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Bobomurat Ahmedov

Development and perspectives of relativistic astrophysics in Uzbekistan

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Abstract Speedy advancement in research in Astrophysics and related area around the globe has led scientists in Central Asia to produce high-quality research and constitute strong research in Relativistic Astrophysics and the Einstein theory of gravitation. Consequently, immense impact of this progress has been seen in central Asian countries in not only in research in Astrophysics and related areas, the academic institutions also received a trickle down effect in terms of the advancement in teaching as well as research. This focus of this paper is to overview the research incentives that are being taken up by the Theoretical Astrophysics Group located at Ulug Beg Astronomical Institute of Uzbekistan Academy of Sciences, Tashkent.

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1 Introduction

Relativistic astrophysics in Uzbekistan has long history [6]. The first steps in its real development in Uzbekistan have been made by Professor Nikolai Vsevolodovich Mitskievich starting the year 1960 in Samarkand State University and by Professor Lennur Arifov in the Institute of Nuclear Physics of Uzbekistan Academy of Sciences starting the year 1964. Moreover Theoretical Physics Chair of the National University of Uzbekistan named after Mirzo Ulugbek (former First Central Asian State University) was established in the year 1935 and has been headed from year 1935 to year 1941 by Professor A.E. Levashov who was leading expert on General Relativity (GR) and Gravitation. The main research team on relativistic astrophysics in Uzbekistan is hosted by Ulugh Beg Astronomical Institute of Uzbekistan Academy of Sciences which was established in the year 1873 being the oldest research institute of modern type in the Central Asia. Starting December of the year 2021 extension of relativistic astrophysics group established in the newly formed Institute of Fundamental and Applied Research, National Research University TIIAME.

One of the most important problems at present is the experimental verification of the modern extended theories of gravity or in other words the validity of general relativity in comparison with the various alternate theories of gravity. Classical gravitational experiments performed in the Solar system in the approximation of the weak gravitational field did not touch on the main property of the gravitational field in strong field regime

B. Ahmedov (✉)
Institute of Fundamental and Applied Research, National Research University TIIAME, Kori Niyoziy 39, Tashkent 100000, Uzbekistan
E-mail: ahmedov@astrin.uz

B. Ahmedov
Ulugh Beg Astronomical Institute, Astronomy St. 33, Tashkent 100052, Uzbekistan

B. Ahmedov
National University of Uzbekistan, Tashkent 100174, Uzbekistan

predicted by general relativity, i.e. its tensorial character. The most famous GR effects as the gravitational redshift, the deviation of the light ray by the gravitational field (of the Sun), the precession of the perihelion of Mercury, the gravitational time delay of the radar echo, the gravitational lensing are described by the diagonal terms of the metric tensor and are subject to the small general relativistic corrections to the Newtonian gravitational field. Investigations of the astrophysical processes in the vicinity of strongly magnetized and neutral gravitational compact objects, which are the relativistic space laboratories, allow and will allow further verification of the basic fundamental principles of general relativity strong gravitational field regime. In particular, we would like to list here the very recent triumphal discoveries in relativistic astrophysics of black holes (BHs) and neutron stars (NSs).

In 2015, the first direct observation of the gravitational waves GW140915 was made while there was a signal generated by the merger of two black holes (having 29 and 36 solar masses) at the distance of roughly 1.3 billion light years. This signal was received by the LIGO gravitational wave detectors located in Livingston and in Hanford. The first detection of gravitational waves has served an important starting point for appearance of new gravitational wave (GW) astronomy.

The first direct detection of the gravitational waves (GWs) and electromagnetic (EM) counterparts emanating from the coalescence of a binary neutron star system by the LIGO and Virgo collaborations as GW170817. The signal, emanating from near the galaxy NGC 4993, was designated as having been linked with the neutron stars merger. This event has given start to the beginning of the Multimessenger Multi-Wavelength Era in the astronomy when the same object/event has been observed through the gravitational waves and the whole spectrum of the electromagnetic waves. This also confirmed the nature of the neutron star of the merged objects and their associated kilonova.

Other extremely important discovery in the relativistic astrophysics is the noticing of the initial image of a black hole, at the center of Messier 87 (M87) announced by the EHT (Event Horizon Telescope) Collaboration on April 10, 2019. The array of radio telescopes performed this high-resolution observation at a wavelength of 1.3mm and with a theoretical resolution of 25 microarcseconds limited by diffraction. More recently, in March 2021, the EHT Collaboration announced more advanced image of M87 based on a polarization analysis which indicates the presence of magnetic field in BH environment and may help better reveal the physical processes giving rise to quasars in the nature.

The precise detection in October 2018 by the near-infrared GRAVITY-Very Large Telescope Interferometer (VLTI) beam-combining instrument (GRAVITY group at ESO) of continuous positional and polarization changes i.e. close orbital motion around the compact gravitational source Sgr A* in the center of Milky Way galaxy in high states (“flares”) of its variable near infrared emission.

IIce Cube, gamma-ray telescopes Fermi and MAGIC together with various other first time ever announced experiments in July 2018, such as detection of high energy particles as neutrinos and photons received from extragalactic blazar TXS 0506+056 are revolutionary. These results constitute and experimentally justify the first ever identification of a possible source of extragalactic neutrinos and high-energy cosmic rays. This is a revolutionary detection associated with a multi-messenger collaboration with detectors and scientists from all over the world. The Follow-up observations by gamma ray, X-ray and optical telescopes got triggered by neutrino alert (in real time) from IceCube on the 22nd September of 2017. High energy cosmic rays with the energy above 10^{18} eV having extragalactic origin were detected by PAO (Argentina) & TA (USA) in September 2017 and February 2018. All these direct observations are responsible to development of Neutrino and Cosmic Ray astronomy.

Main research fields in Uzbekistan are in general related to the development of astrophysical applications via effects of general relativity (GR) and alternate theories of gravity to the modern problems of relativistic astrophysics. The results obtained in the relativistic astrophysics are devoted to relativistic astrophysics, electromagnetic and scalar fields, gravitational waves and ionospheric studies of positioning systems based on satellites.

Relativistic astrophysics Triumphal revolutionary discoveries happened during the past decades developed and enriched our knowledge of objects of relativistic astrophysics and observational cosmology, the physics of gravitationally compact objects and precise gravitational experiments. As a result of recent rapid development of the area, relativistic astrophysics and cosmology became a practically useful, interesting and current subject of research in the World and in its turn the Central Asia is not exemption.

Magnetized neutron stars, magnetars and radio pulsars Relativistic stars as compact neutron stars holding the extremal strong gravitational and electromagnetic fields observed as pulsars and magnetars play an important role of astrophysical laboratory which may provide the best available precision tests of general relativity versus alternate theories of gravity in strong-field regime. It is one of the main subjects of the current rapidly growing research area in the Central Asia.

Isolated and binary black Holes Due to the current status of studies and research, BHs over the past several decades are known exciting astrophysical objects for analytic, numeric and observational studies. Essentially, BHs are relativistic objects which do not have Newtonian equivalents. To analyze BHs and understand their properties demands sophisticated mathematical tools/apparatus to deal with the physics of curved geometry in their close vicinity. Due to this reason, the study of BH is quite a challenging topic in both Central Asia as well as in the rest of the world. Current study of BH physics and, in particular, close BH binary systems, has received immense interest because of which these objects are currently known as most suitable candidates for the detection gravitational wave as remnant of merger in close binary. When two astrophysical BHs in close binary systems fuse together, they discharge a large amount of energy as gravitational waves. GW radiation released in very short millisecond timescale make them the bright most objects in the Universe. Due to fast progress in this direction, a search for the possible electromagnetic counterparts phenomena has a crucial significance from the point of view of observational understanding. Apart from its importance, its astrophysical importance for describing the binary evolution of BHs, a detectable electromagnetic signals counter parting the binary black hole merger may offer new wave of information, which may help understand gravitational wave signals more correctly.

Relativistic Astrophysics group in Tashkent is well experienced in studying astrophysical processes in the framework of the curved space-time and in applying numerical and analytical methods for solving the differential equations to describe the optical, energetic, electromagnetic and other observational properties of relativistic stars and BHs. More particularly, the group proposed formalism to learn about effects on properties of both electromagnetic fields (both within and outside) of magnetized relativistic stars as well as BHs because of extremely curved space-time geometry. Also studied by the group, is the force-free plasma magnetosphere of oscillating and rotating magnetized neutron stars. This was based on suggesting a model that explains phenomenology of the recurring pulsars in terms of stellar oscillations which may possibly be excited periodically by the glitches of the star. The group also studied conditions leading to radio emission in magnetars with focus on processes defining radio-loudness or quietness of magnetars. The current observations demonstrate close relationship during the burst activity of magnetars as well as radio emission generation of magnetar's magnetosphere. A more generic procedure is developed which illustrated BH shadow as an arbitrary polar curve expressed in terms of a Legendre expansion. It is shown that the developed formalism provides accurate description of noisy observational radio-interferometric data, with negligible variances error in comparison with prior measures of distortion. Gravitational lensing produced by compact astrophysical objects is rigorously studied. in this brief review, I summarize current status of progress made in the area of relativistic astrophysics in Uzbekistan over the last several years.

2 Exact analytical solutions for black holes

Exact analytical solutions to the gravitational field equations play an important role in describing space-time geometry of astrophysical black holes and relativistic neutron stars. In spite of the fact that thousands of exact analytical solutions are known in the literature, there is great interest to find new original exact solutions to the gravitational field equations.

An exact solution for black hole (regular) for source belonging to nonlinear electrodynamics given by Ayón-Beato and García (ABG) [10] to generate an exact solution for regular rotating BH, known as ABG and Kerr space-time metrics in the special limiting cases is published in [26]. The possible connection in the vacuum and non-vacuum solutions of the Einstein equations offers new avenues to understand physics of new solutions as with a nonlinear electro-dynamic source. Consequently, in the limiting case of zero electric charge, the solution becomes vacuum. Furthermore, it is shown that inside the rotating regular ABG black hole the weak energy condition is violated due to the appearance of the rotation in space-time. The derived exact solution provides an opening to study both causal and geometric structures, and to test motions of test particles and dynamics in the neighborhood of the regular rotating regular BH [24].

We know that the Newman-Janis algorithm (NJA) is a powerful complexification method to deal with new analytic solutions to the gravitational field equations. Interesting rotating quintessential black hole solution is constructed in paper [29] based on the framework of the modified NJA. Main features of the rotating quintessential BH solution are studied, in detail, for simplicity choosing a particular value of the quintessential parameter $\omega_q = -2/3$. For the quintessence parameter c that corresponds to the BH space-time geometry in the rotating case has been proved as equal to $c \leq 1/6$. This is significantly large as compared with the one in nonrotating case where the same limits equals $c \leq 1/8$ (see, e.g., [37]). It is also observed that limiting existence of static radius for circular geodesics is independent of the rotation parameter of the space-time. A discussion of parameters (a and c) of the quintessential rotating space-times is given as well as obtained. Also, the regions of the stable circular geodesics have also been determined for both black hole as well as the naked singularity.

Paper [28] introduces rotating coupled black hole solutions in GR. Also, the spherically symmetric black hole solution [15] is extended using a method of alternate NJA. Following our model, it is possible to construct exact BH solutions with appropriate values of the parameters μ and ν similar to that of a non-rotating case. The already existing solutions could be obtained for certain specific values of $\nu = 2$ – Bardeen-like black holes, $\nu = 3$ – Hayward-like black holes, and $\nu = 1$ – new type of Maxwellian black hole solution resembling Maxwell field in a weak field approximation. Additionally, it is shown that certain principal properties of the rotating BHs are almost similar to those that occur in non-rotating situation [15]. Specifically, when BH with vanishing gravitational mass , $M = 0$, the space-times with $\mu \geq 3$ are regular (nonsingular) ones. In addition, it is shown that the energy conditions are not effected by the presence of gravitational mass while energy conditions near the center of space-time are violated the obtained solutions. Also, whereas increase in μ reduces the depth of violation of energy conditions, these violations are always preserved even when μ is large.

In the paper [33], it is pointed out that the parameters for the constructed black hole solutions which are coupled to nonlinear electrodynamics (NED) in [15] are not correctly explained despite formalism itself being correct.

It is stressed that due to these mis-interpretations, the acquired BH solutions and the Lagrangian densities are somewhat not consistent. BH solutions for the given Lagrangian densities are found and physical significance of parameters is corrected showing that they lead to correct treatment and weak field limits of physical parameters of the built solutions.

3 Analytical and numerical quasinormal modes of static spherically symmetric black holes

The process of combination of BHs (or magnetized neutron stars) in close binary systems can be canonically splitted three principal stages as **inspiral**, **merger** and **ringdown** with each calculated using different mathematical approaches. The inspiral signifies early evolution of close binary system and because binary components are far enough from each other, one can completely deal with by analytic (analytic) methods in post-Newtonian approximation with the high degree of precision. In the next phase of when merging takes place, as a result strong and dynamic gravitational fields are produced for highly nonsymmetric system and this is calculable only via numerical relativity complex simulations. However, in the last, ringdown phase the remnant resulting from the merger of two compact gravitational objects rapidly settles to its final equilibrium state due to radiation of gravitational waves. These frequencies are called quasinormal modes because they are complex and subject to decay through imaginary part and are fully characterized by the space-time parameters only. The ringdown is calculable (semi-) analytically via perturbation theory. One can obtain the wave equation in the perturbation theory and solving the gravitational field equations in the linear order of perturbations. Thus, the ringdown portion of the gravitational wave signal may be used to distinguish between black holes and other possible gravitational merging sources. If one can detect in the LIGO-VIRGO observations of GW events both the fundamental and the first overtone of quasinormal mode in the ringdown, it will be possible to do more fundamental tests of general relativity versus to alternate in strong field regime gravity theories.

Our main interest was concentrated on the scalar, electromagnetic and gravitational perturbations of black holes in general relativity coupled to nonlinear electrodynamics and calculations of their characteristic oscillation frequencies, so-called quasinormal modes. The scalar, electromagnetic, axial and polar gravitational perturbations of the BH similar to Reissner-Nordström BH, with tidal charge in the Randall-Sundrum braneworld are discussed in [31]. The quasinormal frequencies of these perturbations are calculated in the framework of the WKB method. With an increase in the parameter associated with tidal charge, the frequency oscillations decreases and damping rate increases contrary to Reissner-Nordström black hole. It is also shown that the localized BH in the Randall-Sundrum brane-world is stable as compared with that scalar, electromagnetic, and gravitational perturbations. Also, it is shown that the due to potential (scalar) barrier the wave probability becomes larger than electromagnetic and gravitational waves. It is to this reason that the gravitational fields are preferred as transmission waves. Similarly, the scalar fields are treated on the same footing in terms of reflection of waves. The reflection properties of the perturbation fields decrease tidal charge parameter increase. Also, the horizon radii and the photon sphere increases as tidal charge increases. It should nevertheless be taken into account that in the absence of no upper limit on tidal charge parameter, the BHs on the brane admit event horizons and photon spheres. A study of absorption cross-section of the massless scalar waves by BH in braneworld is also studied in the low- and high-frequency regimes by showing that the BHs in braneworld have larger absorption cross-sections than the ones related to the Schwarzschild and Reissner-Nordström black holes. It is observed that in high-frequency regime, distinction of BHs from particle emission spectrum is almost not possible. In the light of this, one may infer that the infact the low-frequency regime is more significant and presumably relevant for astronomical observations.

Reference [25] studies in sixth order WKB approximation, the quasinormal modes of linear "axial" scalar, electromagnetic and gravitational perturbations in Hayward, Bardeen and ABG regular black hole space-time geometry. It is found that when there is an increase in space-time charge parameter, there appears correspondingly an increase (monotonic) in real part of the quasinormal frequency. In turn, monotonic decrease of the imaginary part of quasinormal frequency, and consequently, the damping rate of the wave decreases. It means that oscillators in the field of regular black holes are better (slowly damped) than that in the field of Schwarzschild black holes. For the scalar fields, the damping of the quasinormal modes is shown to be always the largest one and, mediates for the electromagnetic fields. For the gravitational fields, it is smallest for Hayward and Bardeen regular BH space-times. However, as regards ABG black hole geometries, the situation is additionally complex because the damping of the scalar fields is smaller than of the electromagnetic fields for charge in these geometries is $d > 0.56$. It is nevertheless discussed stability of all regular BHs against axial electromagnetic, gravitational and scalar perturbations.

Paper [32] studies dynamics of the scalar fields (massive or massless) in Schwarzschild-de Sitter geometry. First, for the waves which have quasinormal frequencies related to effective potential barriers (between the black hole and the cosmological horizons of the Schwarzschild-de Sitter space-time) lower limit on mass of the scalar field is obtained. It enables the wave to reach far observer who is located at infinity without any intersection with other potential wall. Inters tingly, the critical mass of the scalar field determining lower limit depends solely on the dimensionless cosmological parameter defined by the $y = \Lambda M^2/3$. In addition, this statement is quite relevant and applicable to the Reissner-Nordström-de Sitter geometry.

Second, the quasinormal frequencies of oscillations which are related to the unstable photon circular orbits in eikonal approximation are found. It is shown that cosmological parameter y of the Schwarzschild-de Sitter space-time increases, both damping rate of the quasinormal modes as well as frequency of oscillations tend towards a decrease.

Third, it is discussed that the large mass approximation of the effective potential of the massive scalar field is based on the space-time function $f(r)$. It is analogous to the eikonal limit governed by the structure of a circular null geodesic. As for the large mass regime is concerned, there is an interesting and relevant new phenomenon of the quasinormal resonance frequencies of the massive scalar fields (which are directly related to the static radius of the black hole space-times) is found. It is explicitly demonstrated the existence of slowly decaying quasinormal resonances in the Schwarzschild-de Sitter space-times with astrophysically relevant cosmological parameter y of very small values. These are similar to the long-lived quasinormal perturbations in ultra-compact objects. These quasi-normal modes correspond to real oscillations whose frequency equals mass of the scalar field for the Schwarzschild BH, $Re(\omega) = \mu$.

For quasinormal resonance relaxation time approaches infinity in Schwarzschild limit, $y \to 0$. The relaxation time of the resonance in Schwarzschild-de Sitter space-time, may be the reason for presence of static radius. Accordingly, its distance from the BH provides an extra information on extension of region bounded by gravitation [23].

In the family of singularity-free non-rotating black hole space-times found in [11,16] scalar and electromagnetic perturbations have been studied in paper [27]. It is observed that the scalar perturbations scaling factor S dependent. However, in a spherically symmetric black hole space-time, the electromagnetic perturbations are not affected by a conformal transformation of the background metric. Consequently, it is concentrated on study of scalar perturbations because the electromagnetic perturbations are identical in the Schwarzschild metric. It is found that in conformal gravity the singularity-free non-rotating black holes are stable under scalar and electromagnetic perturbations. It is found that the quasi-normal modes spectrum of scalar perturbations depends on the conformal gravity parameters L and N. It is found that for the dominant mode $\ell = 2$ for a single black hole remnant produced as result of the merger of a close binary black holes, the impact of parameters L and N can be large enough and in principle detectable.

It is well known that the wavelength is almost negligible relative to the horizon scale of the black hole in the high energy regime. Therefore, massless scalar waves follow the null geodesics in this regime and the classical capture cross-section of null geodesics is calculated to describe the absorption cross-section of massless fields. It has been shown in [11] that in the non-singular black hole space-times in conformal gravity, the classical capture cross section of null geodesics does not depend on the conformal factor. Therefore, one may conclude that the conformal factor, that is in the high energy regime, the parameters L and N, do not effect the dynamics of the massless scalar waves in environment of regular black holes in conformal gravity.

The paper [30], gives electrically and magnetically charged BH solutions (singular and regular ones) in GR coupled to the NED. It is shown that the derived family of new singular black hole solutions (in nonlinear electrodynamics with the Lagrangian density supposed in [15]) tend to the linear (Maxwell) electrodynamics in the weak field limit. The derived black hole solutions are singular at $r = 0$. These black hole solutions are convertible to the regular ones by the special condition at $M = q^3/\alpha$ unlike other singular black hole solutions. These Maxwellian regular black hole space-times as typical regular black hole space-times depending on the values of the gravitational mass and nonlinear electrodynamic field parameters also represent either black hole, extremal black hole, and no horizon space-times.

Assuming that the electromagnetic perturbations do not change the space-time geometry, the axial electromagnetic perturbations of black hole solutions in nonlinear electrodynamics are studied. Different potentials, and hence different results, are obtained for the quasinormal frequencies for electromagnetic perturbations of black holes in nonlinear electrodynamics and results compared with those related to eissner-Nordström black holes in standard electrovacuum theory based on the fact that electromagnetic perturbations of the electrically and magnetically charged black holes in linear electrodynamics (Reissner-Nordström ones) are isospectral. As a consequence the effective potentials and quasi-normal frequencies, there are same. In contrast, electrically and magnetically charged black holes have different potentials and different quasinormal modes in nonlinear electrodynamics. By normalizing charge as $Q = q/q_{ext}$, as a special case, the quasi-normal modes of magnetically charged Maxwellian regular BS with $\mu = 3$ have been worked out and results tallied with Reissner-Nordström one. Study of the quasi-normal frequencies and the time domain profile suggests that the Maxwellian regular BHs are stable against electromagnetic perturbations.

The paper [13] shows that in eikonal (high energy or large multipole number) limit quasinormal modes are characterized by unstable circular null geodesics. It is shown in paper [30] by the electromagnetic perturbations of the nonlinear electrodynamics black holes that this claim is correct in the standard linear electrodynamics, however, it does not work in the nonlinear electrodynamics, since in the nonlinear electrodynamic field photon does not follow the null geodesics, instead it follows the null geodesics of an effective metric. Thus, one can claim that in the eikonal regime, the quasinormal modes of nonlinear electrodynamic black holes are characterized by the unstable circular photon orbits determined by the effective geometry.

It is obtained the crucial result that the electromagnetic perturbations of highly charged black holes in general relativity coupled to nonlinear electrodynamics oscillate extremely slowly or in other words radiate never damping electromagnetic waves. If such electromagnetic signal is detected in electromagnetic counterparts observations in black holes post-merger phase, one can easily extract distinguishable information on the gravitational source of remnant formed as result of black holes merger.

4 Particles dynamics in close black hole environment

Extensive study of motion of test (neutral, charged, magnetized) particles around various gravitational compact objects has been performed due to its significance in the relativistic astrophysics.

The accretion disc in a BH environment contains electron positron plasma which in motion generates the magnetic field playing a crucial role in the high energy astrophysical processes. It is believed that magnetic field generated in the surroundings of BH does not effect the BH geometry and only effects the dynamics of charged particles. Static electrically charged BH can produce the electric field which in turn can generate magnetic field if it is rotating. The presence of gravitational and electromagnetic fields both in the space-time of rotating charged BH makes the motion of particles extremely complicated.

Dynamics of the particles in the environment of an Einstein–Gauss–Bonnet (EGB) black hole (in four dimensions) immersed in external asymptotically uniform magnetic field is explored in [22]. Due to the Lorenz force the magnetic field can strongly affect charged particles motion in the black hole environment. It is found that the Gauss–Bonnet (GB) coupling leads to an analogous effect, moving the innermost stable circular orbit (ISCO) towards the central object. Consequently, ISCO radius is smaller with respect to that around the typical Schwarzschild BH. Furthermore, particle collisions in the black hole vicinity are studied to determine the center of mass energy. It is shown that due to the effect of the GB term this energy is increased with respect to the Schwarzschild case. As a powerful tool to test the predictions of the model against astrophysical observations, epicyclic motion, its frequencies and resonance are studied. In particular, it is tested for low-mass X-ray binaries at which values of the parameters of the theory best fit the 3 : 2 resonance of high-frequency quasi-periodic oscillations .

In [12], the properties of the electromagnetic field generated in the background of a static, axially symmetric vacuum solution of Einstein's field equations assuming it is immersed in an external magnetic field. The exact solution, usually called as the (or Zipoy-Voorhees), is extended from the Schwarzschild space-time through a real positive parameter γ being responsible for deformation that describes its departure from spherical symmetry. It is studied the motion of charged and neutral particles in this space-time and particle collision in the vicinity of the singular surface. The obtained results are compared with the corresponding results for Schwarzschild space-time. A sharp contrast of γ metric results with the black hole case is observed. In particular, particle collision in the prolate case ($\gamma < 1$) can occur with an arbitrarily high center of mass energy. This mechanism could, in principle, provide an opportunity to distinguish such a source from a black hole.

A quasi-Kerr compact object is embedded in an external asymptotically uniform magnetic field in [17]. The paper [36] studies properties of generic singular black hole solutions which are of spherically symmetric magnetically charged coupled to nonlinear electrodynamics. The circular orbits and associated epicyclic motion and its frequencies are discussed for characteristic values of the space-time parameters and the dimensionless parameter characterizing ratio of gravitational and electromagnetic forces acting on an electrically charged particle. It is shown that the off-equatorial circular orbits are allowed, however, equatorial circular orbits are forbidden. To fit the data of the twin high-frequency quasiperiodic oscillations of X-rays observed in microquasars, the possible resonance phenomena of the epicyclic frequencies and the orbital frequency of the electrically charged particles are studied. Furthermore, the dynamics of magnetized particles around the magnetically charged generic black hole are also explored. It is shown that the innermost stable circular orbit (ISCO) radius is inversely proportional to the magnetic charge of black hole. Consequently, increase of magnetic charge and magnetic moment parameters decreases ISCO which disappears at critical value of the magnetic moment parameter. The magnetar PSR J1745-2900 orbiting around Sagittarius (Sgr) A* in the center of Milky Way galaxy is treated as a magnetized particle and used as an astrophysical application. Based on exactly the same value of the ISCO radius calculated, it is observed that the magnetic charge of black hole can mimic black hole spin up to $a/M = 0.865694$ at $\nu = 2$. On the other hand, the spin parameter of BH can mimic the magnetic charge parameter up to $q/M = 0.578575$ at $\nu = 1$. It is predicted that no magnetar with the surface magnetic field of the order of $10^{14} - 10^{15}$G can follow stable orbits, while ordinary standard neutron stars could be observed as recycled radio pulsars in the close environment of Sgr A*.

It is believed that the weak cosmic censorship conjecture (WCCC) for linear order accretion could be violated. The Myers-Perry five dimensional rotating black hole with two rotations could be overspun by test particle accretion, while it cannot do so for a single rotation as proven in [20,21].

5 Optical properties of black holes

Possibility of the detection of the "shadow" to distinguish a black hole from an alternate gravitational compact object is explored in [1]. Properties of a class of vacuum static axially symmetric space-times that is continuously linked to the Schwarzschild space-time metric implementing the single parameter, that can be treated as the deformation of the gravitational source, are studied. With the aim of comparison of the expected theoretically verified image with the shadow of a Schwarzschild black hole, the motion of photons, gravitational lensing effect and shadow produced by the gravitational source studied are investigated. It is concluded that it may not be possible to distinguish an exotic source with small deformation parameter from a black hole in the framework of astrophysical black holes. However, as the deformation increases, noticeable and measurable effects arise. Therefore, the future more precise measurements of the shadow of astrophysical black hole candidates would provide powerful tool to put constraints on the nature and deviation of the gravitating object from spherical symmetry.

The shadow and optical properties of a spinning Kerr-Taub-NUT black hole holding nonzero gravitomagnetic charge are explored in [2]. The shape of the Kerr-Taub-NUT black hole shadow is deformed by the gravitomagnetic charge in addition to the angular momentum of black hole. It is claimed based on the numerical results for a fixed value of the BH spin, that the presence of a gravitomagnetic charge increases the shadow's size and reduces its deformation with respect to the Kerr space-time without gravitomagnetic charge. It is also explored the capture cross section for massive particles by a spinning Kerr-Taub-NUT black hole.

The shadow and optical properties of spinning non-Kerr black hole are studied in the paper [8]. The shape of the shadow of non-Kerr black hole is valuably distorted by the deformation parameter of non-Kerr space-time in addition to the specific angular momentum a of BH. The presence of a deformation parameter reduces the shadow's size and simultaneously enlarges its deformation with respect to the one in the Kerr space-time (for a fixed value of the black hole spin a). Optical features (based on the treatment of rotation of the polarization vector along null congruences) of the spinning non-Kerr black hole are also explored.

In paper [9], the shadow and optical properties of spinning Hořava-Lifshitz black hole are studied. The shape of the shadow of spinning Hořava-Lifshitz black hole is essentially deformed by parameters of space-time in addition to the specific angular momentum a of BH. Radius of the Hořava-Lifshitz BH's shadow and its distortion parameter strongly depend on the coupling parameters of the modified gravity.

In the presence of a plasma with radial power-law density, shadow and optical properties of axially symmetric spinning Kerr black hole are investigated in [7]. In the presence of an astrophysically relevant plasma, the observed shape and size of the shadow of Kerr BH are subject to be changed depending on the (i) plasma main parameters, (ii) black hole spin, and (iii) inclination angle between the observer plane and the axis of rotation of the Kerr black hole. The particular case of the Schwarzschild black hole is investigated to extract the pure effect of the plasma effects on the black hole's observable image. It is found that the plasma does not modify the photon sphere around the spherically symmetric black hole. However, the refraction of the electromagnetic radiation in the plasma environment of the black hole reduces the size of Schwarzschild black hole shadow in the plasma. The presence of the plasma decreases the maximal energy emission rate from the black hole which is based on study of the energy emission from the black hole in plasma environment.

The shadow of the spinning black hole with quintessential energy is investigated in [4] i) in vacuum and ii) in the presence of plasma with radial power-law density. The quintessential field parameter of the spinning black hole essentially modifies the shape of the BH shadow in vacuum case. The radius of the BH shadow also increases with the increase of the quintessential field parameter. The quintessential field parameter produces decrease of the distortion of the shape of BH shadow with the increase of the radius of the shadow of the spinning black hole. The shadow of rapidly spinning black hole starts to become more close to circle under the influence of the quintessential field parameter. The shape and size of shadow of quintessential spinning black hole surrounded by plasma depends on i) plasma main parameters, ii) black hole spin and iii) quintessential field parameter. The apparent radius of the BH shadow increases with the increase of the plasma refraction index. However, the modification of the black hole shadow's shape produced by the influence of plasma is not valuable for the big values of the quintessential field parameter. Obviously, the effect of the quintessential field parameter on optical properties of BH becomes more dominant with compare to the effect of plasma.

The optical properties (including shadow's cast) of the various types of spinning regular black holes as Ayon-Beato-Garcia (ABG), Hayward, and Bardeen is studied in paper [5]. In addition to the total mass (M) and spin parameter (a) the black holes studied contain different parameters as electric charge (Q), magnetic charge ($g*$) and deviation parameter (g). Interestingly, that in addition to the spin parameter the size of the BH shadow is affected by these parameters. It is found that the distortion parameter increases when the values of these parameters increase while the radius of the shadow of BH in each case decreases monotonically. It is also investigated a comparison with the results being valid for the Kerr black hole. The influence of the plasma environment on photon motion around regular black holes is also studied to discuss their shadows. The plasma effects increase the apparent size of the shadow of regular black hole due to two main effects: (i) gravitational redshift of the photons and (ii) radial dependence of plasma density.

Dynamics of the massless particle motion around spinning wormhole in the presence of plasma environment is explored in the paper [3]. It is demonstrated that the inner radius of the circular orbits of photons around spinning wormhole is decreased in the presence of the plasma. It is explored the shadow cast by spinning wormhole surrounded by inhomogeneous plasma with the radial power-law density. The size and shape of the wormhole's shadow is modified and distorted depending on i) plasma's main parameters, ii) wormhole rotation parameter and iii) inclination angle between observer plane and axis of rotation of wormhole. As an example, an inverse radial distribution of the plasma density and different types of the wormhole solutions are explored.

The gravitational lensing in the weak field approximation near the regular Bardeen, Hayward and Ayon-Beato-Garcia (ABG) black holes surrounded by plasma is extensively studied in paper [3]. The exact expressions for the deflection angle of the photons due to the effects of the plasma and the gravitational field are obtained. In the weak field approximation, the gravitational lensing assuming the presence of a magnetic field and a plasma around a compact gravitational source is studied in [34]. The surrounding test magnetic field leads

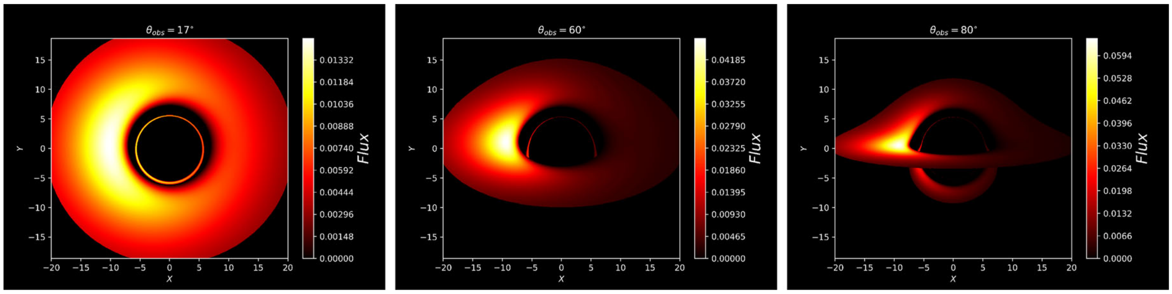

Fig. 1 Images of shadow of Schwarzchild black hole under different angles of view with respect to the plane of the accretion disc

to the split of the gravitationally lensed image, as the counterpart of the Zeeman effect. In addition magnetic field affects the magnification of images, creating additional components. It is also explored the time delay of an electromagnetic signal due to the geometry and the gravitational field in close environment of the source. It is shown that the time delay strongly depends on the plasma parameters. The effects of an inhomogeneous plasma in close environment of source on the gravitational lensing are also explored.

Relativistic astrophysics group in Tashkent developed different codes as ray tracing code for shadow of black holes and codes for numerical images of black holes with illuminating accretion disc. As example, we provide synthetic images of shadow of standard Schwarzchild black hole under different angles of view with respect to the plane of the accretion disc in Fig. 1.

6 Energetic properties of rotating magnetized neutron stars and black holes

Research carried out by Tashkent Relativistic Astrophysics group has been playing a pivotal role in understanding the large scale structure of space-time to test BH parameters in strong gravity regime. This research has helped in explanation of highly energetic processes in the universe occurring in the vicinity of the event horizon of astrophysical and super-massive BHs such as formation of jets (involving escaping charged particles in presence of magnetic field), accretion disks, cosmic-rays and gamma-rays acceleration as being highly interesting areas of research in the subject.

As we know the active galactic nuclei (AGN) can be seen as best gravitational sources for acceleration of particles with ultra high energy and can provide important information about the formation of large-scale structure, evolution and clustering of galaxies and the universe as a whole.

The presence of the strong electromagnetic fields is considered as one of the most important properties of rotating highly magnetized NSs observed as radio pulsars, soft gamma ray repeaters, anomalous X-ray pulsars and magnetars. On the other hand, it is believed that astrophysical BHs do not have their own intrinsic magnetic field. However, one can assume that the BH is located in an external asymptotically uniform magnetic field created by a source nearby, such as a neighboring NS or magnetar or surrounding plasma motion in accretion disc. Wald in 1971 obtained an exact solution of Maxwell's vacuum equations for an asymptotically uniform magnetic field. During the last decades, to study the issue of extraction of BH energy towards high luminocity, the properties of BHs in an external magnetic field were studied in detail by numerous authors, in particular, through the classical Blandford-Znajeck (BZ) effect and Magnetic Penrose process (MPP).

The different processes such as, Bonados-Silk-West (BSW), Penrose, and collisional Penrose processes are applied to explore important issue of energy extraction from the spinning regular ABG black hole. It is found that two colliding neutral particles which are at rest at infinity with the different angular momenta can give enormously huge value for the center of mass energy. The potential of the gravitational field is decreased by the electric charge Q of spinning regular black hole. Consequently, the particle needs less bound energy to be at the circular geodesics. It leads to increase of efficiency of the energy extraction from spinning regular black hole through BSW process due to the strong influence of the electric charge Q. It is found that energy extraction from the extremal spinning ABG black hole through the collisional Penrose process has smaller efficiency than that from the spinning Kerr black hole. Furthermore, it is demonstrated that it takes place not only in IN+ scenario but also in OUT+ one. However, the IN+ scenario is the most favourite one to extract energy with high efficiency. OUT+ scenario has maximum efficiency with $\sim 112.6\%$, that is smaller than that for IN+ with $\sim 145.5\%$. Obviously, it takes place in an extremely narrow restricted region of the values of the spin parameter a and the electric charge Q and of the extremal spinning ABG black hole. The efficiency of the energy extraction from the spinning regular black hole through the Penrose process is decreased with increase of the electric charge Q. Consequently, it is smaller in comparison to 20.7 % which is the efficiency for the extreme maximally spinning Kerr black hole with the specific angular momentum $a = 1$. The ergosphere of the spinning black hole is decreased under influence of the nonvanishing electric charge Q and ergoregion vanishes for the limiting value of the electric charge $Q > 0.634$. The energy extraction can not be performed after vanishing of the ergosphere of BH.

Magnetic Penrose process was first invented by Naresh Dadhich and his coauthors in mid-1980s for discrete particle accretion. It is observed in [14] that feature that distinguish MPP is its super high-efficiency exceeding 100 per cent of extraction of rotational energy of a spinning black hole electromagnetically for a magnetic field of even milli Gauss order. Another similar and important process is Blandford-Znajek mechanism (BZ) which could be envisaged as high magnetic field limit MPP, because it requires threshold magnetic field of order 10^4G. Current studies regarding fully relativistic magnetohydrodynamic (MHD) flows provide super high-efficiency signature of the BZ process for high magnetic field regime. It is predicted that such simulation studies of MHD flows for low magnetic field MPP regime would also have high super-efficiency.

The acceleration mechanisms and generation of ultrahigh-energy cosmic rays (UHECRs) of energy > 10^{20}eV, clearly beyond the GZK cutoff limit, still remain unclear as mystery. It is clear indication to the exotic nature of the interesting unsolved phenomena. The source of UHECRs is an extragalactic supermassive black hole (SMBH) according to recent observations of extragalactic neutrinos. It is demonstrated in [35] that ultraefficient energy extraction from a spinning SMBH driven by the magnetic Penrose process (MPP) could indeed fit the bill. For an SMBH of mass $10^9 M_\odot$ and magnetic field 10^4G ionization of neutral particles, such as neutron beta decay, skirting close to the black hole horizon energizes protons to over 10^{20}eV. For SMBH SgrA* in the center of Milky Way galaxy, a proton energy is of order $10^{15.6}$eV that exactly coincides with the knee of the cosmic-ray spectra. High-energy particles accelerated along the escaping directions can reach large γ_z factors only in the presence of an induced charge of the black hole (which is known as the Wald charge in the case of an asymptotically uniform magnetic field). It is remarkable that the process does not require either fine-tuning of accreting-matter parameters or an extended acceleration zone. Moreover, as the source of UHECRs this leads to certain verifiable constraints on the SMBH's mass and magnetic field strength. These crucial results make the ultraefficient regime of the MPP one of the most promising mechanisms for fueling the UHECR powerhouse.

The dissipation mechanism is extremely dominant in damping stellar oscillations. Hence, determination of this dissipation mechanism extremely important in aster-seismology of compact and magnetized neutron start. These mechanism are limited either gravitational waves or electromagnetic losses in a regime which is linear in low-multi-polarity modes. The energy losses in the form of electromagnetic Poynting fluxes with dipola magnetic field in vacuum have been looked into [19]. This explains electric and magnetic fields which are produced by common modes of oscillation both in the vicinity of the star as well as far away from despite that this formalism is not particularly realistic fast spinning stars. Thus estimates of McDermott are revisited and extended in relativistic context. In-fact, it is found that the relativistic corrections lead to electromagnetic damping time-scales to one order of magnitude smaller as compared with Newtonian gravity. It is also found that f (fundamental), p (pressure), I (interface) and s (shear) modes are suppressed more efficiently by gravitational losses as compared with those of the electromagnetic ones.

7 Conclusion

Very recently good progress have been made in development of post-graduate study and postdoctoral research on relativistic astrophysics in Uzbekistan. In the year 2018, four PhD dissertations were successfully defended by the postgraduate students enrolled in relativistic astrophysics study: i) Farruh Atamurotov, *"Optical and energetic processes in vicinity of spinning relativistic compact objects"*, March; ii) Arman Tursunov, *"Astrophysical processes in the field of compact objects"*, June; iii) Bobir Toshmatov, *"Particles and fields around compact objects"*, August; iv) Sanjar Shaymatov, *"General relativistic astrophysical processes in the vicinity of compact gravitational objects in the presence of electromagnetic field"*, December. Javlon Rayimbayev in the year 2020, Ozodbek Rahimov and Abdullo Hakimov in the year 2021 have successfully defended PhD dissertations entitled as *"Energetic properties of rotating neutron stars"*, *"Motion of charged and magnetized particles with angular momentum in general theory of relativity"* and *"Relativistic astrophysical processes in vicinity of axial symmetric objects in alternate theories of gravity"*, respectively.

Table 1 Author-level metric of the scientific productivity and citation impact of the publications by holders of scientific degrees and specialized in theoretical astrophysics in Ulugh Beg Astronomical Institute

No	Name	H-index WoS Scopus	H-index Google Scholar	Citations Google Scholar	First paper publ	Experience years	H-index growth rate
1	Bobomurat Ahmedov	34	40	4700	1993	29	39/29 = 1.3
2	Ahmadjon Abdujabbarov	30	33	3000	2009	13	33/13 = 2.5
3	Bobir Toshmatov	18	18	1100	2014	8	18/8 = 2.2
4	Arman Tursunov	14	14	850	2013	9	14/9 = 1.5
5	Askar Abdikamalov	12	14	700	2018	4	14/4 = 3.5
6	Sanjar Shaymatov	11	14	450	2011	11	14/11 = 1.2
7	Farruh Atamurotov	12	13	800	2013	9	13/9 = 1.4
8	Shuhrat Mardonov	6	8	150	2011	11	8/11 = 0.7
9	Javlon Rayimbaev	12	13	400	2013	8	13/8 = 1.6
10	Abdullo Hakimov	6	7	200	2010	12	7/12 = 0.6
11	Bakhtiyor Narzilloev	6	6	150	2017	5	6/5 = 1.2
12	Ozodbek Rahimov	3	4	130	2011	10	4/11 = 0.4
	Average	14	16	1500			1.5

Furthermore, during the last years, five habilitation DSc dissertations were also successfully defended by i) Ahmadjon Abdujabbarov, *"Particles and electromagnetic fields around axial - symmetric compact gravitating objects"* in the year 2016; ii) Arman Tursunov, *"Astrophysical processes in black hole environment and high - energy cosmic rays"* in the year 2019; iii) Askar Abdikamalov, *"RELXILL-NK: a relativistic reflection model for testing strong gravity"* in the year 2020; iv) Bobir Toshmatov, *"Dynamics of scalar, electromagnetic and gravitational perturbations and particles around relativistic compact objects"* in the year 2021; v) Bakhtiyor Narzilloev, *"Properties of astrophysical black hole solutions in various theories of gravity"* in the year 2021.

In the next year 2022, four PhD dissertations are expected to be defended by the postgraduate students enrolled in relativistic astrophysics study: i) Husan Eshkuvatov, *"Ionospheric and magnetospheric perturbations associated with the Earth, atmospheric and astrophysical phenomena"*; ii) Pulat Tadjimuratov, *"Electromagnetic signatures and thermal evolution of neutron stars"*; iii) Sardor Tojiev, *"Properties of boson stars and black holes in anti-de Sitter space-time"*; iv) Nozima Juraeva, *"Particles in astrophysical black holes environment"*.

Author-level metric of the scientific productivity and citation impact of the publications by holders of scientific degrees who specialized in Theoretical Astrophysics in Ulugh Beg Astronomical Institute are summarized in Table 1.

Numerous PhD and DSc defences successfully performed in very recent years strongly justify the rapid growth and support promising perspectives of relativistic astrophysics in Uzbekistan.

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Declarations

Conflict of interest The author declares no conflict of interest.

Informed Consent Statement Not applicable.

Data availability statement Not applicable.

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