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# Antinormality and $m$ -isometry of shift on weighted trees

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**Abstract** The shift operator and its various generalizations are amongst the most widely studied operators on a Hilbert space. In this paper, we characterize antinormal and  $m$ -isometric shift operator  $S$  on the Hilbert space  $L^2(\mathcal{T}, \lambda)$  associated with a locally finite directed weighted tree  $\mathcal{T}$ . We also discuss interesting connections between antinormality and  $m$ -isometry of  $S$ .

**Mathematics Subject Classification** 47B37 · 47A05 · 05C05 · 05C63

## 1 Introduction

Throughout this paper, we denote the set of all natural numbers, the set of all integers, the set of all real numbers and the set of all complex numbers by  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{R}$  and  $\mathbb{C}$  respectively. We denote the set of all non-negative integers and set of all non-positive integers by  $\mathbb{Z}^+$  and  $\mathbb{Z}^-$ , respectively. For a given set  $A$ ,  $\text{card}(A)$  denotes the cardinality of  $A$ . Let  $H$  denote a separable complex Hilbert space and,  $\mathcal{B}(H)$  and  $\mathcal{N}$ , respectively, be the set of all linear bounded operators on  $H$  and the set of all normal operators on  $H$ . For a  $T \in \mathcal{B}(H)$ ,  $T^*$  denotes Hilbert adjoint of  $T$ .

A fascinating problem in the Hilbert space operator theory is to determine the distance between a particular operator and a collection of operators. Important examples of such collections are the set of all unitary operators, the set of all self adjoint operators, the set of all normal operators and the set of all compact operators. In this setting, a deep and interesting case is whether there exists a best approximation. Holmes [8] studied best normal approximation and introduced the notion of “antinormal operator”. A particularly simple but interesting example of an antinormal operator is the classical right shift operator of multiplicity one on the Hilbert space  $l^2(\mathbb{N})$ . Antinormal operators have been extensively studied by several authors. For the historical details, we refer to [9, 11–13, 15, 16].

Jabłoński et al. [10] introduced weighted shifts on directed trees. Martínez-Avendaño [14] analyzed the dynamical property of shift operator on weighted directed trees. In this paper, we completely characterize the antinormality of shift operator on weighted directed trees.

The natural generalization of an isometric operator is the  $m$ -isometric operator on Hilbert space. The notion of  $m$ -isometric operator was introduced by Agler [5]. As stated by Agler and Stankus [3, 4, 6] the concept of  $m$ -isometries in case when  $m > 1$  is interesting and deep as it requires ideas from many areas of mathematics such as distribution theory and function theory. Abdullah and Le [1] gave a result on  $m$ -isometry of unilateral

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weighted shift. We extend their result on the shift on a weighted directed tree. We also relate antinormality and  $m$ -isometry of the shift on a weighted directed tree. We now state various definitions and results relevant to our study.

**Definition 1.1** [8] An operator  $T \in \mathcal{B}(H)$  is said to be *antinormal* if  $d(T, \mathcal{N}) = \inf_{N \in \mathcal{N}} \|T - N\| = \|T\|$ .

Observe that an operator  $T \in \mathcal{B}(H)$  is antinormal if and only if  $0 \in \mathcal{B}(H)$  is the best normal approximation. We note that antinormality of an operator  $T \in \mathcal{B}(H)$  is necessary as well as sufficient for antinormality of  $T^*$ .

**Definition 1.2** [7] An operator  $T \in \mathcal{B}(H)$  is said to be *Fredholm operator* if  $\text{range}(T)$  is closed and dimension of both  $\ker(T)$  and  $\ker(T^*)$  are finite.

**Definition 1.3** [2] The *essential spectrum* of an operator  $T \in \mathcal{B}(H)$  is defined as  $\sigma_e(T) = \{\alpha \in \mathbb{C} : T - \alpha I \text{ is not Fredholm}\}$ .

**Definition 1.4** The *essential minimum modulus* of an operator  $T \in \mathcal{B}(H)$  is defined as  $m_e(T) = \inf\{\alpha \geq 0 : \alpha \in \sigma_e(|T|)\}$ , where  $|T| = (T^*T)^{1/2}$ .

**Definition 1.5** For an operator  $T \in \mathcal{B}(H)$ , *index* of  $T$  is defined as

$$i(T) = \begin{cases} \dim(\ker(T)) - \dim(\ker(T^*)), & \text{if } \dim(\ker(T)) < \infty \text{ or } \dim(\ker(T^*)) < \infty \\ 0, & \text{otherwise.} \end{cases}$$

*Remark 1.6* [9] If  $i(T) = 0$  then  $T$  is not antinormal.

**Theorem 1.7** [9] Let  $T \in \mathcal{B}(H)$  with  $i(T) < 0$ . Then the following conditions are equivalent:

- (1)  $T$  is antinormal.
- (2)  $m_e(T) = \|T\|$ .
- (3)  $d(T, \mathcal{U}) = 1 + \|T\|$ , where  $\mathcal{U}$  is the class of all unitary operators in  $\mathcal{B}(H)$ .

**Definition 1.8** [5] Let  $m$  be a natural number and  $T \in \mathcal{B}(H)$ . Then  $T$  is said to be an  $m$ -isometry if  $T$  satisfies the following equation:

$$\sum_{k=0}^m (-1)^k \binom{m}{k} T^{*k} T^k = 0. \quad (1.1)$$

Moreover,  $T$  is said to be *strictly m-isometry* if  $m$  is the smallest natural number for which Eq. (1.1) holds.

**Definition 1.9** Let  $V$  be a non-empty set and  $E$  be a subset of  $V \times V$ . Then  $(V, E)$  is known as *directed graph*. An element of  $V$  is called a *vertex* and an element of  $E$  is called an *edge*.

In this paper we restrict to the case when  $V$  is a countable set. In a directed graph  $(V, E)$ ,  $\text{card}(\{u \in V : (u, v) \in E\})$  and  $\text{card}(\{u \in V : (v, u) \in E\})$  are respectively called *indegree* of  $v$  and *outdegree* of  $v$ . A directed graph  $(V, E)$  is said to be *locally finite* if for every vertex in  $V$  both the indegree and the outdegree are finite. A directed graph  $(V, E)$  is said to be *connected* if its underlying graph is connected [17]. A finite sequence  $\{u_1, u_2, \dots, u_k\}$  for some  $k \geq 2$ , of distinct vertices in  $V$  is called a *directed circuit* of  $(V, E)$  if  $(u_i, u_{i+1}) \in E$  for each  $i = 1, 2, \dots, k-1$  and  $(u_k, u_1) \in E$ .

**Definition 1.10** A directed graph  $\mathcal{T} := (V, E)$  is called a *directed tree* if  $\mathcal{T}$  does not have any directed circuits,  $\mathcal{T}$  is connected and the indegree of every vertex in  $\mathcal{T}$  is at most one.

**Definition 1.11** In a directed tree  $\mathcal{T}$  a vertex is called a *root* if its indegree is zero. If the outdegree of a vertex is zero then it is called a *leaf*.

We have adopted various notations introduced in [14]. Let  $\mathcal{T}$  be a directed tree. For an edge  $(u, v) \in E$ , we define  $u = \text{par}(v)$  and call  $v$  as a child of  $u$ . For a vertex  $v \in V$ ,  $\text{par}^2(v)$  denotes  $\text{par}(\text{par}(v))$  and for  $n \geq 3$   $\text{par}^n(v) = \text{par}(\text{par}^{n-1}(v))$  whenever  $\text{par}^{n-1}(v)$  is not a root. For  $u \in V$  define  $\text{chi}(u) = \{v \in V : u = \text{par}(v)\}$  and for  $n \geq 2$ ,  $\text{chi}^n(u) = \{u \in V : \text{par}^n(v) = u\}$ . Also, for  $u \in V$ ,  $\gamma(u)$  denotes the cardinality of  $\text{chi}(u)$ .

*Remark 1.12* Let  $u, v, w \in V$ . If  $\text{chi}(u) = \{v\}$ , then  $v$  is also denoted by  $\text{chi}(u)$ . Similarly, if  $\text{chi}^n(u) = \{w\}$  for some  $n \geq 2$ , then  $w$  is also denoted by  $\text{chi}^n(u)$ .



**Definition 1.13** Two directed trees  $\mathcal{T} = (V, E)$  and  $\tilde{\mathcal{T}} = (\tilde{V}, \tilde{E})$  are said to be isomorphic if there exists a one-to-one correspondence  $\psi$  from  $V$  onto  $\tilde{V}$  such that  $(u, v) \in E$  if and only if  $(\psi(u), \psi(v)) \in \tilde{E}$ .

**Definition 1.14** Let  $G = (V, E)$  be a directed graph and  $W \subseteq V$ . Then,  $G \setminus W$  denotes the directed graph by removing all vertices of  $W$  from  $G$  and all edges whose at least one end vertex is in  $W$ .

**Definition 1.15** Let  $\mathcal{T}$  be a directed tree and  $\lambda = \{\lambda_v \in \mathbb{R} : \lambda_v > 0, v \in V\}$  be a set. We call  $\lambda$  weight on the vertex set  $V$ . We denote by  $L^2(\mathcal{T}, \lambda)$  the space of complex valued functions  $f : V \rightarrow \mathbb{C}$  such that

$$\sum_{v \in V} |f(v)|^2 \lambda_v < \infty.$$

This is a Hilbert space endow with the inner product

$$\langle f, g \rangle = \sum_{v \in V} f(v) \overline{g(v)} \lambda_v, \quad f, g \in L^2(\mathcal{T}, \lambda).$$

Throughout this paper by “tree”, we mean a locally finite directed tree. Also  $\mathbb{Z}^+$  and  $\mathbb{Z}^-$ , respectively, denote the trees  $(\mathbb{Z}^+, \{(i, i + 1) : i \in \mathbb{Z}^+\})$  and  $(\mathbb{Z}^-, \{(i, i - 1) : i \in \mathbb{Z}^-\})$ . For a tree  $\mathcal{T} = (V, E)$  we define  $V_{\text{leaves}}$  as the set of all leaves,  $V_{\geq 2}$  as the set  $\{u \in V : \gamma(u) \geq 2\}$  and  $\text{chi}(V_{\geq 2})$  as the set  $\bigcup_{v \in V_{\geq 2}} \text{chi}(v)$ . Also, we assign a integer value  $\delta_{\mathcal{T}}$  to the tree  $\mathcal{T}$  as follows:

$$\delta_{\mathcal{T}} = \begin{cases} 1, & \text{if tree has a root} \\ 0, & \text{otherwise.} \end{cases}$$

Let  $\Delta : V \rightarrow (0, \infty)$  be such that  $\Delta(u) = \left(\frac{1}{\lambda_u} \sum_{v \in \text{chi}(u)} \lambda_v\right)^{\frac{1}{2}}$ , for every  $u \in V$ . And a sequence in  $\Delta$  is a sequence of the form  $\{\Delta(\mu(n))\}_{n=1}^{\infty}$ , where  $\mu : \mathbb{N} \rightarrow V$  is an injective function.

**Definition 1.16** Let  $\mathcal{T}$  be a tree and  $\lambda$  be weight on  $V$ . Then the shift  $S : L^2(\mathcal{T}, \lambda) \rightarrow L^2(\mathcal{T}, \lambda)$  is the operator defined as

$$(Sf)(u) = \begin{cases} f(\text{par}(u)), & u \text{ is not a root} \\ 0, & \text{otherwise,} \end{cases}$$

where  $f \in L^2(\mathcal{T}, \lambda)$ .

**Theorem 1.17** [14] *Let  $\mathcal{T}$  be a tree and  $\lambda$  be weight on  $V$ . The shift  $S : L^2(\mathcal{T}, \lambda) \rightarrow L^2(\mathcal{T}, \lambda)$  is bounded if and only if*

$$\sup_{u \in V} \left(\frac{1}{\lambda_u} \sum_{v \in \text{chi}(u)} \lambda_v\right) < \infty.$$

*In either case,*

$$\|S\| = \sup_{u \in V} \left(\frac{1}{\lambda_u} \sum_{v \in \text{chi}(u)} \lambda_v\right)^{1/2}.$$

**Hilbert adjoint of  $S$**  [14]: The Hilbert adjoint  $S^*$  of the shift operator  $S$  on  $L^2(\mathcal{T}, \lambda)$  is given by

$$(S^*g)(u) = \frac{1}{\lambda_u} \sum_{v \in \text{chi}(u)} g(v) \lambda_v,$$

for each  $g \in L^2(\mathcal{T}, \lambda)$  and  $u \in V$ .

## 2 Antinormality of the shift operator

Prior to our investigations on antinormality of  $S$ , we state some results. From the definition of  $S$  and  $S^*$ , we have  $\ker(S) = \{f \in L^2(\mathcal{T}, \lambda) : f(u) = 0, \text{ for each } u \in V \setminus V_{\text{leaves}}\}$  and  $\ker(S^*) = \{f \in L^2(\mathcal{T}, \lambda) : \sum_{v \in \text{chi}(u)} f(v)\lambda_v = 0, \text{ for each } u \in V\}$ . Thus,  $\ker(S)$  is collection of all  $f \in L^2(\mathcal{T}, \lambda)$  such that  $f = \sum_{u \in V_{\text{leaves}}} f(u)\chi_u$ . Hence,  $\dim(\ker(S))$  is finite if and only if  $\text{card}(V_{\text{leaves}})$  is finite. Further, if  $\text{card}(V_{\text{leaves}})$  is finite then  $\dim(\ker(S)) = \text{card}(V_{\text{leaves}})$ .

**Lemma 2.1** *Dimension of  $\ker(S^*)$  is finite if and only if  $\text{card}(V_{\geq 2})$  is finite. Moreover, if  $\text{card}(V_{\geq 2})$  is finite, then  $\dim(\ker(S^*)) = \text{card}(\text{chi}(V_{\geq 2})) - \text{card}(V_{\geq 2}) + \delta_{\mathcal{T}}$ .*

*Proof* Let  $f = \sum_{v \in V} f(v)\chi_v \in L^2(\mathcal{T}, \lambda)$ . Then,

$$\begin{aligned} S^*(f) &= \sum_{v \in V} f(v)S^*(\chi_v) \\ &= \sum_{v \in V \setminus \{\text{root}\}} f(v) \frac{\lambda_v}{\lambda_{\text{par}(v)}} \chi_{\text{par}(v)} \\ &= \sum_{v \in V} \left( \frac{1}{\lambda_v} \sum_{w \in \text{chi}(v)} \lambda_w f(w) \right) \chi_v. \end{aligned} \quad (2.1)$$

If  $v \in V_{\geq 2}$ , then  $\gamma(v) \geq 2$ . Let  $\text{chi}(v) = \{w_1, w_2, \dots, w_l\}$  and  $f = \sum_{i=1}^l f(w_i)\chi_{w_i} \in L^2(\mathcal{T}, \lambda)$  satisfying  $\sum_{i=1}^l f(w_i)\lambda_{w_i} = 0$  then from Eq. (2.1)  $f \in \ker(S^*)$ . Clearly, the collection of all such  $f$  forms a subspace of dimension  $\gamma(v) - 1 \geq 1$ , which is contained in  $\ker(S^*)$ . Also, for  $u_1 \neq u_2$ ,  $\text{chi}(u_1)$  and  $\text{chi}(u_2)$  are disjoint sets. Therefore,  $\dim(\ker(S^*)) \geq \text{card}(V_{\geq 2})$ . Hence, if  $\text{card}(V_{\geq 2})$  is not finite, then  $\dim(\ker(S^*))$  is also not finite.

Now assume that  $\text{card}(V_{\geq 2})$  is finite. If  $V_{\geq 2}$  is empty, then the result obviously holds. Let  $V_{\geq 2} = \{v_1, v_2, \dots, v_k\}$  and for each  $v_j \in V_{\geq 2}$ ,  $\text{chi}(v_j) = \{w_1^j, w_2^j, \dots, w_{m_j}^j\}$ . We now consider the case when the tree has a root. Let  $f = \sum_{v \in V} f(v)\chi_v \in \ker(S^*)$ . Then, from Eq. (2.1),  $f(w) = 0$  whenever  $\text{par}(w)$  has unique child viz.  $w$ . Thus,  $f(w) = 0$ , whenever  $w \in V \setminus (\text{chi}(V_{\geq 2}) \cup \{\text{root}\})$ . Hence

$$\begin{aligned} f &= \sum_{u \in V} f(u)\chi_u \\ &= f(\text{root})\chi_{\text{root}} + \sum_{u \in V \setminus (\text{chi}(V_{\geq 2}) \cup \{\text{root}\})} f(u)\chi_u + \sum_{v \in \text{chi}(V_{\geq 2})} f(v)\chi_v \\ &= f(\text{root})\chi_{\text{root}} + 0 + \sum_{v \in \text{chi}(V_{\geq 2})} f(v)\chi_v. \end{aligned}$$

Therefore,  $f$  is linear combination of  $\{\chi_v : v \in \text{chi}(V_{\geq 2}) \cup \{\text{root}\}\}$  with  $k (= \text{card}(V_{\geq 2}))$  extra conditions

$$\sum_{i=1}^{m_j} f(w_i^j)\lambda_{w_i^j} = 0, \quad 1 \leq j \leq k.$$

This implies that  $\ker(S^*) = \{f \in L^2(\mathcal{T}, \lambda) : f = f(\text{root})\chi_{\text{root}} + \sum_{v \in \text{chi}(V_{\geq 2})} f(v)\chi_v \text{ and } \sum_{i=1}^{m_j} f(w_i^j)\lambda_{w_i^j} = 0, 1 \leq j \leq k\}$ , where  $V_{\geq 2} = \{v_1, v_2, \dots, v_k\}$  and for each  $v_j \in V_{\geq 2}$ ,  $\text{chi}(v_j) = \{w_1^j, w_2^j, \dots, w_{m_j}^j\}$ . Consequently,

$$\dim(\ker(S^*)) = 1 + \text{card}(\text{chi}(V_{\geq 2})) - \text{card}(V_{\geq 2}). \quad (2.2)$$

Preceding arguments also implies that if  $\text{card}(V_{\geq 2})$  is finite and the tree has no root then  $\ker(S^*) = \{f \in L^2(\mathcal{T}, \lambda) : f = \sum_{u \in \text{chi}(V_{\geq 2})} f(v)\chi_v \text{ and } \sum_{i=1}^{m_j} f(w_i^j)\lambda_{w_i^j} = 0, 1 \leq j \leq k\}$ , where  $V_{\geq 2} = \{v_1, v_2, \dots, v_k\}$  and for each  $v_j \in V_{\geq 2}$ ,  $\text{chi}(v_j) = \{w_1^j, w_2^j, \dots, w_{m_j}^j\}$ . Therefore,

$$\dim(\ker(S^*)) = \text{card}(\text{chi}(V_{\geq 2})) - \text{card}(V_{\geq 2}). \quad (2.3)$$



Hence from Eqs. (2.2) and (2.3), we get

$$\dim(\ker(S^*)) = \text{card}(\text{chi}(V_{\geq 2})) - \text{card}(V_{\geq 2}) + \delta_{\mathcal{T}}.$$

□

*Remark 2.2* (1) It is worth noting that, if tree is not locally finite, then

$$\dim(\ker(S^*)) = \sum_{u \in \text{chi}(V_{\geq 2})} [\text{card}(\text{chi}(V_{\geq 2})) - 1] + \delta_{\mathcal{T}}.$$

(2) In view of preceding definitions and the above lemma, the following statements follow easily.

- (i) If  $\text{card}(V_{\geq 2})$  is finite then  $\text{card}(V_{\text{leaves}})$  is also finite.
- (ii) The  $\text{card}(V_{\geq 2})$  is finite if and only if  $\text{card}(\text{chi}(V_{\geq 2}))$  is finite.
- (iii)  $2 \cdot \text{card}(V_{\geq 2}) \leq \text{card}(\text{chi}(V_{\geq 2}))$ .

We now give a relation between  $\text{card}(V_{\geq 2})$ ,  $\text{card}(\text{chi}(V_{\geq 2}))$  and  $\text{card}(V_{\text{leaves}})$ .

**Proposition 2.3** *In a tree  $\mathcal{T}$ ,  $\text{card}(V_{\text{leaves}}) + \text{card}(V_{\geq 2}) \leq \text{card}(\text{chi}(V_{\geq 2})) + 1$ .*

*Proof* First consider the case when the tree  $\mathcal{T}$  is finite. In this case, using induction on number of vertices we prove that the above inequality is an equality. Clearly equality holds for the tree with 1 and 2 vertices. Let the assertion be true for the tree with  $k(\geq 2)$  vertices. Now, consider a tree  $\mathcal{T}$  with  $k + 1$  vertices. Then  $\mathcal{T}$  has a leaf  $v^*$ . Consequently, the tree  $\tilde{\mathcal{T}} = (\tilde{V}, \tilde{E}) = \mathcal{T} \setminus \{v^*\}$  has  $k$  vertices with two possibilities:

- (1) If  $v^* \in \text{chi}(V_{\geq 2})$ , then  $\text{card}(\tilde{V}_{\text{leaves}}) = \text{card}(V_{\text{leaves}}) - 1$ . Further, in this case, we have following two possibilities:
  - (a) If  $\gamma(\text{par}(v^*)) = 2$  then  $\text{card}(\tilde{V}_{\geq 2}) = \text{card}(V_{\geq 2}) - 1$  and  $\text{card}(\text{chi}(\tilde{V}_{\geq 2})) = \text{card}(\text{chi}(V_{\geq 2})) - 2$ . Since equality holds for a tree with  $k$  vertices, therefore

$$\text{card}(\tilde{V}_{\text{leaves}}) + \text{card}(\tilde{V}_{\geq 2}) = \text{card}(\text{chi}(\tilde{V}_{\geq 2})) + 1.$$

This implies

$$\text{card}(V_{\text{leaves}}) - 1 + \text{card}(V_{\geq 2}) - 1 = \text{card}(\text{chi}(V_{\geq 2})) - 2 + 1.$$

Thus,

$$\text{card}(V_{\text{leaves}}) + \text{card}(V_{\geq 2}) = \text{card}(\text{chi}(V_{\geq 2})) + 1.$$

- (b) If  $\gamma(\text{par}(v^*)) \geq 3$  then  $\text{card}(\tilde{V}_{\geq 2}) = \text{card}(V_{\geq 2})$  and  $\text{card}(\text{chi}(\tilde{V}_{\geq 2})) = \text{card}(\text{chi}(V_{\geq 2})) - 1$ . Therefore,

$$\text{card}(\tilde{V}_{\text{leaves}}) + \text{card}(\tilde{V}_{\geq 2}) = \text{card}(\text{chi}(\tilde{V}_{\geq 2})) + 1.$$

Which gives

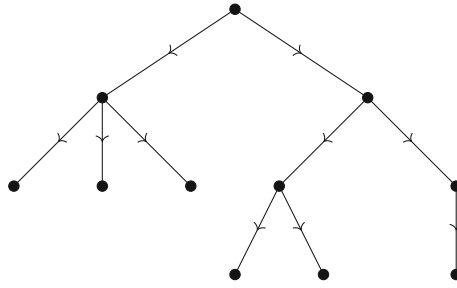
$$\text{card}(V_{\text{leaves}}) + \text{card}(V_{\geq 2}) = \text{card}(\text{chi}(V_{\geq 2})) + 1.$$

- (2) If  $v^* \notin \text{chi}(V_{\geq 2})$  then  $\text{card}(\tilde{V}_{\text{leaves}}) = \text{card}(V_{\text{leaves}})$ ,  $\text{card}(\tilde{V}_{\geq 2}) = \text{card}(V_{\geq 2})$  and  $\text{card}(\text{chi}(\tilde{V}_{\geq 2})) = \text{card}(\text{chi}(V_{\geq 2}))$ . Therefore  $\text{card}(V_{\text{leaves}}) + \text{card}(V_{\geq 2}) = \text{card}(\text{chi}(V_{\geq 2})) + 1$ .

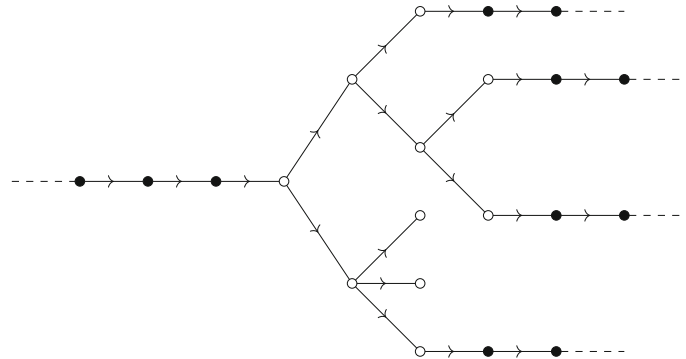
Thus, equality also holds for a tree with  $k + 1$  vertices. Hence, it holds for any finite tree.

Now consider the case when the tree  $\mathcal{T}$  is not finite. Let us split this case into two parts as follows:

- (1) If  $\text{card}(V_{\geq 2})$  is not finite then  $\text{card}(\text{chi}(V_{\geq 2}))$  is also not finite. Thus in this case the result holds.
- (2) If  $\text{card}(V_{\geq 2})$  is finite (which implies  $\text{card}(V_{\text{leaves}})$  is also finite) then we can choose a finite subset  $W \subseteq V$  such that  $\tilde{\mathcal{T}} = (W, E \cap (W \times W))$  is a tree with the properties that  $V_{\geq 2}$ ,  $\text{chi}(V_{\geq 2})$  and  $V_{\text{leaves}}$  are all contained in  $W$  and the directed graph  $\mathcal{T} \setminus W$  is collection of finite number of linear trees each of which is isomorphic to  $\mathbb{Z}_{\mathcal{T}}^+$  or  $\mathbb{Z}_{\mathcal{T}}^-$ . Further, let  $W_{\geq 2} = \{w \in W : \gamma(w) > 1\}$  and  $W_{\text{leaves}} = \{w \in W : w \text{ is leaf of } \tilde{\mathcal{T}}\}$ . Then  $W_{\geq 2} = V_{\geq 2}$ ,  $\text{chi}(W_{\geq 2}) = \text{chi}(V_{\geq 2})$  and we have exactly two scenarios:
  - (a)  $\mathcal{T} \setminus W$  contains a linear tree which is isomorphic to  $\mathbb{Z}_{\mathcal{T}}^+$  and  $\text{card}(W_{\text{leaves}}) > \text{card}(V_{\text{leaves}})$ .



**Fig. 1** A Finite tree



**Fig. 2** Infinite tree in which  $\text{card}(V_{\geq 2})$  is finite

(b)  $T \setminus W$  does not contain any linear tree which is isomorphic to  $\mathbb{Z}_T^+$  (in this case  $T \setminus W$  is isomorphic to  $\mathbb{Z}_T^-$ ) and  $\text{card}(W_{\text{leaves}}) = \text{card}(V_{\text{leaves}})$ . Thus, we have  $\text{card}(V_{\text{leaves}}) \leq \text{card}(W_{\text{leaves}})$ . Therefore,

$$\begin{aligned} \text{card}(V_{\text{leaves}}) + \text{card}(V_{\geq 2}) &\leq \text{card}(W_{\text{leaves}}) + \text{card}(W_{\geq 2}) \\ &= \text{card}(\text{chi}(W_{\geq 2})) + 1 \\ &= \text{card}(\text{chi}(V_{\geq 2})) + 1. \end{aligned}$$

Hence, the result holds for every tree  $T$ . □

*Remark 2.4* Above proposition can be visualized in the following trees. Figure 1 is a finite tree for which the inequality of Proposition 2.3 becomes an equality. In Figure 2 tree  $T$  is not finite but  $\text{card}(V_{\geq 2})$  is finite. In this tree, if we take  $W$  as collection of all the vertices such that  $V_{\geq 2}$ ,  $\text{chi}(V_{\geq 2})$  and  $V_{\text{leaves}}$  are all contained in  $W$ , then  $T \setminus W$  is collection of finite number of linear trees each of which is isomorphic to  $\mathbb{Z}_T^+$  or  $\mathbb{Z}_T^-$ .

**Corollary 2.5** For the  $S : L^2(\mathcal{T}, \lambda) \rightarrow L^2(\mathcal{T}, \lambda)$ ,  $i(S) \leq 1$ .

*Proof* Consider the case when  $\dim(\ker(S))$  and  $\dim(\ker(S^*))$  are both infinite then by definition,  $i(S) = 0$ . The Remark 2.2 precludes the possibility that  $\dim(\ker(S))$  is infinite and  $\dim(\ker(S^*))$  is finite. If  $\dim(\ker(S))$  is finite but  $\dim(\ker(S^*))$  is infinite then  $i(S) = -\infty$ . Now if  $\dim(\ker(S))$  and  $\dim(\ker(S^*))$  both are finite, then

$$\begin{aligned} i(S) &= \dim(\ker(S)) - \dim(\ker(S^*)) \\ &= \text{card}(V_{\text{leaves}}) - [\text{card}(\text{chi}(V_{\geq 2})) - \text{card}(V_{\geq 2}) + \delta_T] \\ &\leq 1 - \delta_T \\ &= \begin{cases} 1, & \text{if the tree is without a root} \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Thus,  $i(S) \leq 1$ . □

*Remark 2.6* If  $i(S) = 1$ , then tree is not finite and by Corollary 2.5  $\delta_{\mathcal{T}} = 0$ . This implies that tree has no root. Again by Corollary 2.5  $\text{card}(V_{\geq 2})$  is finite and  $\text{card}(V_{\text{leaves}}) = \text{card}(\text{chi}(V_{\geq 2})) - \text{card}(V_{\geq 2}) + 1$ . Now repeating the arguments of Proposition 2.3 above, it follows that there exists  $W \subseteq V$  such that  $\mathcal{T} \setminus W$  is isomorphic to  $\mathbb{Z}_{\mathcal{T}}^-$ . In all other cases, the  $i(S) \leq 0$ .

We now have the following propositions.

**Proposition 2.7** *If  $i(S) < 0$  then  $S$  is antinormal if and only if for every  $\epsilon > 0$ , there exists a finite subset  $U \subset V$  such that  $|\Delta(v) - \|S\|| < \epsilon$  for each  $v \in V \setminus U$ .*

*Proof* Let  $i(S) < 0$ . Then,  $V_{\text{leaves}}$  must be finite. For if,  $V_{\text{leaves}}$  is an infinite set then  $V_{\geq 2}$  is also an infinite set. Hence both the  $\dim(\ker(S))$  and  $\dim(\ker(S^*))$  are infinite. Consequently  $i(S) = 0$ , which is a contradiction. Now consider  $T = (S^*S)^{\frac{1}{2}}$ . Since

$$(S^*S)\chi_u = \begin{cases} \left(\frac{1}{\lambda_u} \sum_{v \in \text{chi}(u)} \lambda_v\right)\chi_u, & u \notin V_{\text{leaves}} \\ 0, & u \in V_{\text{leaves}}, \end{cases}$$

therefore,

$$T\chi_u = \begin{cases} \left(\frac{1}{\lambda_u} \sum_{v \in \text{chi}(u)} \lambda_v\right)^{\frac{1}{2}}\chi_u, & u \notin V_{\text{leaves}} \\ 0, & u \in V_{\text{leaves}}. \end{cases}$$

Thus,  $T$  is a diagonal operator. Hence, essential minimum modulus of  $S$  is

$$\begin{aligned} m_e(S) &= \inf\{\alpha \geq 0 : \alpha \in \sigma_e(T)\} \\ &= \inf\{\alpha \geq 0 : T - \alpha I \text{ is not Fredholm}\} \\ &= \inf\{\alpha \geq 0 : \text{either } \dim(\ker(T - \alpha I)) \text{ is not finite or } \text{range}(T - \alpha I) \text{ is not closed}\} \\ &= \inf\{\alpha \geq 0 : \alpha \text{ is a limit of some convergent sequence in } \Delta\}, \end{aligned}$$

Now  $\|S\| = \sup_{u \in V} \left(\frac{1}{\lambda_u} \sum_{v \in \text{chi}(u)} \lambda_v\right)^{\frac{1}{2}}$  and  $i(S) < 0$ . Hence  $S$  is antinormal if and only if  $m_e(S) = \|S\|$ . This is equivalent to  $\Delta$  has a convergent sequence and every convergent sequence in  $\Delta$  converge to  $\|S\|$ . As  $\sup_{v \in V} \Delta(v) = \|S\|$ , therefore  $S$  is antinormal if and only if for every  $\epsilon > 0$ , there exists a finite subset  $U \subset V$  such that  $|\Delta(v) - \|S\|| < \epsilon$  for each  $v \in V \setminus U$ . □

**Proposition 2.8** *If  $i(S) = 1$  then  $S$  is antinormal if and only if for every  $\epsilon > 0$ , there exists a finite subset  $U \subset V$  such that  $|\Delta(v) - \|S\|| < \epsilon$  for each  $v \in V \setminus U$ .*

*Proof* Let  $i(S) = 1$ . Then by Remark 2.6, the tree  $\mathcal{T}$  has no root and there exists a set of finite vertices  $W = \{w_1, w_2, \dots, w_l\}$  such that the tree  $\mathcal{T} \setminus W$  is isomorphic to the  $\mathbb{Z}_{\mathcal{T}}^-$ . Hence any sequence in  $\left\{\left(\frac{1}{\lambda_u} \sum_{v \in \text{chi}(u)} \lambda_v\right)^{1/2} : u \in V\right\}$  will lie eventually in  $\left\{\left(\frac{1}{\lambda_u} \sum_{v \in \text{chi}(u)} \lambda_v\right)^{1/2} : u \in V \setminus W\right\}$ . Also  $i(S^*) = -i(S) = -1 < 0$ . Consider  $T = (SS^*)^{\frac{1}{2}}$ . A simple computation shows that

$$(SS^*)\chi_u = \frac{\lambda_u}{\lambda_{\text{par}(u)}} \cdot \left(\sum_{v \in \text{chi}(\text{par}(u))} \chi_v\right), \text{ for each } u \in V.$$

Therefore,

$$T\chi_u = \frac{\lambda_u}{\left(\lambda_{\text{par}(u)} \sum_{v \in \text{chi}(\text{par}(u))} \lambda_v\right)^{1/2}} \cdot \left(\sum_{v \in \text{chi}(\text{par}(u))} \chi_v\right), \text{ for each } u \in V.$$

The above equation can also be expressed as

$$T\chi_u = \begin{cases} \frac{\lambda_u}{\left(\sum_{v \in \text{chi}(\text{par}(u))} \lambda_v\right)^{1/2}} \cdot \sum_{v \in \text{chi}(\text{par}(u))} \chi_v, & u \in W \\ \left(\frac{\lambda_u}{\lambda_{\text{par}(u)}}\right)^{\frac{1}{2}} \cdot \chi_u, & u \in V \setminus W. \end{cases}$$

Using the fact that  $W$  is finite set and  $T$  is diagonal operator on  $V \setminus W$ , we get

$$\begin{aligned} m_e(S^*) &= \inf\{\alpha \geq 0 : \alpha \in \sigma_e(T)\} \\ &= \inf\{\alpha \geq 0 : T - \alpha I \text{ is not Fredholm}\} \\ &= \inf\{\alpha \geq 0 : \text{either } \dim(\ker(T - \alpha I)) \text{ is not finite or } \text{range}(T - \alpha I) \text{ is not closed}\} \\ &= \inf\left\{\alpha \geq 0 : \alpha \text{ is a limit of some convergent sequence in } \left\{\left(\frac{1}{\lambda_u} \sum_{v \in \text{chi}(u)} \lambda_v\right)^{\frac{1}{2}} : u \in V \setminus W\right\}\right\}, \\ &= \inf\{\alpha \geq 0 : \alpha \text{ is a limit of some convergent sequence in } \Delta\}, \end{aligned}$$

Since  $i(S^*) < 0$ , therefore  $S^*$  is antinormal if and only if  $m_e(S^*) = \|S^*\|$ . Also  $S$  is antinormal if and only if  $S^*$  is antinormal. Hence,  $S$  is antinormal if and only if  $m_e(S^*) = \|S\|$ . Consequently  $S$  is antinormal if and only for every  $\epsilon > 0$ , there exists a finite subset  $U \subset V$  such that  $|\Delta(v) - \|S\|| < \epsilon$  for each  $v \in V \setminus U$ .  $\square$

Combining above two propositions and Remark 1.6 we obtain the following result.

**Theorem 2.9** *S is antinormal if and only if  $i(S) \neq 0$ , for every  $\epsilon > 0$ , there exists a finite subset  $U \subset V$  such that  $|\Delta(v) - \|S\|| < \epsilon$  for each  $v \in V \setminus U$ .*

### 3 m-isometry of the shift operator

In this section, we investigate the  $m$ -isometry of the shift operator and relate it to its antinormality. From the definition of  $m$ -isometry we have the following result.

**Proposition 3.1** *S is an m-isometry if and only if for every  $u \in V$  the following holds:*

$$\lambda_u - \binom{m}{1} \sum_{v \in \text{chi}(u)} \lambda_v + \binom{m}{2} \sum_{v \in \text{chi}^2(u)} \lambda_v - \dots + (-1)^m \binom{m}{m} \sum_{v \in \text{chi}^m(u)} \lambda_v = 0. \tag{3.1}$$

*Remark 3.2* If  $S$  is an  $m$ -isometry for any  $m \geq 1$ , then the tree is without leaves.

**Proposition 3.3** *S is an m-isometry if and only if for every  $u \in V$ , there exists a polynomial  $P_u$  of degree at most  $m - 1$  such that  $\sum_{v \in \text{chi}^n(u)} \lambda_v = P_u(n)$ , for every  $n \in \mathbb{N} \cup \{0\}$ .*

*Proof* Assume that  $S$  is an  $m$ -isometry. Let  $u \in V$ , define,  $b_n^u = \sum_{v \in \text{chi}^n(u)} \lambda_v$ ,  $n \geq 1$  and  $b_0^u = \lambda_u$ . Then, Eq. (3.1) becomes

$$b_0^u - \binom{m}{1} b_1^u + \binom{m}{2} b_2^u - \dots + (-1)^m \binom{m}{m} b_m^u = 0.$$

More generally for every  $n \geq 0$ , we have

$$b_n^u - \binom{m}{1} b_{n+1}^u + \binom{m}{2} b_{n+2}^u - \dots + (-1)^m \binom{m}{m} b_{n+m}^u = 0. \tag{3.2}$$

Now using theory of difference equations, the auxiliary equation of the recursive Eq. (3.2) is

$$1 - \binom{m}{1} r + \binom{m}{2} r^2 - \dots + (-1)^m \binom{m}{m} r^m = 0.$$



Above equation has 1 as the only zero with multiplicity  $m$ . Therefore, the general solution of Eq. (3.2) is

$$b_n^u = A_0(1)^n + A_1n(1)^n + A_2n^2(1)^n + \dots + A_{m-1}n^{m-1}(1)^n.$$

Thus,

$$b_n^u = \sum_{v \in \text{chi}^n(u)} \lambda_v = A_0 + A_1n + A_2n^2 + \dots + A_{m-1}n^{m-1}, \text{ for each } n \geq 0, \tag{3.3}$$

where  $A_0, A_1, \dots, A_{m-1}$  are real constants which can be determined by  $m$  initial conditions

$$b_i^u = \sum_{v \in \text{chi}^i(u)} \lambda_v, \quad 0 \leq i \leq m - 1.$$

Conversely, let  $u \in V$ . Then, there exists a polynomial  $P_u$  of degree at most  $m - 1$  such that  $b_n^u = \sum_{v \in \text{chi}^n(u)} \lambda_v = P_u(n)$  for every  $n \in \mathbb{N} \cup \{0\}$ . Let

$$P_u(n) = C_0 + C_1n + C_2n^2 + \dots + C_kn^k, \quad C_k \neq 0 \text{ and } 0 \leq k \leq m - 1.$$

This implies

$$b_n^u = C_0 + C_1n + C_2n^2 + \dots + C_kn^k, \quad C_k \neq 0 \text{ and } 0 \leq k \leq m - 1.$$

Then, the sequence  $\{b_n^u\}_{n=0}^\infty$  can be recursively obtained by

$$b_n^u - \binom{k+1}{1}b_{n+1}^u + \binom{k+1}{2}b_{n+2}^u - \dots + (-1)^{k+1} \binom{k+1}{k+1}b_{n+k+1}^u = 0.$$

Since  $k + 1 \leq m$ , therefore for every  $n \geq 0$

$$b_n^u - \binom{m}{1}b_{n+1}^u + \binom{m}{2}b_{n+2}^u - \dots + (-1)^m \binom{m}{m}b_{n+m}^u = 0.$$

In particular,

$$b_0^u - \binom{m}{1}b_1^u + \binom{m}{2}b_2^u - \dots + (-1)^m \binom{m}{m}b_m^u = 0.$$

Thus,  $S$  is an  $m$ -isometry. □

**Corollary 3.4** *Suppose  $S$  is an  $m$ -isometry. Then,  $S$  is strictly  $m$ -isometry if and only if  $\max\{\deg(P_u) : u \in V\} = m - 1$ .*

Now onwards, whenever we talk about a polynomial  $P_u$  of a vertex  $u$  then it is presumed that  $S$  is an  $m$ -isometry for some  $m \geq 1$ .

*Remark 3.5* If  $u \in V$  and  $v_1, v_2, \dots, v_l \in \text{chi}(u)$ , then  $P_u(1) = \sum_{i=1}^l \lambda_{v_i} = \sum_{i=1}^l P_{v_i}(0)$ . Similarly,  $P_u(n + 1) = \sum_{i=1}^l P_{v_i}(n)$ .

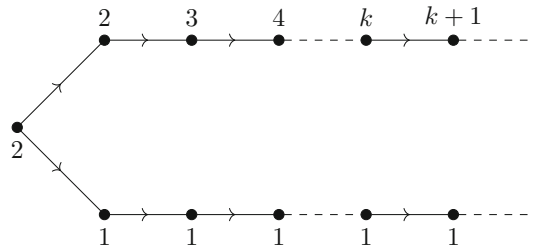
In [10, Proposition 8.1.6] Zenon Jan Jabłoński et al. characterized the normality of weighted shift operator on trees. We state an equivalent form of this result as Proposition 3.6.

**Proposition 3.6**  *$S$  is normal if and only if the following conditions are satisfied.*

- (1) *Tree is rootless and has no leaves,*
- (2) *for every  $u \in V$ ,  $\gamma(u) = 1$  and*
- (3) *for every  $u \in V$ ,  $\frac{\lambda_u}{\lambda_{\text{par}(u)}} = \beta$  for some positive real number  $\beta$ .*

*Remark 3.7* Suppose  $S$  is an  $m$ -isometry and for a vertex  $u$ ,  $\deg(P_u) = k$ . Then, the following statements hold.

- (1) For every  $n \in \mathbb{N}$  and for every  $v \in \text{chi}^n(u)$ ,  $\deg(P_v) \leq k$ .



**Fig. 3** Tree on which shift operator is strictly 2-isometry but not antinormal

(2) For every  $n \in \mathbb{N}$  there exists a  $v \in \text{chi}^n(u)$ ,  $\text{deg}(P_v) = k$ .

Using the above remarks, we have the following result.

**Proposition 3.8** *If  $S$  is an  $m$ -isometry and there exists  $u_0 \in V$  such that  $P_{u_0}$  is constant polynomial, then there are infinitely many  $u$  in  $V$  such that  $P_u$  is constant polynomial (in other words we have a sequence  $\{\Delta(\mu(n))\}_{n=1}^\infty$  in  $\Delta$  with  $\Delta(\mu(n)) = 1$  for each  $n \geq 1$ ).*

We now investigate a connection between antinormality and  $m$ -isometry of  $S$ . Recall that, if  $S$  is antinormal then  $i(S) \neq 0$ . Hence, either  $i(S) = 1$  or  $i(S) < 0$ . If  $i(S) = 1$ , then the tree must have a leaf. Consequently  $S$  can not be  $m$ -isometry for any  $m \geq 1$ .

**Theorem 3.9** *If  $S$  is antinormal and  $i(S) = -k$  for some  $k \in \mathbb{N}$  then  $S$  is an  $m$ -isometry for some  $m \geq 2$  if and only if it is an isometry.*

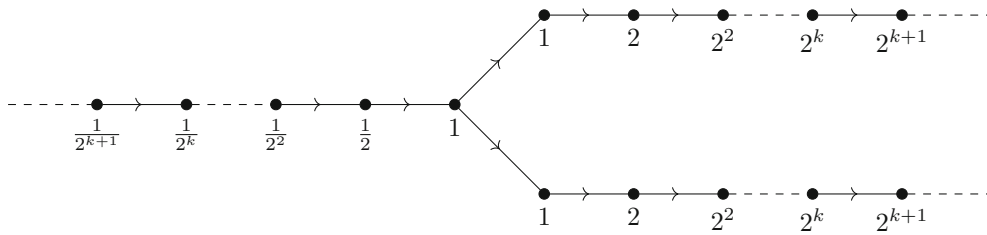
*Proof* One way the implication is trivial. On the other hand, suppose that  $S$  is antinormal and an  $m$ -isometry for some  $m \geq 2$ . As  $S$  is antinormal, there is a sequence  $\{\Delta(\mu(n))\}_{n=1}^\infty$  in  $\Delta$  which converges to  $\|S\|$ . Also, since  $i(S) = -k$ , therefore  $\text{card}(V_{\geq 2})$  is finite, and the tree has no leaves. This implies that we can choose a finite set  $W \subseteq V$  such that  $\mathcal{T} \setminus W$  is finite collection of trees each of which is isomorphic to  $\mathbb{Z}_{\mathcal{T}}^+$  or  $\mathbb{Z}_{\mathcal{T}}^-$ . Therefore there exists a tree  $\tilde{\mathcal{T}} = (\tilde{V}, \tilde{E})$  in  $\mathcal{T} \setminus W$  such that a subsequence of  $\{\Delta(\mu(n))\}_{n=1}^\infty$  which lies in  $\{(\frac{1}{\lambda_u} \sum_{v \in \text{chi}(u)} \lambda_v)^{\frac{1}{2}} : u \in \tilde{V}\}$ . Without loss of generality, let us assume that  $\{\Delta(\mu(n))\}_{n=1}^\infty$  lie in  $\{(\frac{1}{\lambda_u} \sum_{v \in \text{chi}(u)} \lambda_v)^{\frac{1}{2}} : u \in \tilde{V}\}$ . Now assume that  $\tilde{\mathcal{T}}$  be isomorphic to  $\mathbb{Z}_{\mathcal{T}}^+$  with root  $u_0$ . Since for every  $m \in \mathbb{N}$ ,  $\frac{P_{u_0}(m+1)}{P_{u_0}(m)} = \frac{\lambda_{\text{chi}^{(m+1)} u_0}}{\lambda_{\text{chi}^m u_0}}$ , therefore for each  $n \in \mathbb{N}$  there exists a  $m_n \in \mathbb{N}$  such that  $\frac{P_{u_0}(m_n+1)}{P_{u_0}(m_n)} = \Delta(\mu(n))^2$ . Further, choose a sequence  $(n_l)_{l=1}^\infty$  in  $\mathbb{N}$  such that  $(m_{n_l})_{l=1}^\infty$  is an increasing sequence in  $\mathbb{N}$ . Since  $\Delta(\mu(n))_{n=1}^\infty$  converges to  $\|S\|^2$ , so  $\frac{P_{u_0}(m_{n_l}+1)}{P_{u_0}(m_{n_l})}$  converge to  $\|S\|^2$ . But  $P_{u_0}$  is a non-zero polynomial, hence  $\frac{P_{u_0}(m_{n_l}+1)}{P_{u_0}(m_{n_l})}$  converges to 1. Therefore,  $\|S\| = 1$ . Now consider the case that  $\tilde{\mathcal{T}}$  is isomorphic to  $\mathbb{Z}_{\mathcal{T}}^-$  with leaf  $w_0$ . Since for each vertex  $w$  in  $\tilde{\mathcal{T}}$ ,  $P_w(n) > 0$  therefore using Remark 3.5 we get  $P_{\text{par}(w)}(n) = P_w(n - 1)$ . Consequently  $P_{w_0}(-n) = \lambda_{\text{par}^n(w_0)}$  for every  $n \in \mathbb{N}$ . Hence,  $\|S\| = 1$ . Thus  $S$  is an isometry.  $\square$

**Remark 3.10** It is worth noting that, if  $i(S) = -k$ , then tree must be locally finite. Therefore in the above theorem, we can relax the condition that tree is locally finite.

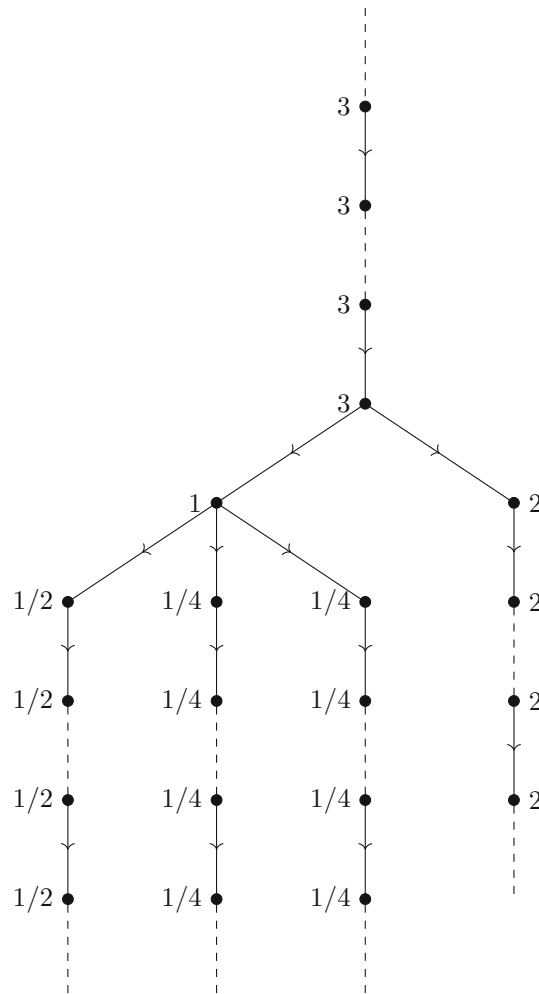
Finally, we give some examples and counterexamples. In the following Figs. 3, 4, 5 and 6, a “•” represents a vertex labeled by their corresponding weights and a line joining two vertices represent an edge with an arrow pointing from vertex  $\text{par}(u)$  to vertex  $u$ .

**Example 3.11** The shift operator on the following weighted tree is strictly 2-isometry but is not antinormal. This follows from Corollary 3.4 and Theorem 2.9.

**Example 3.12** Consider the shift operator  $S$  on the following weighted tree. It can be readily seen from Theorem 2.9 that  $S$  is antinormal. However, since weights of vertices grow exponentially, so  $S$  is not an  $m$ -isometry for any  $m \in \mathbb{N}$ .



**Fig. 4** Tree on which shift operator is antinormal but not  $m$ -isometry ( $\forall m \geq 1$ )



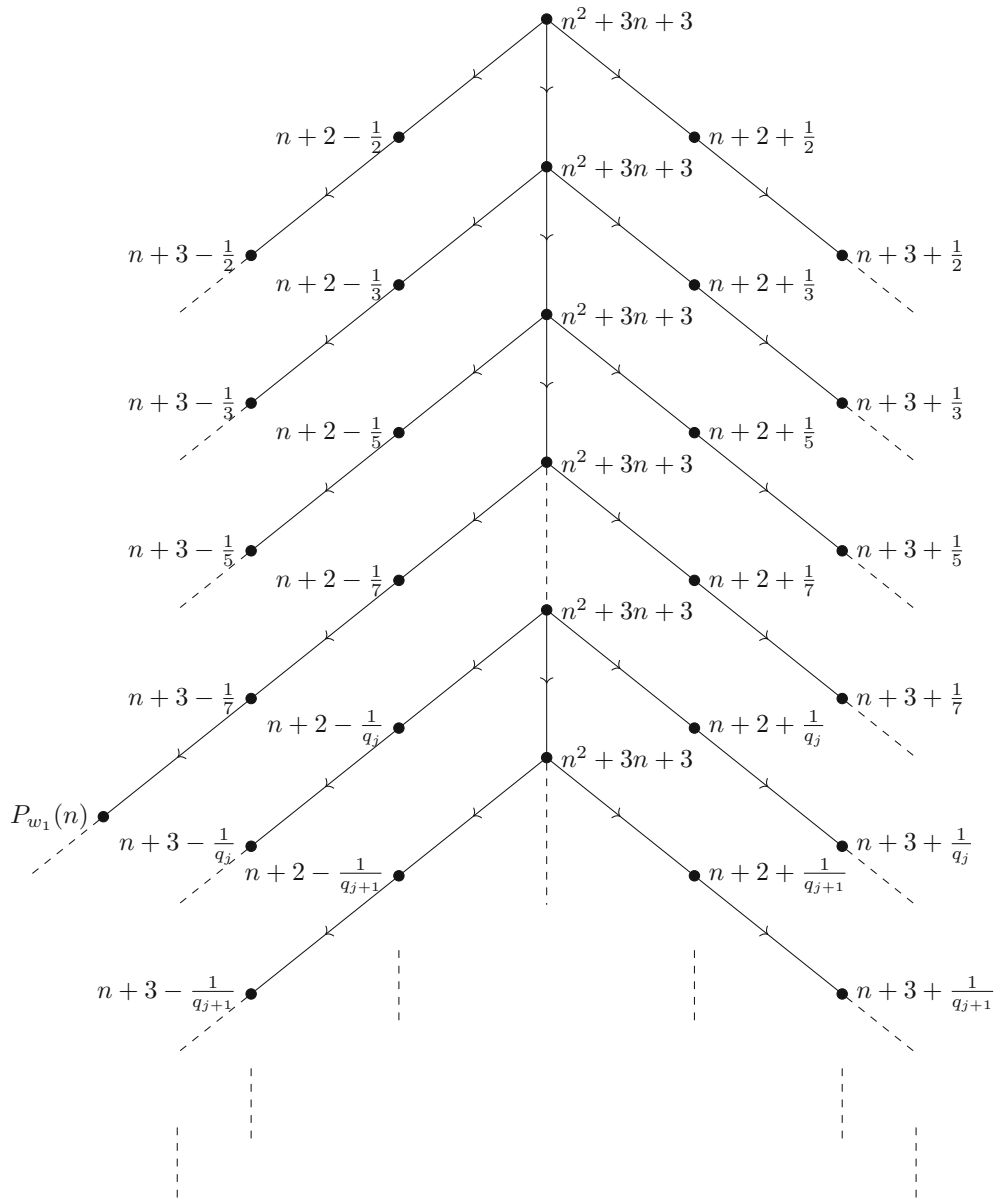
**Fig. 5** Tree on which shift operator is antinormal as well as an isometry

*Example 3.13* The shift operator on the following weighted tree is antinormal as well as an isometry. This is because  $\sum_{v \in \text{chi}(u)} \lambda_v = \lambda_u$ , for every vertex  $u$  in the following weighted tree.

The following example shows that when  $i(S) = -\infty$  and  $S$  is antinormal as well as an  $m$ -isometry then it need not be an isometry.

*Example 3.14* In the following figure a vertex  $u \in V$  is labeled by  $P_u(n)$ , where  $P_u$  is the polynomial corresponding to  $u$  and  $q_j$  denotes the  $j$ th prime number.

Since the above tree has no leaves, therefore  $\dim(\ker(S)) = 0$ . Also  $\text{card}(V_{\geq 2}) = \infty$ . Hence  $i(S) = -\infty$ . For a vertex  $w \in V$  such that  $\text{card}(\text{chi}(\text{par}(w))) = 1$ , its corresponding polynomial  $P_w$  is determined by Remark 3.5. For example  $P_{w_1}(n) = (n + 1) + 3 - \frac{1}{7} = n + 4 - \frac{1}{7}$ . Since  $\max_{u \in V} \deg(P_u) = 2$ , therefore  $S$  is



**Fig. 6** Tree on which shift operator is antinormal as well as strictly 3-isometry

strictly 3-isometry. We also note that  $P_u(0)$  is the weight of the vertex  $u$ . Further, from Theorem 2.9 it follows that the shift operator  $S$  on the above tree is antinormal.

Finally, we obtain the following proposition followed by its corollary.

**Proposition 3.15** *If  $S$  is antinormal and strictly  $m$ -isometry for  $m \geq 2$  then for every  $u \in V$   $\deg(P_u) \geq 1$ .*

*Proof* On the contrary, suppose there exists a  $u_0$  in  $V$  such that  $\deg(P_{u_0}) = 0$ . Since  $S$  is strictly  $m$ -isometry for  $m \geq 2$ . Therefore  $\|S\| > 1$ . As  $\deg(P_{u_0}) = 0$ , by Proposition 3.8 there exists a sequence  $\{\Delta(\mu(n))\}_{n=1}^\infty$  in  $\Delta$  with  $\Delta(\mu(n)) = 1$  for each  $n \geq 1$ . Consequently, from Theorem 2.9  $\|S\| = 1$ . This is a contradiction.  $\square$

**Corollary 3.16** *If  $S$  is an antinormal  $m$ -isometry for some  $m \geq 2$  and  $i(S) = -\infty$  then  $S$  is an isometry if and only if there exists a vertex  $u_0$  in  $V$  such that  $\lambda_{u_0} = \sum_{v \in \text{chi}^n(u_0)} \lambda_v$  for every  $n \in \mathbb{N}$ .*

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#### Declarations

**Conflict of interest** The first author declares that he has no conflict of interest. The second author declares that he has no conflict of interest.

**Ethical approval** This article does not contain any studies with human participants or animals performed by any of the authors.

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