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Numerical solution for a class of parabolic integro-differential equations subject to integral boundary conditions

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Abstract Many physical phenomena can be modelled through nonlocal boundary value problems whose boundary conditions involve integral terms. In this work we propose a numerical algorithm, by combining second-order Crank-Nicolson schema for the temporal discretization and Legendre-Chebyshev pseudo-spectral method (LC-PSM) for the space discretization, to solve a class of parabolic integrodifferential equations subject to nonlocal boundary conditions. The approach proposed in this paper is based on Galerkin formulation and Legendre polynomials. Results on stability and convergence are established. Numerical tests are presented to support theoretical results and to demonstrate the accuracy and effectiveness of the proposed method

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1 Introduction

In the last decades, the theory of integrodifferential equations has been extensively investigated by many researchers, and it has become a very active research area. The study of this class of equations ranges from the theoretical aspects of solvability and well-posedness to the analytic and numerical methods for obtaining solutions. A strong motivation for studying integrodifferential equations of PDEs type comes from the fact that they could serve as mathematical models for many problems in physics, mechanics, biology and other fields of sciences.

In this work, we are concerned with the numerical solution of the following parabolic integrodifferential equation:

$$\partial_t u(x,t) - \partial_x^2 u(x,t) = \int_0^t a(t-s)u(x,s)\mathrm{d}s + f(x,t), \quad x \in \Lambda, t \in J. \tag{1.1}$$

with the initial condition

$$u(x,0) = u_0(x), \quad x \in \Lambda. \tag{1.2}$$

subject to integral boundary conditions

$$\begin{aligned} \partial_x u(-1,t) &= \int_{-1}^1 u(x,t)K_1(x)\mathrm{d}x, \\ \partial_x u(1,t) &= \int_{-1}^1 u(x,t)K_2(x)\mathrm{d}x. \end{aligned} \tag{1.3}$$

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where Λ and J stand for the space domain $(-1, 1)$ and time interval $[0, T]$ with $T > 0$, respectively. The functions a, f, u_0, K_1 and K_2 are well-defined functions. Assume that the kernel a in the integral part of Eq. (1.1) is bounded, namely

$$|a(t-s)| \leq a_0, \quad t, s \in J. \tag{1.4}$$

Integrodifferential equations of the form (1.1), and other similar variants, arise in the mathematical modelling of many physical phenomena and practical engineering problems, such as nonlocal reactive flows in porous media [8,9], heat transfer in materials with memory [13,17], phenomena of visco-elasticity [7,19], gas diffusion problems [18], spatio-temporal development of epidemics [21], and so on.

Considerable work has been made on the area of nonlocal boundary values problems in the numerical and theoretical aspects. Indeed theoretical studies devoted to these classes of problems are usually connected with some difficulties due to the presence of an integral terms in the boundary conditions, this promoted researchers to perform some modifications and improvements on the classical methods to overcome this issue (see, e.g., [2,3,12,16]).

On the other hand, integrodifferential equations are usually too complicate to be solved analytically; this made the use of numerical methods is required to obtain approximate solutions. Many efforts have been undertaken to design and develop efficient numerical approaches for solving differential and integrodifferential equations with nonlocal boundary conditions. In [15], Merad and Martín-Vaquero presented a computational study for two-dimensional hyperbolic integrodifferential equations with purely integral conditions, in which, they demonstrated the existence and uniqueness of the solution and proposed a numerical approach based on Galerkin method. Authors in [11], utilized reproducing kernels approach to solve parabolic and hyperbolic integrodifferential equations subject to integral and weighted integral conditions. More recently, Bencheikh et al. [1] implemented numerical method, based on operational matrices of orthonormal Bernstein polynomials, to approximate the solution of an integrodifferential parabolic equation with purely nonlocal integral conditions. The problem under consideration in this paper has been well studied in [10], where the authors proved the existence and uniqueness of the solution using energy inequalities method, and for the numerical resolution, a numerical algorithm based on superposition principle is presented, where the original nonlocal problem was replaced by three auxiliary standard boundary value problems that solved using finite difference method.

As for the numerical methods, Spectral and pseudo-spectral methods [4,20] have gained increasing popularity in the numerical resolution of many types of problems. In the context of spectral methods, Legendre approximation has been used widely, and this Legendre–Galerkin spectral method has been shown to be computationally efficient and highly accurate with exponential rate of convergence. While plenty of papers have devoted to discussing the use of spectral methods for solving problems with classical boundary conditions. Surprisingly, a limited number of authors touched upon the implementation and analysis of the spectral methods for problems with nonlocal boundary conditions [6].

The primary aim in this paper is to present a suitable way to analysis and implement Legendre–Chebyshev pseudo-spectral method for the numerical resolution of a class of parabolic integrodifferential equations subject to non-local boundary conditions. The proposed approach is based on Galerkin formulation and used Legendre polynomials as a basis for the spatial discretization, followed by, temporal discretization using the trapezoidal method. Both efficiency and accuracy are achieved using the presented method, and the numerical experiments showed that (LC–PSM) can realize better accuracy compared to other existing methods and with less computational time.

This paper is organized as follows. In the next section, we briefly describe the way to implement Legendre–Chebyshev pseudo-spectral method for discretizing the parabolic integrodifferential equation (1.1). In Sect. 3, we first recall some lemmas and results related to spectral methods, and then, the stability and convergence of the method are established in L^2-norms. In Sect. 4, we provide some numerical tests to confirm the effectiveness and robustness of (LC–PSM) presented in this paper. Finally, in Sect. 5, we summarize some remarks on the main features of our method and cite some possible extensions.

2 Legendre–Galerkin spectral method

In the next subsections, we shall briefly describe the way to implement Legendre–Chebyshev pseudo-spectral method to approximate the solution of the nonlocal boundary value problem considered in this paper. As a starting point, we formulate the nonlocal problem (1.1)–(1.3) in weak formulation: find $u : J \to H^1(\Lambda)$ such that for any $v \in H^1(\Lambda)$

$$\begin{cases} (\partial_t u, v) + (\partial_x u, \partial_x v) - \mathcal{K}(u, v) = \int_0^t a(t-s)(u(s), v)\mathrm{d}s + (f, v), \quad t \in J, \\ u(0) = u_0. \end{cases} \tag{2.1}$$

where the functional $\mathcal{K}(\cdot,\cdot)$ is defined as follows:

$$\mathcal{K}(z, v) = v(1)\left(\int_\Lambda K_2(x)z(x)\mathrm{d}x\right) - v(-1)\left(\int_\Lambda K_1(x)z(x)\mathrm{d}x\right), \quad v, z \in H^1(\Lambda). \tag{2.2}$$

Here and in what follows, we use the notation $(\cdot,\cdot)$ to denote the L^2-inner product and $\|\cdot\|$ for the induced norm on the space $L^2(\Lambda)$. Denote by $H^m(\Lambda)$ the standard Sobolev space with norm and semi-norm denoted by $\|\cdot\|_m$ and $|\cdot|_m$, respectively. Solvability of the above variational problem is addressed in the following theorem [10].

Theorem 2.1 *Assume that a_0 satisfies rm (1.4). Then the variational problem (2.1) admits a unique weak solution in $L^2(J; H^1(\Lambda))$.*

2.1 Space discretization: LC–PSM

Let $\mathbb{P}_N(\Lambda)$ be the space consisting of all algebraic polynomials of degree at most N and denote by $I_N^C : L^2(\Lambda) \to \mathbb{P}_N(\Lambda)$ the operator of interpolation at Chebyshev–Gauss–Lobatto points $\xi_i = cos\left(\frac{i\pi}{N}\right), 0 \le i \le N$ defined as

$$I_N^C v(\xi_i) = v(\xi_i), \quad 0 \le i \le N, \quad v \in H^1(\Lambda).$$

Based on the above weak formulation, we pose the semi-discrete Legendre–Chebyshev Galerkin schema as: find $u_N : J \to \mathbb{P}_N(\Lambda)$ such that for any $v \in \mathbb{P}_N(\Lambda)$

$$\begin{cases} (\partial_t u_N, v) + (\partial_x u_N, \partial_x v) - \mathcal{K}(u_N, v) = \int_0^t a(t-s)(u_N(s), v)\mathrm{d}s + (I_N^C f, v), \\ u_N(0) = I_N^C u_0. \end{cases} \tag{2.3}$$

Let L_k be the kth degree Legendre polynomial defined by the following three-term recurrence formula:

$$L_0(x) = 1, \quad L_1(x) = x, \quad L_{k+1}(x) = \frac{2k+1}{k+1} x L_k(x) + \frac{k}{k+1} L_{k-1}(x), k \ge 1.$$

We recall that the set of Legendre polynomials is mutually orthogonal in $L^2(\Lambda)$, namely

$$(L_k, L_j) = \int_\Lambda L_k(x) L_j(x)\mathrm{d}x = \frac{2}{2k+1}\delta_{j,k}.$$

Let N be a positive integer, we define [5]

$$\begin{aligned} \varphi_k(x) &= \frac{1}{\sqrt{4k+6}}\left(L_k(x) - L_{k+2}(x)\right), \quad 0 \le k \le N-2, \\ \varphi_{N-1}(x) &= \frac{1}{2}\left(L_0(x) + L_1(x)\right), \\ \varphi_N(x) &= \frac{1}{2}\left(L_0(x) - L_1(x)\right). \end{aligned} \tag{2.4}$$

The following lemma is the key technique in our algorithm.

Lemma 2.2 *[22] For two integer $j, k \in \mathbb{N}$, let us denote,*

$$m_{j,k} = m_{k,j} = (\varphi_j, \varphi_k) = \int_{-1}^{1} \varphi_j(x)\varphi_k(x)\mathrm{d}x,$$
$$p_{j,k} = p_{k,j} = (\varphi'_j, \varphi'_k) = \int_{-1}^{1} \varphi'_j(x)\varphi'_k(x)\mathrm{d}x.$$

Then, for $0 \le j, k \le N-2$

$$m_{j,k} = m_{k,j} = \begin{cases} \frac{1}{4k+6}\left(\frac{2}{2k+1} + \frac{2}{2k+5}\right), & j = k, \\ -\frac{1}{\sqrt{4k+6}} \cdot \frac{1}{\sqrt{4(k+2)+6}} \cdot \frac{2}{2k+5}, & j = k \pm 2, \\ 0, & \textit{otherwise}. \end{cases}$$

and

$$p_{j,k} = p_{k,j} = \delta_{jk} = \begin{cases} 1, & j = k, \\ 0, & \textit{otherwise}. \end{cases}$$

Thanks to linear algebra arguments on can easily prove that

$$\mathbb{P}_N(\Lambda) = span\,\{\varphi_k, 0 \le k \le N\},$$

Consequently, the numerical solution u_N of (2.3) can be expanded in terms of $(\varphi_k)_{k=0}^N$ with time-dependent coefficients, namely

$$u_N(x,t) = \sum_{k=0}^{N} \alpha_k(t)\varphi_k(x), \quad (x,t) \in \Lambda \times J. \tag{2.5}$$

Inserting (2.5) into (2.3) and taking $v = \varphi_j$, $0 \le j \le N$, we obtain the following system of ODEs

$$\sum_{k=0}^{N} m_{jk}\alpha'_k(t) + \sum_{k=0}^{N} (p_{jk} - q_{jk})\alpha_k(t) = \sum_{k=0}^{N} m_{jk}A_k(t) + (I_N^C f, \varphi_j), \quad 0 \le j \le N. \tag{2.6}$$

where

$$m_{jk} = (\varphi_k, \varphi_j), \quad p_{jk} = (\varphi'_k, \varphi'_j), \quad q_{jk} = \mathcal{K}(\varphi_k, \varphi_j), \quad A_k(t) = \int_0^t a(t-s)\alpha_k(s)\mathrm{d}s.$$

with initial conditions

$$\sum_{k=0}^{N} p_{jk}\alpha_k(0) = (I_N^C u_0, \varphi_j), \quad 0 \le j \le N. \tag{2.7}$$

Denote

$$\mathbf{A}(t) = (A_0(t), A_1(t), \ldots, A_N(t))^t,$$
$$\mathbf{U}(t) = (\alpha_0(t), \alpha_1(t), \ldots, \alpha_N(t))^t,$$
$$\mathbf{U}_0 = (u_0^0, u_1^0, \ldots, u_N^0)^t, \quad u_j^0 = \int_\Lambda I_N^C u_0(x)\phi_j(x)\mathrm{d}x,$$
$$\mathbf{F}(t) = (f_0(t), f_1(t), \ldots, f_N(t))^t, \quad f_j(t) = \int_\Lambda I_N^C f(x,t)\phi_j(x)\mathrm{d}x$$
$$\mathbf{M} = [m_{jk}]_{0 \le j,k \le N}, \quad \mathbf{P} = [p_{jk}]_{0 \le j,k \le N}, \quad \mathbf{Q} = [q_{jk}]_{0 \le j,k \le N}.$$

Then, the initial value problem (2.6) and (2.7) can be written in matrix formulation as follows:

$$\begin{aligned} &\mathbf{M}\mathbf{U}'(t) + (\mathbf{P} - \mathbf{Q})\mathbf{U}(t) = \mathbf{M}\mathbf{A}(t) + \mathbf{F}(t), \\ &\mathbf{U}(0) = \mathbf{U}_0. \end{aligned} \tag{2.8}$$

The coefficients m_{jk} and p_{jk} are already determined in Lemma (2.2). For the matrix $\mathbf{Q}$, one can uses the values of $\phi_j(\pm 1)$ to determinate its entries. In fact, since $\phi_j(\pm 1) = 0$ for $0 \leq j \leq N-2$, hence $\mathbf{Q}$ is almost-null matrix except the two last rows whose entries

$$q_{N-1,k} = \int_{\Lambda} K_2(x)\varphi_k(x)\mathrm{d}x, \quad q_{N,k} = \int_{\Lambda} K_1(x)\varphi_k(x)\mathrm{d}x, \quad 0 \leq k \leq N.$$

2.2 Fully-discretization schema

For time advancing, we use the second-order Crank–Nicolson scheme to discretize the differential system (2.8). For a given positive integer M, we define the step time $\Delta t = \frac{T}{M}$. Let $t_i = i\Delta t$, $(i = 0 \cdots, M)$, we denote by α_k^i and A_k^i the approximations of $\alpha_k(t_i)$ and $A_k(t_i)$, respectively.
The fully discretization LC–PSM/CN for (1.1)–(1.3) leads to the following recurrent algebraic system

$$\begin{aligned} &(\mathbf{M} + \Delta t(\mathbf{P} - \mathbf{Q}))\,\mathbf{U}^{i+1} = (\mathbf{M} - \Delta t(\mathbf{P} + \mathbf{Q}))\,\mathbf{U}^i + \Delta t(\mathbf{F}^{i+1} + \mathbf{F}^i) + \Delta t\mathbf{M}(\mathbf{A}^{\mathbf{i}}) \quad, i \geq 1 \\ &\mathbf{M}\mathbf{U}^0 = \mathbf{U}(0) \end{aligned}$$

where

$$\mathbf{U}^i = (\alpha_0^i, \alpha_1^i, \ldots, \alpha_N^i)^t, \;\; \mathbf{F}^i = (f_0(t_i), f_1(t_i), \ldots, f_N(t_i))^t, \;\; \mathbf{A}^i = (A_0^i, A_1^i, \ldots, A_N^i)^t.$$

The above algebraic system can be solved easily using either direct or iterative methods. As a choice, on can use QR factorization method, given its accurate results and ease of implementation.

3 Error analysis

In this section, we derive L^2-error estimate for the error $e_N(t) = u_N(t) - u(t)$. For this purpose, we first, in the next subsection, recall a sequence of lemmas that will be needed to perform the error analysis.

3.1 Preliminaries

Now, we introduce two projection operators and their approximation properties. First, let $P_N : L^2(\Lambda) \to \mathbb{P}_N(\Lambda)$ be the L^2-orthogonal projection, namely

$$(P_N v, \varphi) = (v, \varphi), \quad \forall \varphi \in \mathbb{P}_N(\Lambda).$$

We also define the operator $P_N^1 : H^1(\Lambda) \to \mathbb{P}_N(\Lambda)$ such that

$$P_N^1 v(x) = v(-1) + \int_{-1}^{x} P_{N-1}\partial_y v(y)\mathrm{d}y.$$

From the definition of P_N^1, one can obtain

$$(\partial_x P_N^1 v - \partial_x v, \partial_x \varphi) = 0, \quad \forall \varphi \in \mathbb{P}_N(\Lambda). \tag{3.1}$$

Next, we give the approximation property of the projection operator P_N^1 and the interpolation operator I_N^C.

Lemma 3.1 *[14] If $v \in H^r(\Lambda)$ with $r \geq 1$, then the following estimate holds*

$$\|v - P_N^1 v\|_l \leq CN^{l-r}\|v\|_r, \quad 0 \leq l \leq 1. \tag{3.2}$$

where $C > 0$ is a positive constant independent on N.

Lemma 3.2 *[14] Let $v \in H^1(\Lambda)$, there exists a positive constant C independent on N such that*

$$N\|I_N^C v - v\| + |I_N^C v|_1 \leq C\|v\|_1. \tag{3.3}$$

Moreover, if $v \in H^r(\Lambda)$ for $r \leq 1$, then the following estimate holds

$$\|v - I_N^C v\|_r \leq C N^{r-s}\|v\|_s, \quad 0 \leq r \leq 1. \tag{3.4}$$

where $C > 0$ is a positive constant independent on N.

Remark 3.3 Under the same assumptions of Lemma (3.2), we can obtain using approximation property (3.3) the following inequality

$$\|I_N^C v\| \leq C\|v\|_1. \tag{3.5}$$

Now, we derive a basic estimate that will be used later in our proofs.

Lemma 3.4 *[5] Let $\mathcal{K}(\cdot,\cdot)$ defined by (2.2). Assume that $K_1, K_2 \in L^2(\Lambda)$. Then, for any $w, v \in H^1(\Lambda)$, the following estimate holds*

$$|\mathcal{K}(w, v)| \leq C_\varepsilon\left(\|w\|^2 + \|v\|^2\right) + \varepsilon|v|_1^2. \tag{3.6}$$

3.2 Error estimates

In this subsection, we consider the stability and convergence of the semi-discrete approximation (2.3). We first state a Gronwall-type inequality that will be used in the proof of our main results.

Lemma 3.5 *Let $E(t)$ and $H(t)$ be two non-negative integrable functions on $[0, T]$ satisfying*

$$E(t) \leq H(t) + C_1\int_0^t E(s)\mathrm{d}s + C_2\int_0^t\int_0^s E(r)\mathrm{d}r\mathrm{d}t, \quad t \in [0, T], \tag{3.7}$$

where $C_1, C_2 \in \mathbb{R}^+$. Then there exists $C > 0$ such that

$$E(t) \leq e^{Ct}H(t), \quad t \in [0, T]. \tag{3.8}$$

Proof For a non-negative function $E(t)$, we perform a permutation of variables to obtain:

$$\int_0^t\int_0^s E(r)\mathrm{d}r\mathrm{d}s = \int_0^t (t - v)E(v)\mathrm{d}v \leq C\int_0^t E(s)\mathrm{d}s.$$

Hence, inequality (3.7) of Lemma (3.5) becomes

$$E(t) \leq H(t) + C\int_0^t E(s)\mathrm{d}s, \quad t \in [0, T].$$

Now, applying the standard Gronwall inequality yields the desired estimate (3.8). □

Theorem 3.6 *Let $u_0 \in H^1(\Lambda)$ and $f \in C^1\left(0, T; H^1(\Lambda)\right)$. Then the solution $u_N(t)$ of (2.3) satisfies*

$$\|u_N(t)\|^2 + \int_0^t |u_N(s)|_1^2\mathrm{d}s \leq C\left(\int_0^t \|f(s)\|_1^2\mathrm{d}s + \|u_0\|_1^2\right), \quad t \in J. \tag{3.9}$$

Proof Let $t \in J$, setting $u_N(t) = v$ in

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_N(t)\|^2 + |u_N(t)|_1^2 = \int_0^t a(t-s)(u_N(s), u_N(t))\mathrm{d}s + (I_N^C f(t), u_N(t)) + \mathcal{K}(u_N(t), u_N(t)) =: I_1 + I_2 + I_3, \quad t \in J. \tag{3.10}$$

We have to estimate the terms on the right-hand side of (3.10). For the first term I_1, we use the hypothesis (1.4) and then apply Cauchy and Young inequalities.

$$\begin{aligned} |I_1| &\le \int_0^t |a(t-s)(u_N(s), u_N(t))\mathrm{d}s \\ &\le a_0 \int_0^t |(u_N(s), u_N(t))\mathrm{d}s \\ &\le \frac{a_0}{2}\left(\|u_N(t)\|^2 + \int_0^t \|u_N(s)\|^2\mathrm{d}s\right). \end{aligned} \tag{3.11}$$

Next, combining Cauchy and Young inequalities with approximation property (3.5) to estimate I_2.

$$\begin{aligned} |I_2| &\le |(I_N^C f(t), u_N(t))| \\ &\le \frac{1}{2}\|I_N^C f(t)\|^2 + \frac{1}{2}\|u_N(t)\|^2 \\ &\le C_1\|f(t)\|_1^2 + \frac{1}{2}\|u_N(t)\|^2. \end{aligned} \tag{3.12}$$

The estimate of I_3 is an immediate consequence of Lemma (3.6), namely

$$|\mathcal{K}(u_N(t), u_N(t))| \le C_\varepsilon\|u_N(t)\|^2 + \varepsilon|u_N(t)|_1. \tag{3.13}$$

Putting things together and choosing $0 < \varepsilon < 1$ yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_N(t)\|^2 + |u_N(t)|_1^2 \le C_2\|u_N(t)\|^2 + C_3\|f(t)\|_1^2 + C_4\int_0^t \|u_N(s)\|^2\mathrm{d}s. \tag{3.14}$$

Integrating both sides of (3.14) form 0 to t, we obtain

$$E(t) \le C_5\int_0^t E(s)\mathrm{d}s + C_6\int_0^t\int_0^s E(r)\mathrm{d}r\mathrm{d}s + H(t), \quad t \in J, \tag{3.15}$$

where

$$E(t) = \|u_N(t)\|^2 + \int_0^t |u_N(s)|_1^2\mathrm{d}s,$$

$$H(t) = \int_0^t \|f(s)\|_1^2\mathrm{d}s + \|u_N(0)\|^2.$$

Thanks to the Gronwall-type inequality (3.5), we get

$$E(t) \le H(t)e^{Ct}, \quad t \in J.$$

Because of $u_N(0) = I_N^C u_0 = (I_N^C u_0 - u_0) + u_0$, we use approximation properties (3.3) and (3.5) to obtain $\|u_N(0)\| \le C\|u_0\|_1^2$. Then it is easy to show the desired result. □

Let $u(t)$ and $u_N(t)$ be the solutions to (2.1) and (2.3), respectively. Denoting

$$\theta_N(t) = u_N(t) - P_N^1 u(t), \text{ and } \rho_N(t) = P_N^1 u(t) - u(t), \quad \forall t \in J$$

Then, we have the following estimate.

Lemma 3.7 *Assume that $u \in C^1(0, T; H^r(\Lambda))$, $r \geq 2$. Then the following estimate holds*

$$\|\theta_N(t)\| \leq CN^{-r}, \quad t \in J. \tag{3.16}$$

where $C > 0$ is a positive constant independent on N.

Proof From (2.1), (2.3) and (3.1) we know that for a fixed $t \in J$ the $\theta_N(t)$ satisfies for all $v \in \mathbb{P}_N(\Lambda)$ the following error equation:

$$(\partial_t\theta_N(t), v) + (\partial_x\theta_N(t), \partial_x v) = \int_0^t a(t-s)(\theta_N(s), v)\mathrm{d}s \; + \\ \int_0^t a(t-s)(\rho_N(s), v)\mathrm{d}s + \left(I_N^C f(t) - f(t), v\right) - (\partial_t\rho_N(t), v) + \mathcal{K}(\theta_N(t) + \rho_N(t), v). \tag{3.17}$$

Setting $v = \theta_N(t)$ in (3.17), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_N(t)\|^2 + |\theta_N(t)|_1^2 \leq I_1 + I_2 + I_3 + I_4 + I_5. \tag{3.18}$$

where

$$I_1 = \int_0^t |a(t-s)(\theta_N(s), \theta_N(t))|\mathrm{d}s, \quad I_2 = \int_0^t |a(t-s)(\rho_N(s), \theta_N(t))|\mathrm{d}s,$$
$$I_3 = |\left(I_N^C f - f, \theta_N(t)\right)|, \quad I_4 = |(\partial_t\rho_N(t), \theta_N(t))|, \quad I_5 = |\mathcal{K}(\theta_N(t) + \rho_N(t), \theta_N(t))|$$

Now, we estimate the terms on the right hand-side of inequality (3.17) using a standard procedure. For the term I_1, we apply Cauchy and Young inequalities and take into account (1.4),

$$\begin{aligned} I_1 &= \int_0^t |a(t-s)(\theta_N(s), \theta_N(t))|\mathrm{d}s \\ &\leq a_0 \int_0^t \|\theta_N(t)\| \cdot \|\theta_N(s)\|\mathrm{d}s \\ &\leq C_1\left(\|\theta_N(t)\|^2 + \int_0^t \|\theta_N(s)\|^2\mathrm{d}s\right) \end{aligned} \tag{3.19}$$

In a similar manner, we can obtain for I_2

$$I_2 \leq C\left(\|\rho_N(t)\|^2 + \int_0^t \|\theta_N(s)\|^2\mathrm{d}s\right)$$

By virtue of approximation property (3.2), we bound I_2 as follows

$$I_2 \leq C_2N^{-2r}\|u(t)\|_r^2 + C_3\int_0^t \|\theta_N(s)\|^2\mathrm{d}s \tag{3.20}$$

For the term I_3

$$\begin{aligned} I_3 &= |(I_N^C f(t) - f(t), \theta_N(t))| \\ &\leq \|I_N^C f(t) - f(t)\| \cdot \|\theta_N(t)\| \\ &\leq C_4N^{-2r}\|f(t)\|_r^2 + \|\theta_N(t)\|^2. \end{aligned} \tag{3.21}$$

Similarly,

$$I_4 \leq C_5N^{-2r}\|\partial_t u(t)\|_r^2 + \|\theta_N(t)\|^2. \tag{3.22}$$

To estimate of the term I_5 we use Lemma (3.4). Setting $w = \theta_N(t) + \rho_N(t)$ and $v = \theta_N(t)$ in (3.6) yields

$$|I_5| = |\mathcal{K}(\theta_N(t) + \rho_N(t), \theta_N(t))|$$

$$\leq C_\varepsilon\left(\|\theta_N(t)+\rho_N(t)\|^2+\|\theta_N(t)\|^2\right)+\varepsilon|\theta_N(t)|_1^2 \tag{3.23}$$

using the triangular inequality

$$|I_5|\leq C_\varepsilon\left(\|\rho_N(t)\|^2+\|\theta_N(t)\|^2\right)+\varepsilon|\theta_N(t)|_1^2 \tag{3.24}$$

hence, due to Lemma (3.1), on can obtain,

$$|I_5|\leq C_\varepsilon\|\theta_N(t)\|^2+\varepsilon|\theta_N(t)|^2+C_5N^{-2r}\|u\|_r^2 \tag{3.25}$$

In virtue of above estimates, the inequality (3.18) becomes

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_N(t)\|^2+|\theta_N(t)|_1^2\leq C_4N^{-2r}\left(\|f\|_r^2+\|\partial_t u\|_r^2+\|u\|_r^2\right)+C_\varepsilon\|\theta_N(t)\|_r^2+\varepsilon|\theta_N(t)|_1^2 \tag{3.26}$$

By taking ε sufficiently small and integrating (3.26) over $(0, t)$, we obtain

$$E(t)\leq H(t)+C\int_0^t E(s)\mathrm{d}s+C'\int_0^t\int_0^s E(r)\mathrm{d}r\mathrm{d}s,\quad t\in J \tag{3.27}$$

where

$$\begin{aligned}E(t)&=\|\theta_N(t)\|^2+\int_0^t|\theta_N(s)|_1^2\mathrm{d}s\\ H(t)&=CN^{-2r}\int_0^t\left(\|f(s)\|_r^2+\|\partial_t u(s)\|_r^2+\|u(s)\|_r^2\right)\mathrm{d}s+\|\theta_N(0)\|^2\end{aligned} \tag{3.28}$$

Gronwall-type inequality (3.5) implies

$$E(t)\leq H(t)e^{Ct},\quad t\in J. \tag{3.29}$$

Take into account,

$$\theta_N(0)=P_N^1u_0-I_N^Cu_0=\left(P_N^1u_0-u_0\right)+\left(u_0-I_N^Cu_0\right)$$

and approximation results (3.2) and (3.4), we obtain

$$\|\theta_N(0)\|^2\leq CN^{-2r}\|u_0\|_r^2 \tag{3.30}$$

Inserting (3.30) into (3.27) yields

$$\|\theta_N(t)\|^2+\int_0^t|\theta_N(s)|^2\mathrm{d}x\leq C\left(\int_0^t\left(\|f(s)\|_r^2+\|\partial_t u(s)\|_r^2+\|u(s)\|_r^2\right)\mathrm{d}s+\|u_0\|_r^2\right)$$

for all $0<t\leq T$, which is the desired result. □

Now, we are in position to state our main result concerning the convergence of the semi-discrete approximation (2.3).

Theorem 3.8 *Let $u(t)$ and $u_N(t)$ be the solution of (2.1) and (2.3), respectively. If $u\in C^1(0,T;H^r(\Lambda))$ with $r\geq 1$, then the following error estimate holds,*

$$\|u(t)-u_N(t)\|\leq CN^{-r},\quad t\in J. \tag{3.31}$$

where $C>0$ is a positive constant independent on N.

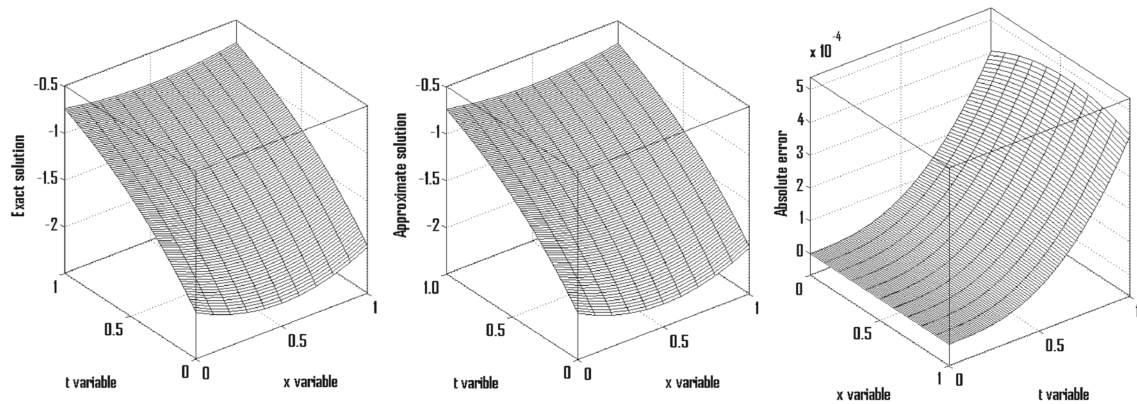
Proof Using triangular inequality, we have

$$\|u(t)-u_N(t)\|\leq\|u_N(t)-P_N^1u(t)\|+\|P_N^1u(t)-u(t)\|=\|\theta_N(t)\|+\|\rho_N(t)\|$$

By the aid Lemmas (3.2) and (3.7), for all $t\in J$ we obtain

$$\|u(t)-u_N(t)\|\leq CN^{-r}+C'N^{-r} \tag{3.32}$$

This completes the proof. □



(A) Exact solution (B) Approximate solution (C) Absolute error

Fig. 1 Profiles of exact and approximate solutions, and the absolute error for step time $\tau = 10^{-3}$

Table 1 L_∞-errors with different discretization parameters for Example (4.1)

Algorithm $A2$ [10]	$M_x = 20$ $N_t = 40$	$M_x = 40$ $N_t = 40$	$M_x = 80$ $N_t = 40$	$M_x = 160$ $N_t = 40$
$\|u^* - u_N\|_\infty$	$6.8285e-003$	$1.7008e-003$	$4.2479e-004$	$1.0617e-004$
Our method	$M_x = 4$ $N_t = 40$	$M_x = 4$ $N_t = 40$	$M_x = 4$ $N_t = 40$	$M_x = 4$ $N_t = 40$
$\|u^* - u_N\|_\infty$	$2.9416e-003$	$7.3521e-004$	$1.8378e-004$	$4.5932e-005$

Table 2 Absolute errors of some numerical solutions at $t = 0.5$ for Example (4.1)

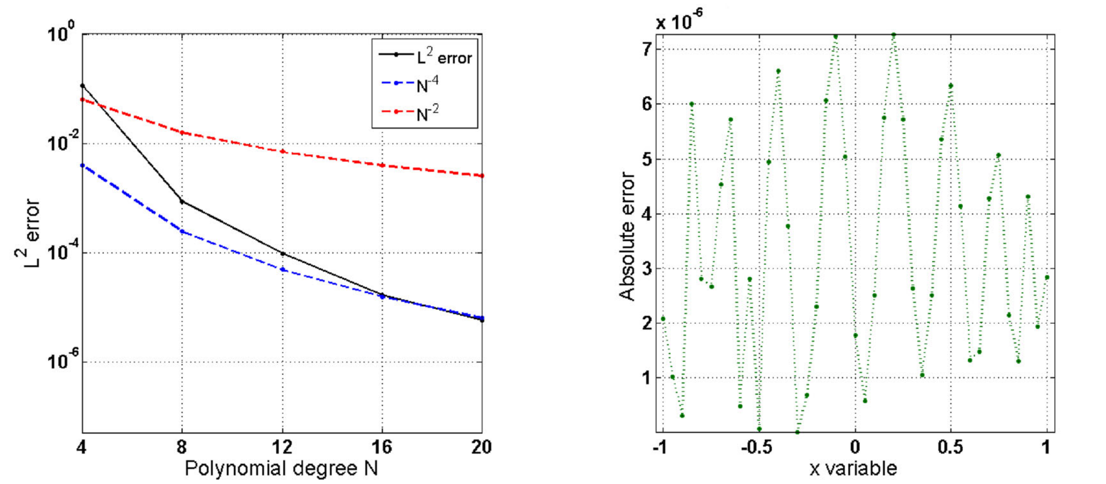
(x, t)	Our method $M_x = 4$, $N_t = 640$	Algorithm $A2$ [10] $M_x = 1280$, $N_t = 640$
(0.2, 0.5)	$1.1040e-004$	$1.5000e-004$
(0.6, 0.5)	$1.1540e-004$	$1.6840e-004$
(1.0, 0.5)	$1.0340e-004$	$1.6220e-004$
CPU time (s)	8.08	438.62

Table 3 Spatial convergence rates at $t = 1$ for Example 4.2

N	L^2-error	order	L^∞-error	order
4	1.1202e−001	–	1.2161e−001	–
8	8.6098e−004	$N^{-7.02}$	1.2361e−003	$N^{-6.62}$
12	9.5949e−005	$N^{-5.41}$	1.4962e−004	$N^{-5.21}$
16	1.6561e−005	$N^{-6.11}$	3.0725e−005	$N^{-5.50}$
20	5.7758e−006	$N^{-4.72}$	7.2971e−006	$N^{-6.44}$
24	5.3747e−006	–	7.0248e−006	–

4 Numerical experiments

In this section, we carry out several numerical experiments to verify the efficiency and accuracy of the proposed (LC–PSM), and we will compare our results against results obtained using other methods.



(A) Convergence rate in L^2-norm at $t = 1$. (B) Point-wise absolute error at $t = 1$.

Fig. 2 **A** L^2-norm versus N. **B** Pointwise absolute errors with $N = 20$, $\Delta t = 10^{-2}$ for Example (4.2) at $t = 1$

Example 4.1 In this first test problem, the following parabolic integrodifferential equation is considered

$$\partial_t u(x,t) - \partial_x^2 u(x,t) = 2\int_0^t e^{t-s} u(x,t) \mathrm{d}s + f(x,t),$$
$$\partial_x u(0,t) = \frac{-6}{13}\int_{-1}^{1} u(x,t)\mathrm{d}x,$$
$$\partial_x u(1,t) = \frac{6}{13}\int_{-1}^{1} u(x,t)\mathrm{d}x.$$

where $f(x,t) = -(x^2 - x - 2)(-3e^{-t} - 4t + 2t^2 + 4) - 2e^{-t}$ and $u_0(x) = x^2 - x - 2$. The exact solution to the above integrodifferential problem is given as

$$u^*(x,t) = (x^2 - x - 2)e^{-t}.$$

Figure 1 presents the computational results obtained by applying (LC–PSM) to the above test problem, where the profiles of exact and approximate solutions as well as the absolute error are plotted.

From the numerical results illustrated in Fig. 1, one can observe that the approximate solution shows a great agreement with the exact solution, which confirms that (LG–PSM) yields a very accurate an efficient numerical method for the numerical resolution of nonlocal boundary value problems of integrodifferential parabolic type.

For comparison purposes, in Tables 1 and 2 we compared our computational results with the results obtained in [10]. Obviously, the proposed (LC–PSM) in this paper gives more accurate solutions with less CPU time than the finite difference schema used in mentioned reference.

Example 4.2 To examine the spatial discretization, we take in this example a test problem that has an analytic solution with limited regularity. Let us consider the following problem:

$$\partial_t u(x,t) - \partial_x^2 u(x,t) = \int_0^t e^{-(t-s)} u(x,t) \mathrm{d}s + f(x,t),$$
$$\partial_x u(-1,t) = \int_{-1}^{1} x(x+1)^{1/2}(1-x)u(x,t)\mathrm{d}x,$$
$$\partial_x u(-1,t) = \int_{-1}^{1} \sqrt{1-x}(x^3 - x)u(x,t)\mathrm{d}1x.$$

The exact solution is given as the following:

$$u^*(x,t) = e^t (x+1)^{\frac{5}{2}} (x-1)^2.$$

We first choose a step time small enough so that the error of the temporal discretization can be eliminated, and make the polynomial degree N varies. Table 3 shows the error in L^2 and L^∞-norms at a selected point $t = 1$ and by going through each line one can observe an increasing accuracy until the error of the temporal discretization becomes dominant.

To examine the theoretical result, we plot in Fig. 2 the decay rates of error in L^2-norm versus N in a log-scale and the lines of decay rates N^{-2} and N^{-4}. As expected, L^2-error of (LC-PSM) for the solved problem in this example has a rate of convergence between N^{-3} and N^{-4} , which supports the results established in Theorem (3.8) since $u \in H^3(\Lambda)$ and $u \notin H^4(\Lambda)$

5 Conclusions

In this paper, we are concerned in the implement and analysis of the spectral method to solve a class of integrodifferential parabolic equations subject to nonlocal boundary conditions of Neumann-type. We combined the Legendre spectral method based on Galerkin formulation to discretize the problem in the spatial direction and the second-order Crank–Nicolson finite difference schema for the temporal discretization. Rigorous error analysis has been carried out in L^2-norm for the proposed method, and the computational results of numerical examples have supported the theoretical results. Moreover, a comparison with fully finite-difference schema clearly shows that the presented method is computationally superior with less required CPU time. It should be noted that other high-order methods can be used for time integration to improve the accuracy of the fully discretization. Convergence and stability of such combinations are still undiscussed.

In future works, we plan to investigate how to implement space–time spectral method for the resolution of this class and other challenging models, such as nonlocal boundary value problems in the two-dimensional case and fractional integrodifferential problems.

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Declarations

Conflict of interest The author declares that he has no competing interests.

References

1. Bencheikh, A.; Chiterb, L.; Lic, T.: Solving parabolic integro-differential equations with purely nonlocal conditions by using the operational matrices of Bernstein polynomials. J. Nonlinear Sci. Appl. **11**(5), 624–634 (2018)
2. Boumaza, N.; Gheraibia, B.: On the existence of a local solution for an integro-differential equation with an integral boundary condition. Bol. Soc. Math. Mex. **26**, 521–534 (2020)
3. Bouziani, A.; Mechri, B.: The Rothe's method to a parabolic integrodifferential equation with a nonclassical boundary conditions. Int. J. Stoch. Anal. **2010**, 1–16 (2010)
4. Canuto, C.; Hussaini, M.Y.; Quarteroni, A.; Zang, T.A.: Spectral Methods: Evolution to Complex Geometries and Applications to Fluid Dynamics. Springer, Berlin (2007)
5. Chattouh, A.; Saoudi, K.: Legendre–Chebyshev pseudo-spectral method for the diffusion equation with non-classical boundary conditions. Moroc. J. Pure Appl. Anal. **6**(2), 303–317 (2020)
6. Chattouh, A.; Saoudi, K.: Error analysis of Legendre–Galerkin spectral method for a parabolic equation with Dirichlet-type non-local boundary conditions. Math. Model. Anal. **26**(2), 287–303 (2021)
7. Christensen, R.M.: Theory of Viscoelasticity. Academic Press, New York (1971)
8. Cushman, J.H.; Ginn, T.R.: Nonlocal dispersion in media with continuously evolving scales of heterogeneity. Transp. Porous Media **13**(1), 123–138 (1993)
9. Dagan, G.: The significance of heterogeneity of evolving scales to transport in porous formations. Water Resour. Res. **30**(12), 3327–3336 (1994)

10. Dehilis, S.; Bouziani, A.; Oussaeif, T.: Study of solution for a parabolic integrodifferential equation with the second kind integral condition. Int. J. Anal. Appl. **16**(4), 569–593 (2018)
11. Fardi, M.; Ghasemi, M.: Solving nonlocal initial-boundary value problems for parabolic and hyperbolic integro-differential equations in reproducing kernel Hilbert space. Numer. Methods Partial Differ. Equ. **33**(1), 174–198 (2017)
12. Guezane-Lakoud, A.; Belakroum, D.: Time-discretization schema for an integrodifferential Sobolev type equation with integral conditions. App. Math. Comput. **218**(9), 4695–4702 (2012)
13. Gurtin, M.E.; Pipkin, A.C.: A general theory of heat conduction with nite wave speed. Arch. Ration. Mech. Anal. **31**(2), 113–126 (1968)
14. Ma, H.P.: Chebyshev–Legendre spectral viscosity method for nonlinear conservation laws. SIAM J. Numer. Anal. **35**(3), 869–892 (1998)
15. Merad, A.; Martin-Vaquero, J.: A Galerkin method for two-dimensional hyperbolic integro-differential equation with purely integral conditions. Appl. Math. Comput. **291**, 386–394 (2016)
16. Merad, A.; Bouziani, A.; Ozel, C.; Kiliçman, A.: On solvability of the integrodifferential hyperbolic equation with purely nonlocal conditions. Acta Math. Sci. **35**(3), 1–9 (2015)
17. Miller, R.K.: An integro-differential equation for grid heat conductors with memory. J. Math. Anal. Appl. **66**, 313–332 (1978)
18. Raynal, M.: On some nonlinear problems of diffusion. Lect. Notes Math. **737**, 251–266 (1979)
19. Rcnardy, M.: Mathematical analysis of viscoelastic flows. Ann. Rev. Fluid Mech. **21**, 21–36 (1989)
20. Shen, J.; Tang, T.; Wang, L.L.: Spectral Methods: Algorithms. Analysis and Applications. Springer, Heidelberg (2011)
21. Thieme, H.R.: A model for the spatio spread of an epidemic. J. Math. Biol. **4**(4), 337–351 (1977)
22. Zhao, T.; Li, C.; Zang, Z.; Wu, Y.: Chebyshev–Legendre pseudo-spectral method for the generalised Burgers–Fisher equation. Appl. Math. Mod. **36**(3), 1046–1056 (2012)