau;\mathcal{H})}\Big].$$

The following proposition gives a characterization of the trace space $TR(s,\beta)$.

Proposition 3.2 *For all $\beta \in (0,1)$, $s \in (0,\tau)$, we have*

$$TR(s,\beta) = [\mathcal{H}; D(A(s))]_{\frac{1-\beta}{2}} \ \textit{with equivalent norms.}$$

Proof The first injection $TR(s,\beta) \hookrightarrow [\mathcal{H}; D(A(s))]_{\frac{1-\beta}{2}}$ is obtained by [4, Proposition 5.1]. The second injection "$\hookleftarrow$" follows by Theorem 3.1. □

The following is our main result.

Theorem 3.3 *Suppose that $\mathcal{A} \in W^{\frac{1}{2},2}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}')) \cap C^{\varepsilon}([0, \tau], \mathcal{L}(\mathcal{V}, \mathcal{V}'))$, with $\varepsilon > 0$, and (2.2) holds. Then for all $u_0 \in [\mathcal{H}; D(A(0))]_{\frac{1-\beta}{2}}$, there exists a unique $u \in W_\beta(D(A(.), \mathcal{H})$ satisfies*

$$u'(t) + \mathcal{A}(t)u(t) = F(t, u),\ t\text{-a.e.},\ u(0) = u_0. \tag{3.2}$$

Moreover, there exists a positive constant $c > 0$ such that

$$\|u\|_{W_\beta(D(A(.),\mathcal{H})} \leq c\Big[\|u_0\|_{[\mathcal{H};D(A(0))]_{\frac{1-\beta}{2}}} + \|F_0\|_{L^2_\beta(0,\tau;\mathcal{H})}\Big]. \tag{3.3}$$

Proof First, let us define the space $W_{\beta,0}(D(A(.), \mathcal{H}) := W_\beta(D(A(.), \mathcal{H})\} \cap W^{1,2}_{\beta,0}(0, \tau; X)$. For $v \in W_\beta(D(A(.), \mathcal{H})$ consider the linear equation

$$w' + A(.)w = F(., v),\ w(0) = 0. \tag{3.4}$$

Thanks to Theorem 3.1, (3.4) has a unique solution $w \in W_{\beta,0}(D(A(.), \mathcal{H})$.
We define

$$\begin{aligned} S : W_{\beta,0}(D(A(.), \mathcal{H}) &\to W_{\beta,0}(D(A(.), \mathcal{H}) \\ v &\mapsto w. \end{aligned}$$

Let $v_1, v_2 \in W_{\beta,0}(D(A(.), \mathcal{H})$. Obviously, $x = Sv_1 - Sv_2$ satisfies $x' + A(.)x = F(., v_1) - F(., v_2)$, $x(0) = 0$ and we have by Theorem 3.1 and Lemma 2.2 that there exists $N > 0$ such that

$$\begin{aligned} \|Sv_1 - Sv_2\|^2_{W_\beta(D(A(.),\mathcal{H})} &\leq N\|F(., v_1) - F(., v_2)\|^2_{L^2_\beta(0,\tau;\mathcal{H})} \\ &\leq N\varepsilon\|v_1 - v_2\|^2_{W_\beta(D(A(.),\mathcal{H})} + NN_\varepsilon\|v_1 - v_2\|^2_{L^2_\beta(0,\tau;\mathcal{H})} \\ &\leq N\varepsilon\|v_1 - v_2\|^2_{W_\beta(D(A(.),\mathcal{H})} + NN_\varepsilon \int_0^\tau s \int_0^s \left\|(v_1 - v_2)'(r)\right\|^2 r^\beta\, dr\, ds \\ &\leq N\varepsilon\|v_1 - v_2\|^2_{W_\beta(D(A(.),\mathcal{H})} + NN_\varepsilon \int_0^\tau s\|v_1' - v_2'\|^2_{L^2_\beta(0,s;\mathcal{H})}\, ds. \end{aligned}$$

Set $K_0 := N\varepsilon$ and $K_1 := NN_\varepsilon$. Then, repeating the above inequality and using the identity

$$\int_0^t s_1 \int_0^{s_1} s_2 \ldots \int_0^{s_{n-1}} s_n\, ds_n \ldots ds_1 = \frac{1}{\Gamma(2n+1)} t^{2n},$$

we obtain

$$\begin{aligned} \left\|S^n v_1 - S^n v_2\right\|^2_{W_\beta(D(A(.),\mathcal{H})} &\leq \sum_{k=0}^n \binom{n}{k} K_0^{n-k}(K_1\tau^2)^k \frac{1}{\Gamma(2k+1)}\|v_1 - v_2\|^2_{W_\beta(D(A(.),\mathcal{H})} \\ &\leq (2K_0)^n \left[\max_{k=0,..,n}\left(\frac{\left(K_0^{-1}\tau^2 K_1\right)^k}{\Gamma(2k+1)}\right)\right]\|v_1 - v_2\|^2_{W_\beta(D(A(.),\mathcal{H})}. \end{aligned}$$

In the second inequality we used

$$\sum_{k=0}^n \binom{n}{k} = 2^n.$$

Observe that $\max_{k=0,..,n}\left(\frac{\left(K_0^{-1}\tau^2 K_1\right)^k}{\Gamma(2k+1)}\right)$ is bounded for all $n \in \mathbb{N}^*$.

Now, we take $\varepsilon < \frac{1}{4N}$ (which gives $K_0 < \frac{1}{4}$) and n sufficiently large to get

$$\left\| S^n v_1 - S^n v_2 \right\|^2_{W_\beta(D(A(.),\mathcal{H})} < \frac{1}{2^n}\left[\max_{k=0,..,n}\left(\frac{\left(K_0^{-1}\tau^2 K_1\right)^k}{\Gamma(2k+1)}\right)\right]\|v_1 - v_2\|^2_{W_\beta(D(A(.),\mathcal{H})}$$
$$< \|v_1 - v_2\|^2_{W_\beta(D(A(.),\mathcal{H})}.$$

Then S^n is a contraction map on $W_\beta(D(A(.),\mathcal{H})$ and this yields the existence and uniqueness of a solution $w \in W_\beta(D(A(.),\mathcal{H})$ to (3.4). Therefore it only remains to prove the a priori estimate (3.3). From the linear equation and (3.1), we have for all $\varepsilon > 0$

$$\begin{aligned}\|w\|^2_{W_\beta(D(A(.),\mathcal{H})} &\le N\|F(.,w)\|^2_{L^2_\beta(0,\tau;\mathcal{H})}\\ &\le N\|F(.,w) - F_0(.)\|^2_{L^2_\beta(0,\tau;\mathcal{H})} + 2N\|F_0(.)\|^2_{L^2_\beta(0,\tau;\mathcal{H})}\\ &\le 2N\varepsilon\|w\|^2_{W_\beta(D(A(.),\mathcal{H})} + 2NN_\varepsilon\|w\|^2_{L^2_\beta(0,s;\mathcal{H})} + 2N\|F_0(.)\|^2_{L^2_\beta(0,s;\mathcal{H})}\\ &\le 2N\varepsilon\|w\|^2_{W_\beta(D(A(.),\mathcal{H})} + 2NN_\varepsilon\int_0^\tau s\|w'\|^p_{L^2_\beta(0,s;\mathcal{H})}\,\mathrm{d}s + 2N\|F_0(.)\|^2_{L^2_\beta(0,s;\mathcal{H})}.\end{aligned}$$

Take $\varepsilon = \frac{1}{8N}$. Gronwall's lemma gives that there exists $C > 0$ such that

$$\|w\|_{W_\beta(D(A(.),\mathcal{H})} \le C\|F_0\|_{L^2_\beta(0,s;\mathcal{H})}.$$

Consider now the non homogeneous equation. Let $u_0 \in [\mathcal{H}; D(A(0))]_{\frac{1-\beta}{2}}$. Proposition 3.2, and the fact that $Tr(0,\beta) = [\mathcal{H}; D(A(0))]_{\frac{1-\beta}{2}}$ shows that there exists $v \in W_\beta(D(A(.),\mathcal{H})$ (with minimal norm) with $v(0) = u_0$ and

$$\|v\|_{W_\beta(D(A(.),\mathcal{H})} = \|u_0\|_{[\mathcal{H};D(A(0))]_{\frac{1-\beta}{2}}}.$$

For $w \in W_\beta(D(A(.),\mathcal{H})$, we define the function

$$G(t,w,w') = F\left(t, w+v, w'+v'\right) - \left(v'(t) + \mathcal{A}(t)v(t)\right),\ t \in (0,\tau).$$

It is easy to check that G satisfies the condition (3.1), $t \mapsto G(t,w,w') \in L^2_\beta(0,\tau;\mathcal{H}), G(t,0,0) = F(t,v,v') - (v'(t) + \mathcal{A}(t)v(t))$. Moreover,

$$\begin{aligned}\|G(.,0,0)\|_{L^2_\beta(0,\tau;\mathcal{H})} &\le \left\|F\left(.,v,v'\right) - F(.,0,0)\right\|_{L^2_\beta(0,\tau;\mathcal{H})} + \|F(.,0,0)\|_{L^2_\beta(0,\tau;\mathcal{H})}\\ &\quad + \left\|v' + \mathcal{A}(.)v\right\|_{L^2_\beta(0,\tau;\mathcal{H})}\\ &\le C_1\|v\|_{W_\beta(D(A(.),\mathcal{H})} + \|F_0\|_{L^2_\beta(0,\tau;\mathcal{H})}\\ &\le C\left[\|F_0\|_{L^2_\beta(0,\tau;\mathcal{H})} + \|u_0\|_{(\mathcal{H};D(A(0)))_{\frac{1-\beta}{2},2}}\right].\end{aligned}$$

Now, we follow the same procedure as before we get the existence and the uniqueness of the solution to the equation

$$w' + A(.)w = G(.,w),\ w(0) = 0.$$

Set $u = v + w$. Hence, u is the unique solution to (3.2). □

4 Applications

This section is devoted to application of our results on existence and maximal regularity to concrete evolution equations. We show how they can be applied to both linear and semilinear evolution equations.

4.1 Elliptic operators.

Define on $\mathcal{H} = L^2(\mathbb{R}^d)$ the sesquilinear forms

$$\mathfrak{a}(t,u,v) = \sum_{k,j=1}^{d} \int_{\mathbb{R}^d} a_{kj}(t,x)\partial_k u \overline{\partial_j v}\,\mathrm{d}x + \sum_{j=1}^{d} \int_{\mathbb{R}^d} b_j(t,x)\partial_j u \bar{v}\,\mathrm{d}x + \int_{\mathbb{R}^d} c(t,x) u\bar{v}\,\mathrm{d}x,\, u, v \in H^1\left(\mathbb{R}^d\right).$$

We assume that $a_{kj}, b_j, c : [0,\tau] \times \mathbb{R}^d \to \mathbb{C}$ such that:

$$a_{kj}, b_j, c \in L^\infty\left([0,\tau] \times \mathbb{R}^d\right) \text{ for } 1 \le k, j \le d,$$

and

$$\operatorname{Re} \sum_{k,j=1}^{d} a_{kj}(t,x)\xi_k \bar{\xi_j} \ge \delta|\xi|^2 \text{ for all } \xi \in \mathbb{C}^d \text{ and a.e. } (t,x) \in [0,\tau] \times \mathbb{R}^d.$$

Here, $\delta > 0$ is a constant independent of t.
It easy to check that $\mathfrak{a}(t,.,.)$ is $H^1(\mathbb{R}^d)$-bounded and quasi-coercive. The associated operator with $\mathfrak{a}(t,.,.)$ is elliptic operator given by the formal expression

$$A(t)u = -\sum_{k,j=1}^{d} \partial_j \left(a_{kj}(t,.)\partial_k u\right) + \sum_{j=1}^{d} b_j(t,.)\partial_j u + c(t,.)u.$$

In addition to the above assumptions, we suppose that $C = (a_{kj})_{k,j} \in W^{\frac{1}{2},2}(0,\tau; L^\infty(\Omega; \mathbb{C}^{n\times n})) \cap C^\varepsilon([0,\tau]; L^\infty(\Omega; \mathbb{C}^{n\times n}))$, with $\varepsilon > 0$, which is equivalent to

$$\int_0^\tau \int_0^\tau \sup_{x\in\Omega} \frac{\|C(t,x) - C(s,x)\|^2_{\mathbb{C}^{n\times n}}}{|t-s|^2}\,\mathrm{d}s\,\mathrm{d}t < \infty,$$
$$\|C(t,x) - C(s,x)\|_{\mathbb{C}^{n\times n}} < C|t-s|^\varepsilon$$

a.e. for $x \in \Omega$ and $t, s \in [0,\tau]$.

Note that

$$\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')} \lesssim \|C(t,.) - C(s,.)\|_{L^\infty(\Omega;\mathbb{C}^{n\times n})}.$$

Hence

$$\mathcal{A} \in W^{\frac{1}{2},2}\left(0,\tau; \mathcal{L}\left(\mathcal{V},\mathcal{V}'\right)\right) \cap C^\varepsilon\left([0,\tau]; \mathcal{L}\left(\mathcal{V},\mathcal{V}'\right)\right).$$

Let $F(t,x) : (0,\tau) \times \mathcal{H} \to \mathcal{H}$ and $F_0(t) = F(t,0)$. Assume that $F_0 \in L^2_\beta(0,\tau;\mathcal{H})$ and F satisfies the following continuity property:

$$\|F(t,x) - F(t,y)\|_{L^2(\mathbb{R}^d)} \le K\|x-y\|_{H^1(\mathbb{R}^d)},\, K > 0,\, x, y \in H^1\left(\mathbb{R}^d\right),\, t \in (0,\tau). \tag{4.1}$$

Therefore, applying Theorem 3.3, we conclude that for every $u_0 \in [\mathcal{H}; D(A(0))]_{\frac{1-\beta}{2}}$ the problem

$$u'(t) - \sum_{k,j=1}^{d} \partial_j \left(a_{kj}(t,.)\partial_k u(t)\right) + \sum_{j=1}^{d} b_j(t,.)\partial_j u(t) + c(t,.)u(t) = F(t,u(t)),\, t-\text{a.e.},\;\; u(0) = u_0$$

has a unique solution $u \in L^2_\beta(0,\tau; H^1(\mathbb{R}^d))$ such that $A(.)u \in L^2_\beta(0,\tau;\mathcal{H})$ and $u' \in L^2_\beta(0,\tau;\mathcal{H})$.

Remark 4.1 Observe that for all $\beta \in [0,1[$ we have

$$\left[L^2\left(\mathbb{R}^d\right); D(A(0))\right]_{\frac{1-\beta}{2}} = \left[L^2\left(\mathbb{R}^d\right); H^1\left(\mathbb{R}^d\right)\right]_{1-\beta} = H^{1-\beta}\left(\mathbb{R}^d\right).$$

The maximal regularity we proved here holds also in the case of elliptic operators on Lipschitz domains with Dirichlet or Neumann boundary conditions. The arguments are the same. One define the previous forms $\mathfrak{a}(t)$ with domain $\mathcal{V} = H_0^1(\Omega)$ (for Dirichlet boundary conditions) or $\mathcal{V} = H^1(\Omega)$ (for Neumann boundary conditions).

As an example of non linearity, we take

$$F(t, y) = f(t, x) + g(t, x)|y(x)|^\alpha + h(t, x) \sum_{i=1}^{i=n} \left| \frac{\partial y(x)}{\partial x_i} \right|^\gamma,$$

such that $\alpha, \gamma \in [0, 1]$ and $f \in L^2_\beta(0, \tau; \mathcal{H})$, $h \in L^\infty(0, \tau; L^{\frac{2}{1-\gamma}}(\mathbb{R}^d))$, $y \in H^1(\mathbb{R}^d)$ with

- $g \in L^\infty(0, \tau; L^2(\mathbb{R}^d))$ for $d = 1$.
- $g \in L^\infty(0, \tau; L^{\frac{2q}{q-1}}(\mathbb{R}^d))$ for $q \geq \frac{1}{\alpha}$ and $d = 2$.
- $g \in L^\infty(0, \tau; L^{\frac{2q}{q-1}}(\mathbb{R}^d))$ for $\frac{1}{\alpha} \leq q \leq \frac{d}{\alpha(d-2)}$ and $d > 2$.

Then, the function F satisfies the condition (3.1).

Funding The authors have not disclosed any funding.

Declarations

Competing interests A competing interests declaration is mandatory for publication in this journal. Please confirm that this declaration is accurate, or provide an alternative.

References

1. Achache, M.: Maximal regularity for the damped wave equations. J. Elliptic Parabol. Equ. **6**, 835–870 (2020)
2. Achache, M.; Ouhabaz, E.M.: Non-autonomous right and left multiplicative perturbations and maximal regularity. Stud. Math. **242**(1), 1–30 (2018)
3. Achache, M.; Ouhabaz, E.M.: Lions' maximal regularity problem with $H^{1/2}$-regularity in time. J. Differ. Equ. **266**, 3654–3678 (2019)
4. Achache, M.; Tebbani, H.: Non-autonomous maximal regularity in weighted space. Electron. J. Differ. Equ. **124**, 1–24 (2020)
5. Achache, M.: Non autonomous maximal regularity for the fractional evolution equations, 2020. To appear in journal of evolution equations. hal-0-2927759 (2020)
6. Arendt, W.; Dier, D.; Fackler, S.: J. L. Lions' problem on maximal regularity. Arch. Math. (Basel) **109**(1), 5972 (2017)
7. Auscher, P.; Axelsson, A.: Remarks on maximal regularity. Progress Nonlinear Differ. Equ. Appl. **80**, 45–55 (2011)
8. Auscher, P.; Tchamitchian, Ph.: Square roots of elliptic second order divergence operators on strongly Lipschitz domains. J. Anal. Math. **90**, 1–12 (2003)
9. Auscher, P.; Egert, M.: On non-autonomous maximal regularity for elliptic operators in divergence form. Arch. Math. (Basel) **107**(3), 271–284 (2016)
10. Cowling, M.; Doust, I.; McIntosh, A.; Yagi, A.: Banach space operators with a bounded H^∞ functional calculus. J. Austral. Math. Soc. Ser. A **60**(1), 51–89 (1996)
11. Colao, V.; Muglia, L.: Solutions to nonlocal evolution equations governed by non-autonomous forms and demicontinuous nonlinearities, arXiv:2110.03565
12. Fackler, S.: J. L. Lions' problem concerning maximal regularity of equations governed by non-autonomous forms. Ann. Inst. H. Poincaré Anal., Non Linéaire **34**(3), 699–709 (2017)
13. Lions, J.L.; Magenes, E.: Non-homogeneous Boundary Value Problems and Applications, vol. 1. Springer, Berlin (1972)
14. Lunardi, A.: *Interpolation theory*. Second. Appunti. Scuola Normale Superiore di Pisa (Nuova Serie). In: Lecture Notes, ScuolaNormale Superiore di Pisa.Edizioni della Normale, Pisa, Italy (2009)
15. Ouhabaz, E.M.: Analysis of Heat Equations on Domains, London Mathematical society monographs series, vol. 31. Princeton University Press, Princeton (2005)

16. Simon, J.: Sobolev, Besov and Nikolskii fractional spaces: imbeddings and comparisons for vector valued spaces on an interval. Ann. Mat. Pura Appl. **157**(4), 117–148 (1990)