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On composite entire functions and their central indexes

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Abstract In this paper, we present a few relations on the composition of two entire functions in terms of their central indexes. Furthermore, we investigate various growth properties of composite entire functions.

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1 Introduction

Let us consider an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then, the maximum modulus, maximum term, and central index of the entire function $f(z)$ are defined as

$$\begin{aligned} M_f(r) &= \max_{|z| \leq r} |f(z)| \text{ for } r > 0, \\ \mu_f(r) &= \max_{n \geq 0} |a_n| r^n \text{ for } r > 0, \\ \nu_f(r) &= \max \{n : \mu_f(r) = |a_n| r^n\} \text{ for } r > 0, \end{aligned}$$

respectively.

It follows immediately that $M_f(r)$, $\mu_f(r)$ and $\nu_f(r)$ are real-valued increasing continuous functions of r . The properties of $M_f(r)$, $\mu_f(r)$ and $\nu_f(r)$ can be found in [4, 5], so we do not discuss them in details. For the necessity of our paper, we first recall the following definitions.

Definition 1.1 The order ρ_f and lower order λ_f of the entire function $f(z)$ are defined as

$$\rho_f = \limsup_{r \rightarrow \infty} \frac{\log \log M_f(r)}{\log r}$$

and

$$\lambda_f = \liminf_{r \rightarrow \infty} \frac{\log \log M_f(r)}{\log r},$$

respectively.

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Definition 1.2 Also from [3], we get

$$\rho_f = \limsup_{r \rightarrow \infty} \frac{\log v_f(r)}{\log r}$$

and

$$\lambda_f = \liminf_{r \rightarrow \infty} \frac{\log v_f(r)}{\log r},$$

respectively.

Recently, Biswas [1] used central index to prove some results of growth of entire functions with respect to L^* -order. Being motivated, we investigate some results of growth of composite entire functions using their central indexes. In this paper, we present a few relations on the composition of two entire functions in terms of their central indexes. Furthermore, we investigate various growth properties of composite entire functions.

In 1970, Clunie [2] discussed the behaviour of the ratios $\frac{\log M_{f \circ g}(r)}{\log M_f(r)}$ and $\frac{\log M_{f \circ g}(r)}{\log M_g(r)}$ for a meromorphic function $f(z)$ and an entire function $g(z)$. Considering the fact that central index is much weaker than the maximum modulus in some sense, we investigate the behaviour of the ratios $\frac{v_{f \circ g}(r)}{v_f(r)}$ and $\frac{v_{f \circ g}(r)}{v_g(r)}$ along with few other conditions for the growth analysis of composite entire functions, where $f(z)$ and $g(z)$ are two entire functions of nonzero finite order.

In 1984, Song and Yang [6] discussed the behaviour of the ratios $\frac{\log \log M_{f \circ g}(r)}{\log \log M_f(r)}$ and $\frac{\log \log M_{f \circ g}(r)}{\log \log M_g(r)}$, where both $f(z)$ and $g(z)$ are entire. We also study the nature of the ratios $\frac{\log v_{f \circ g}(r)}{\log v_f(r)}$ and $\frac{\log v_{f \circ g}(r)}{\log v_g(r)}$, where in the first ratio, $f(z)$ is an entire function of nonzero finite order and $g(z)$ is an entire function of nonzero finite lower order, and in the second case, $f(z)$ is an entire function of nonzero finite lower order and $g(z)$ is an entire function of nonzero finite order.

Furthermore, under some suitable conditions, for an entire function $f(z)$ of nonzero finite order and an entire function $g(z)$ of nonzero finite lower order, we investigate the behaviour of the ratio $\frac{\log v_{f \circ g}(r)}{v_f(r)}$.

Throughout this paper, we use the notations $\log_1 r = \log r$, $\log_{i+1} r = \log(\log_i r)$ for $i = 1, 2, \dots$ and for sufficiently large r , also $\exp_1 r = e^r$, $\exp_{i+1} r = \exp(\exp_i r)$ for $0 \leq r < \infty$ and $i = 1, 2, \dots$

2 Preliminary lemmas

In this section, we present following known lemmas which are useful to prove the main results.

Lemma 2.1 [6] Let $\lambda(g) < \infty$. Then, for any $\varepsilon > 0$ and sufficiently large r

$$M_{f \circ g}(r^{1+\varepsilon}) \geq M_f(M_g(r)). \quad (2.1)$$

Lemma 2.2 [2] If $f(z)$ and $g(z)$ are two entire functions with $g(0) = 0$, then for $r > 0$

$$M_{f \circ g}(r) \geq M_f(c(\alpha) M_g(\alpha r)), \quad (2.2)$$

where α satisfy $0 < \alpha < 1$ and take $c(\alpha) = \frac{(1-\alpha)^2}{4\alpha}$.

Furthermore, if $g(z)$ is any entire function, then with $\alpha = \frac{1}{2}$, for sufficiently large values of r

$$M_{f \circ g}(r) \geq M_f\left(\frac{1}{8} M_g\left(\frac{r}{2}\right) - |g(0)|\right). \quad (2.3)$$

Again, by the definition, it follows immediately that:

$$M_{f \circ g}(r) \leq M_f(M_g(r)). \quad (2.4)$$

Lemma 2.3 [3] Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function. Then

(i) for $a_0 \neq 0$,

$$\log \mu_f(r) = \log |a_0| + \int_0^r \frac{v_f(t)}{t} dt,$$

(ii) for $r < R$,

$$M_f(r) < \mu_f(r) \left\{ v_f(R) + \frac{R}{R-r} \right\}.$$

Lemma 2.4 [1] *Let $f(z)$ be an entire function with nonzero finite order. Also, let $g(z)$ be an entire function with nonzero finite lower order. If $0 < \alpha < \lambda_g$, then for all sufficiently large values of r*

$$v_{f \circ g}(r) > v_f(\exp(r^\alpha)).$$

Lemma 2.5 [1] *Let $f(z)$ be an entire function with nonzero finite lower order. Also, let $g(z)$ be an entire function with nonzero finite order. If $0 < \alpha < \lambda_g$, then for all sufficiently large values of r*

$$v_{f \circ g}(r) > v_g(\exp(r^\alpha)).$$

3 Main results

In this section, we prove our main results.

Theorem 3.1 *Let $f(z)$ and $g(z)$ be two entire functions with nonzero finite order. If $\rho_g < \beta$, then for all sufficiently large values of r*

$$v_{f \circ g}(r) \leq v_f(\exp r^\beta).$$

Proof For an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, let us assume that $|a_0| \neq 0$.

Now, from (i) of Lemma 2.3, we have

$$\log \mu_f(r) = \log |a_0| + \int_0^r \frac{v_f(t)}{t} dt,$$

that is

$$\begin{aligned} \log \mu_f(2r) &= \log |a_0| + \int_0^{2r} \frac{v_f(t)}{t} dt \\ &\geq \log |a_0| + \int_r^{2r} \frac{v_f(t)}{t} dt \\ &\geq \log |a_0| + v_f(r) \int_r^{2r} \frac{1}{t} dt, \quad (\text{By second MVT}) \\ &\geq \log |a_0| + v_f(r) \log 2. \end{aligned} \tag{3.1}$$

Again, using Cauchy inequality, we have

$$\mu_f(2r) \leq M_f(2r). \tag{3.2}$$

Hence, using (3.1) and (3.2), we get

$$v_f(r) \log 2 \leq \log M_f(2r) + c_1, \text{ for some suitable constant } c_1.$$

The above implies

$$v_f(r) \leq c_2 \log M_f(2r) + c_3, \text{ for some suitable constants } c_2 \text{ and } c_3.$$

In view of the above, we get for all sufficiently large values of r

$$v_{f \circ g}(r) \leq c_2 \log M_{f \circ g}(2r) + c_3,$$

that is

$$\log v_{f \circ g}(r) \leq \log \log M_{f \circ g}(2r) + O(1). \quad (3.3)$$

Again, for any given $\varepsilon > 0$ and for all sufficiently large values of r , by Definition 1.2, for the entire function $f(z)$ in terms of central index, we have

$$\log v_f(r) > (\lambda_f - \varepsilon) \log r. \quad (3.4)$$

If there exists a constant β , such that $\beta > \rho_g$, then (3.4) implies

$$\begin{aligned} \log v_f(\exp r^\beta) &> (\lambda_f - \varepsilon) \log(\exp r^\beta) \\ &= (\lambda_f - \varepsilon) r^\beta. \end{aligned} \quad (3.5)$$

From (3.3) and (3.5), we get

$$\frac{\log v_{f \circ g}(r)}{\log v_f(\exp r^\beta)} \leq \frac{\log \log M_{f \circ g}(2r) + O(1)}{(\lambda_f - \varepsilon) r^\beta}. \quad (3.6)$$

Now, by (2.4), we have

$$\begin{aligned} \log \log M_{f \circ g}(2r) &\leq \log \log M_f(M_g(2r)) \\ &< (\rho_f + \varepsilon) \log M_g(2r) \\ &< (\rho_f + \varepsilon) (2r)^{(\rho_g + \varepsilon)}. \end{aligned} \quad (3.7)$$

Using (3.7) in (3.6), we thus have for sufficiently large r and for any $\varepsilon > 0$

$$\frac{\log v_{f \circ g}(r)}{\log v_f(\exp r^\beta)} \leq \frac{(\rho_f + \varepsilon) (2r)^{(\rho_g + \varepsilon)} + O(1)}{(\lambda_f - \varepsilon) r^\beta}. \quad (3.8)$$

As it is given $\beta > \rho_g$, we can choose $\varepsilon > 0$, such that $\beta > \rho_g + \varepsilon$.

Using this condition in (3.8), we get that

$$\lim_{r \rightarrow \infty} \frac{\log v_{f \circ g}(r)}{\log v_f(\exp r^\beta)} = 0. \quad (3.9)$$

Therefore, by (3.9), for sufficiently large values of r and for a small constant $A (> 0)$, we obtain

$$\log v_{f \circ g}(r) < A \log v_f(\exp r^\beta),$$

that is

$$\log v_{f \circ g}(r) < \log \{v_f(\exp r^\beta)\}^A,$$

which implies that

$$v_{f \circ g}(r) \leq v_f(\exp r^\beta).$$

This completes the proof. $\square$

Corollary 3.2 Let $f(z)$ and $g(z)$ be two entire functions with nonzero finite order. If $\rho_g < \beta$, then for all sufficiently large values of r

$$v_{f \circ g}(r) \leq v_g(\exp r^\beta).$$

Definition 3.3 We introduce a new function $\zeta_f(r)$ for an entire function $f(z)$, which is defined as

$$\zeta_f(r) = \frac{\log v_f(r)}{\log \log r}.$$

Theorem 3.4 Let $f(z)$ be an entire function of nonzero finite order and $g(z)$ be an entire function of nonzero finite lower order. Furthermore, let $\limsup_{r \rightarrow \infty} \zeta_f(r) = \infty$ with the condition $0 < \alpha < \lambda_g$, and then

$$\rho_{f \circ g} = \infty.$$

Proof For any $\varepsilon > 0$ and for sufficiently large r

$$\rho_{f \circ g} = \limsup_{r \rightarrow \infty} \frac{\log v_{f \circ g}(r)}{\log r}.$$

Using Lemma 2.4, the above definition implies

$$\begin{aligned} \rho_{f \circ g} &\geq \limsup_{r \rightarrow \infty} \frac{\log v_f(\exp(r^\alpha))}{\log r} \\ &\geq \limsup_{r \rightarrow \infty} \frac{\log v_f(\exp(r^\alpha))}{\log \log(\exp(r^\alpha))} \cdot \frac{\log \log(\exp(r^\alpha))}{\log r} \\ &\geq \limsup_{r \rightarrow \infty} \frac{\log v_f(\exp(r^\alpha))}{\log \log(\exp(r^\alpha))} \cdot \limsup_{r \rightarrow \infty} \frac{\log r^\alpha}{\log r} \\ &\geq \alpha \cdot \limsup_{r \rightarrow \infty} \frac{\log v_f(\exp(r^\alpha))}{\log \log(\exp(r^\alpha))} \\ &= \infty, \end{aligned}$$

since $\exp(r^\alpha)$ is increasing, continuous, and unbounded in r for $\alpha > 0$. $\square$

Theorem 3.5 Suppose $f(z)$ and $g(z)$ are two transcendental entire functions of nonzero finite order with the condition $\beta > \rho_g$. Furthermore, let

$$\eta = \limsup_{r \rightarrow \infty} \zeta_f(r).$$

If $\eta < \infty$, then

$$\rho_{f \circ g} \leq \eta\beta.$$

Proof For any $\varepsilon > 0$ and for sufficiently large r , from $\eta = \limsup_{r \rightarrow \infty} \zeta_f(r)$, we obtain

$$\zeta_f(r) < \eta + \varepsilon.$$

Now, by definition of order of $\rho_{f \circ g}$ and using Theorem 3.1, we get

$$\begin{aligned} \rho_{f \circ g} &= \limsup_{r \rightarrow \infty} \frac{\log v_{f \circ g}(r)}{\log r} \\ &\leq \limsup_{r \rightarrow \infty} \frac{\log v_f(\exp(r^\beta))}{\log r} \\ &\leq \limsup_{r \rightarrow \infty} \frac{\log v_f(\exp(r^\beta))}{\log \log(\exp(r^\beta))} \cdot \frac{\log \log(\exp(r^\beta))}{\log r} \\ &\leq \limsup_{r \rightarrow \infty} \zeta_f(\exp(r^\beta)) \cdot \frac{\log r^\beta}{\log r} \\ &\leq \limsup_{r \rightarrow \infty} (\eta + \varepsilon) \cdot \beta \\ &= (\eta + \varepsilon) \beta. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, therefore, we get

$$\rho_{f \circ g} \leq \eta\beta.$$

$\square$

In view of the Theorem 3.5, one can prove the following theorem; hence, we omit the proof.

Theorem 3.6 Suppose $f(z)$ and $g(z)$ are two transcendental entire functions of nonzero finite order with the condition $\beta > \rho_g$. Furthermore, let

$$\eta = \limsup_{r \rightarrow \infty} \zeta_f(r).$$

If $\eta < \infty$, then

$$\lambda_{f \circ g} \leq \eta\beta.$$

Theorem 3.7 Suppose $f(z)$ is an entire function of nonzero finite order and $g(z)$ is an entire function of nonzero finite lower order with the condition $0 < \alpha < \lambda_g$. Furthermore, let

$$\eta = \limsup_{r \rightarrow \infty} \zeta_f(r).$$

If $\eta < \infty$, then

$$\rho_{f \circ g} \geq \eta \alpha.$$

Proof By definition of order of $\rho_{f \circ g}$ and using Lemma 2.4, we get

$$\begin{aligned} \rho_{f \circ g} &= \limsup_{r \rightarrow \infty} \frac{\log v_{f \circ g}(r)}{\log r} \\ &\geq \limsup_{r \rightarrow \infty} \frac{\log v_f(\exp(r^\alpha))}{\log r} \\ &\geq \limsup_{r \rightarrow \infty} \frac{\log v_f(\exp(r^\alpha))}{\log \log(\exp(r^\alpha))} \cdot \frac{\log \log(\exp(r^\alpha))}{\log r} \\ &\geq \eta \alpha. \end{aligned}$$

Hence, the result follows. $\square$

Remark 3.8 In Theorem 3.7, if $\eta = \infty$, then $\rho_{f \circ g} = \infty$ for a positive constant α as stated in the above theorem.

In view of Theorems 3.5 and 3.7, the following result can be proved.

Theorem 3.9 Let $f(z)$ and $g(z)$ be two transcendental entire functions of nonzero finite order. Suppose there exist two constants α and β satisfying $0 < \alpha < \lambda_g \leq \rho_g < \beta$. Furthermore, let

$$\eta = \limsup_{r \rightarrow \infty} \zeta_f(r).$$

If $\eta < \infty$, then

$$\eta \alpha \leq \rho_{f \circ g} \leq \eta \beta.$$

Theorem 3.10 Let $f(z)$ be an entire function of nonzero finite order and $g(z)$ be an entire function of nonzero finite lower order. If there exists a positive constant α with the condition $0 < \alpha < \lambda_g$, then

$$\lambda_{f \circ g} = \infty.$$

Proof For any $\varepsilon > 0$ and for sufficiently large r , the lower order of composition of two entire functions in terms of central index is

$$\lambda_{f \circ g} = \liminf_{r \rightarrow \infty} \frac{\log v_{f \circ g}(r)}{\log r},$$

which implies that there exists an α , such that $0 < \alpha < \lambda_g$, and by Lemma 2.4, we have

$$\lambda_{f \circ g} \geq \liminf_{r \rightarrow \infty} \frac{\log v_f(\exp(r^\alpha))}{\log r}. \quad (3.10)$$

Again, by Definition 1.2, for any $\varepsilon > 0$ and for sufficiently large r , we have

$$\log v_f(r) > (\lambda_f - \varepsilon) \log r,$$

that is

$$\begin{aligned} \log v_f(\exp(r^\alpha)) &> (\lambda_f - \varepsilon) \log \exp(r^\alpha) \\ &> (\lambda_f - \varepsilon) r^\alpha. \end{aligned} \quad (3.11)$$

Using (3.11) in (3.10), we thus have

$$\lambda_{f \circ g} \geq \liminf_{r \rightarrow \infty} \frac{(\lambda_f - \varepsilon) r^\alpha}{\log r},$$

which implies that $\lambda_{f \circ g} = \infty$, since $\alpha > 0$ and $\frac{r^\alpha}{\log r} \rightarrow \infty$ as $r \rightarrow \infty$.

Therefore, the theorem is proved. $\square$

Theorem 3.11 *Let $f(z)$ be an entire function of nonzero finite lower order and $g(z)$ be an entire function of nonzero finite order. Furthermore, let*

$$\liminf_{r \rightarrow \infty} \frac{\log v_g(r)}{(\log r)^{\alpha_1}} = a > 0,$$

for any positive number α_1 . Then, $\lambda_{f \circ g} = \infty$.

Proof For any $\varepsilon > 0$ and for sufficiently large r , we have by given condition

$$\log v_g(r) \geq (a - \varepsilon) (\log r)^{\alpha_1}. \quad (3.12)$$

If there exists a constant α , such that $0 < \alpha < \lambda_g$, then by (3.12)

$$\begin{aligned} \log v_g(\exp(r^\alpha)) &\geq (a - \varepsilon) (\log \exp(r^\alpha))^{\alpha_1} \\ &= (a - \varepsilon) (r^\alpha)^{\alpha_1} = (a - \varepsilon) r^{\alpha\alpha_1}. \end{aligned} \quad (3.13)$$

Now, for any chosen $\varepsilon > 0$ and for sufficiently large r , by definition of the order of $\lambda_{f \circ g}$, we get

$$\begin{aligned} \lambda_{f \circ g} &= \liminf_{r \rightarrow \infty} \frac{\log v_{f \circ g}(r)}{\log r} \\ &\geq \liminf_{r \rightarrow \infty} \frac{\log v_g(\exp(r^\alpha))}{\log r}, \quad (\text{By using Lemma 2.5}) \\ &\geq \liminf_{r \rightarrow \infty} \frac{(a - \varepsilon) r^{\alpha\alpha_1}}{\log r}, \quad (\text{Using (3.13)}). \end{aligned} \quad (3.14)$$

Since for chosen $\varepsilon > 0$, $a - \varepsilon > 0$ and $\alpha\alpha_1 > 0$, hence $\frac{r^{\alpha\alpha_1}}{\log r} \rightarrow \infty$ as $r \rightarrow \infty$.

Therefore, by (3.14), we have $\lambda_{f \circ g} = \infty$ and the theorem is proved. $\square$

Remark 3.12 If $f(z)$ is an entire function of nonzero finite lower order and $g(z)$ is an entire function of nonzero finite order. Then, in view of the above theorem assuming

$$\limsup_{r \rightarrow \infty} \frac{\log v_g(r)}{(\log r)^{\alpha_1}} = c > 0,$$

for any positive number α_1 , we obtain $\rho_{f \circ g} = \infty$.

In view of Theorem 3.11, one can easily prove the following theorem.

Theorem 3.13 *Let $f(z)$ be an entire function of nonzero finite order and $g(z)$ be an entire function of nonzero finite lower order. Furthermore, let*

$$\liminf_{r \rightarrow \infty} \frac{\log v_f(r)}{(\log r)^{\alpha_1}} = a > 0,$$

for any positive number α_1 . Then, $\lambda_{f \circ g} = \infty$.

Theorem 3.14 *Let $f(z)$ be an entire function of nonzero finite order and $g(z)$ be an entire function of nonzero finite lower order. Furthermore, let*

$$\liminf_{r \rightarrow \infty} \frac{\zeta_f(r)}{(\log r)^{\beta_1}} = b > 0,$$

for any positive number β_1 . Then, $\lambda_{f \circ g} = \infty$.

Proof For any chosen $\varepsilon > 0$ and for sufficiently large r , we have from the given condition

$$\zeta_f(r) \geq (b - \varepsilon) (\log r)^{\beta_1}. \quad (3.15)$$

If there exists a constant α , such that $0 < \alpha < \lambda_g$, then by (3.15)

$$\begin{aligned} \zeta_f(\exp(r^\alpha)) &\geq (b - \varepsilon) (\log \exp(r^\alpha))^{\beta_1} \\ &= (b - \varepsilon) (r^\alpha)^{\beta_1} = (b - \varepsilon) r^{\alpha\beta_1}. \end{aligned} \quad (3.16)$$

By definition of the order of $\lambda_{f \circ g}$, we get for sufficiently large r

$$\begin{aligned}\lambda_{f \circ g} &= \liminf_{r \rightarrow \infty} \frac{\log v_{f \circ g}(r)}{\log r} \\ &\geq \liminf_{r \rightarrow \infty} \frac{\log v_f(\exp(r^\alpha))}{\log r}, \quad (\text{By using Lemma 2.4}) \\ &\geq \liminf_{r \rightarrow \infty} \frac{\log v_f(\exp(r^\alpha))}{\log \log(\exp(r^\alpha))} \cdot \frac{\log \log(\exp(r^\alpha))}{\log r} \\ &\geq \liminf_{r \rightarrow \infty} \frac{\log v_f(\exp(r^\alpha))}{\log \log(\exp(r^\alpha))} \cdot \liminf_{r \rightarrow \infty} \frac{\log r^\alpha}{\log r} \\ &\geq \alpha \liminf_{r \rightarrow \infty} \zeta_f(\exp(r^\alpha)) \\ &\geq \alpha(b - \varepsilon)r^{\alpha\beta_1}, \quad (\text{By using (3.16)}).\end{aligned}$$

Since for chosen $\varepsilon > 0$, $b - \varepsilon > 0$ and $\alpha\beta_1 > 0$, hence $\alpha(b - \varepsilon)r^{\alpha\beta_1} \rightarrow \infty$ as $r \rightarrow \infty$.

Therefore, $\lambda_{f \circ g} = \infty$ and the theorem is proved. $\square$

The following theorem can be proved in view of Theorem 3.14.

Theorem 3.15 Let $f(z)$ be an entire function of nonzero finite lower order and $g(z)$ be an entire function of nonzero finite order. Furthermore, let

$$\liminf_{r \rightarrow \infty} \frac{\zeta_g(r)}{(\log r)^{\beta_1}} = b > 0,$$

for any positive number β_1 . Then, $\lambda_{f \circ g} = \infty$.

Theorem 3.16 Let $f(z)$ be an entire function of nonzero finite lower order and $g(z)$ be an entire function of nonzero finite order. Furthermore, let

$$\liminf_{r \rightarrow \infty} \frac{\log_k v_g(r)}{(\log_k r)^{\alpha_1}} = a > 0,$$

for any positive integer $k \geq 1$ and α_1 is any positive number. Then, $\lambda_{f \circ g} = \infty$.

Proof For any $\varepsilon > 0$ and for sufficiently large r , by given condition, we have

$$\log_k v_g(r) \geq (a - \varepsilon)(\log_k r)^{\alpha_1},$$

that is

$$\log v_g(r) \geq \exp_{k-1}[(a - \varepsilon)(\log_k r)^{\alpha_1}]. \quad (3.17)$$

If there exists a constant α , such that $0 < \alpha < \lambda_g$, then by (3.17)

$$\begin{aligned}\log v_g(\exp(r^\alpha)) &\geq \exp_{k-1}[(a - \varepsilon)(\log_k \exp(r^\alpha))^{\alpha_1}] \\ &= \exp_{k-1}[(a - \varepsilon)(\log_{k-1}(r^\alpha))^{\alpha_1}].\end{aligned} \quad (3.18)$$

Now, for any chosen $\varepsilon > 0$ and for sufficiently large r , by definition of the order of $\lambda_{f \circ g}$, we get

$$\begin{aligned}\lambda_{f \circ g} &= \liminf_{r \rightarrow \infty} \frac{\log v_{f \circ g}(r)}{\log r} \\ &\geq \liminf_{r \rightarrow \infty} \frac{\log v_g(\exp(r^\alpha))}{\log r}, \quad (\text{By using Lemma 2.5}) \\ &\geq \liminf_{r \rightarrow \infty} \frac{\exp_{k-1}[(a - \varepsilon)(\log_{k-1}(r^\alpha))^{\alpha_1}]}{\log r}, \quad (\text{Using (3.18)}).\end{aligned} \quad (3.19)$$

Since for chosen $\varepsilon > 0$, $a - \varepsilon > 0$ and $\alpha\alpha_1 > 0$, hence $\frac{\exp_{k-1}[(a - \varepsilon)(\log_{k-1}(r^\alpha))^{\alpha_1}]}{\log r} \rightarrow \infty$ as $r \rightarrow \infty$.

Therefore, by (3.19), we have $\lambda_{f \circ g} = \infty$ and the theorem is proved. $\square$

Remark 3.17 If $f(z)$ is an entire function of nonzero finite lower order and $g(z)$ is an entire function of nonzero finite order. Then, assuming

$$\limsup_{r \rightarrow \infty} \frac{\log_k v_g(r)}{(\log_k r)^{\alpha_1}} = c_1 > 0,$$

for any positive integer $k \geq 1$ and $\alpha_1 > 0$, we can get $\rho_{f \circ g} = \infty$.

In view of Theorem 3.16, one can easily prove the following theorem, and hence, its proof is omitted.

Theorem 3.18 Let $f(z)$ be an entire function of nonzero finite order and $g(z)$ be an entire function of nonzero finite lower order. Furthermore, let

$$\liminf_{r \rightarrow \infty} \frac{\log_k v_f(r)}{(\log_k r)^{\alpha_1}} = a > 0,$$

for any positive integer $k \geq 1$ and α_1 is any positive number. Then, $\lambda_{f \circ g} = \infty$.

Theorem 3.19 Let $f(z)$ be an entire function of nonzero finite order and $g(z)$ be an entire function of nonzero finite lower order. Furthermore, let

$$\liminf_{r \rightarrow \infty} \frac{\log_{k-1} \zeta_f(r)}{(\log_k r)^{\beta_1}} = b > 0,$$

for any positive integer $k \geq 1$ and β_1 is any positive number. Then, $\lambda_{f \circ g} = \infty$.

Proof For any chosen $\varepsilon > 0$ and for sufficiently large r , we have from the given condition

$$\log_{k-1} \zeta_f(r) \geq (b - \varepsilon) (\log_k r)^{\beta_1},$$

that is

$$\zeta_f(r) \geq \exp_{k-1} \left[(b - \varepsilon) (\log_k r)^{\beta_1} \right]. \quad (3.20)$$

If there exists a constant α , such that $0 < \alpha < \lambda_g$, then by (3.20), we have

$$\begin{aligned} \zeta_f(\exp(r^\alpha)) &\geq \exp_{k-1} \left[(b - \varepsilon) (\log_k \exp(r^\alpha))^{\beta_1} \right] \\ &= \exp_{k-1} \left[(b - \varepsilon) (\log_{k-1}(r^\alpha))^{\beta_1} \right]. \end{aligned} \quad (3.21)$$

By definition of the order of $\lambda_{f \circ g}$, we get for sufficiently large r

$$\begin{aligned} \lambda_{f \circ g} &= \liminf_{r \rightarrow \infty} \frac{\log v_{f \circ g}(r)}{\log r} \\ &\geq \liminf_{r \rightarrow \infty} \frac{\log v_f(\exp(r^\alpha))}{\log r}, \quad (\text{By using Lemma 2.4}) \\ &\geq \liminf_{r \rightarrow \infty} \frac{\log v_f(\exp(r^\alpha))}{\log \log(\exp(r^\alpha))} \cdot \frac{\log \log(\exp(r^\alpha))}{\log r} \\ &\geq \liminf_{r \rightarrow \infty} \frac{\log v_f(\exp(r^\alpha))}{\log \log(\exp(r^\alpha))} \cdot \liminf_{r \rightarrow \infty} \frac{\log r^\alpha}{\log r} \\ &\geq \alpha \liminf_{r \rightarrow \infty} \zeta_f(\exp(r^\alpha)) \\ &\geq \alpha \liminf_{r \rightarrow \infty} \exp_{k-1} \left[(b - \varepsilon) (\log_{k-1}(r^\alpha))^{\beta_1} \right], \quad (\text{By using (3.21)}). \end{aligned}$$

Since for chosen $\varepsilon > 0$, $b - \varepsilon > 0$ and $\alpha \beta_1 > 0$, hence $\alpha \exp_{k-1} \left[(b - \varepsilon) (\log_{k-1}(r^\alpha))^{\beta_1} \right] \rightarrow \infty$ as $r \rightarrow \infty$.

Therefore, $\lambda_{f \circ g} = \infty$ and the theorem is proved. $\square$

Remark 3.20 If $f(z)$ is an entire function of nonzero finite order and $g(z)$ is an entire function of nonzero finite lower order. Then, assuming

$$\limsup_{r \rightarrow \infty} \frac{\log_{k-1} \zeta_f(r)}{(\log_k r)^{\beta_1}} = d > 0,$$

for any positive integer $k \geq 1$ and $\beta_1 > 0$, we can get $\rho_{f \circ g} = \infty$.

The following theorem can be proved in view of Theorem 3.19, and hence, its proof is omitted.

Theorem 3.21 Let $f(z)$ be an entire function of nonzero finite lower order and $g(z)$ be an entire function of nonzero finite order. Furthermore, let

$$\liminf_{r \rightarrow \infty} \frac{\log_{k-1} \zeta_g(r)}{(\log_k r)^{\beta_1}} = b > 0,$$

for any positive integer $k \geq 1$ and β_1 is any positive number. Then, $\lambda_{f \circ g} = \infty$.

Theorem 3.22 Let $f(z)$ be an entire function of nonzero finite lower order and $g(z)$ be an entire function of nonzero finite order. Then, $\lambda_{f \circ g} = \infty$ if one of the following conditions (I) and (II) is satisfied:

$$(I) \quad \liminf_{r \rightarrow \infty} \frac{\log_k v_g(r)}{(\log_k r)^{\alpha_1}} = a_1 < \infty;$$

$$(II) \quad \liminf_{r \rightarrow \infty} \frac{\log_k v_g(r)}{(\log_{k+1} r)^{\alpha_2}} = a_2 < \infty,$$

for any two positive numbers $\alpha_1, \alpha_2 (> 1)$ and k a positive integer.

Proof Case (I) : In view of Theorem 3.16, first part of the theorem can easily be proved.

Case (II) : Using the same method as in (3.18), for sufficiently large r and for any chosen $\varepsilon > 0$, we get from condition (II)

$$\log v_g(\exp(r^\alpha)) \geq \exp_{k-1}[(a_2 - \varepsilon)(\log_k(r^\alpha))^{\alpha_2}]. \quad (3.22)$$

By definition of the lower order of $\lambda_{f \circ g}$, we get for any chosen $\varepsilon > 0$ and for sufficiently large r

$$\begin{aligned} \lambda_{f \circ g} &= \liminf_{r \rightarrow \infty} \frac{\log v_{f \circ g}(r)}{\log r} \\ &\geq \liminf_{r \rightarrow \infty} \frac{\log v_g(\exp(r^\alpha))}{\log r}, \quad (\text{By using Lemma 2.5}) \\ &\geq \liminf_{r \rightarrow \infty} \frac{\exp_{k-1}[(a_2 - \varepsilon)(\log_k(r^\alpha))^{\alpha_2}]}{\log r}, \quad (\text{By using (3.22)}). \end{aligned} \quad (3.23)$$

Since for chosen $\varepsilon > 0$, $a_2 - \varepsilon > 0$ and $\alpha_2 - 1 > 0$, hence $\frac{\exp_{k-1}[(a_2 - \varepsilon)(\log_k(r^\alpha))^{\alpha_2}]}{\log r} \rightarrow \infty$ as $r \rightarrow \infty$.

Therefore, by (3.23), we have $\lambda_{f \circ g} = \infty$ and second part of the theorem is proved. $\square$

The following theorem can be proved in view of Theorem 3.22, and hence, its proof is omitted.

Theorem 3.23 Let $f(z)$ be an entire function of nonzero finite order and $g(z)$ be an entire function of nonzero finite lower order. Then, $\lambda_{f \circ g} = \infty$ if one of the following conditions (I) and (II) is satisfied:

$$(I) \quad \liminf_{r \rightarrow \infty} \frac{\log_k v_f(r)}{(\log_k r)^{\alpha_1}} = a_1 < \infty;$$

$$(II) \quad \liminf_{r \rightarrow \infty} \frac{\log_k v_f(r)}{(\log_{k+1} r)^{\alpha_2}} = a_2 < \infty,$$

for any two positive numbers $\alpha_1, \alpha_2 (> 1)$ and k a positive integer.

Theorem 3.24 Let $f(z)$ be an entire function of nonzero finite order and $g(z)$ be an entire function of nonzero finite lower order. Then, $\lambda_{f \circ g} = \infty$, if one of the following conditions (I) and (II) is satisfied :

$$(I) \quad \liminf_{r \rightarrow \infty} \frac{\log_{k-1} \zeta_f(r)}{(\log_k r)^{\beta_1}} = b_1 < \infty;$$

$$(II) \quad \liminf_{r \rightarrow \infty} \frac{\log_k \zeta_f(r)}{(\log_k r)^{\beta_2}} = b_2 < \infty,$$

for any two positive numbers $\beta_1, \beta_2 (> 1)$ and k a positive integer.

Proof Case (I) : In view of Theorem 3.19, first part of the theorem can easily be proved.

Case (II) : Using the same method as in (3.20), for sufficiently large r and for any chosen $\varepsilon > 0$, we get from condition (II)

$$\zeta_f(r) \geq \exp_k \left[(b_2 - \varepsilon) (\log_k r)^{\beta_2} \right]. \quad (3.24)$$

If there exists a constant α such that $0 < \alpha < \lambda_g$, then by (3.24), we have

$$\zeta_f(\exp(r^\alpha)) \geq \exp_k \left[(b_2 - \varepsilon) (\log_{k-1}(r^\alpha))^{\beta_2} \right]. \quad (3.25)$$

By definition of the lower order of $\lambda_{f \circ g}$, we get for sufficiently large r

$$\begin{aligned} \lambda_{f \circ g} &= \liminf_{r \rightarrow \infty} \frac{\log v_{f \circ g}(r)}{\log r} \\ &\geq \liminf_{r \rightarrow \infty} \frac{\log v_f(\exp(r^\alpha))}{\log r}, \quad (\text{By using Lemma 2.4}) \\ &\geq \liminf_{r \rightarrow \infty} \frac{\log v_f(\exp(r^\alpha))}{\log \log(\exp(r^\alpha))} \cdot \frac{\log \log(\exp(r^\alpha))}{\log r} \\ &\geq \liminf_{r \rightarrow \infty} \frac{\log v_f(\exp(r^\alpha))}{\log \log(\exp(r^\alpha))} \cdot \liminf_{r \rightarrow \infty} \frac{\log r^\alpha}{\log r} \\ &\geq \alpha \liminf_{r \rightarrow \infty} \zeta_f(\exp(r^\alpha)) \\ &\geq \alpha \liminf_{r \rightarrow \infty} \exp_k \left[(b_2 - \varepsilon) (\log_{k-1}(r^\alpha))^{\beta_2} \right], \quad (\text{By using (3.25)}) \end{aligned}$$

Since for some chosen $\varepsilon > 0$, $b_2 - \varepsilon > 0$ and $\alpha\beta_2 > 0$, hence $\exp_k \left[(b_2 - \varepsilon) (\log_{k-1}(r^\alpha))^{\beta_2} \right] \rightarrow \infty$ as $r \rightarrow \infty$.

Therefore, $\lambda_{f \circ g} = \infty$ and the second part of the theorem is proved. $\square$

Remark 3.25 If $f(z)$ be an entire function of nonzero finite order and $g(z)$ be an entire function of nonzero finite lower order. Then, $\rho_{f \circ g} = \infty$, if we consider one of the following two conditions:

$$\begin{aligned} (I) \quad &\limsup_{r \rightarrow \infty} \frac{\log_{k-1} \zeta_f(r)}{(\log_k r)^{\beta_1}} = d_1 < \infty; \\ (II) \quad &\limsup_{r \rightarrow \infty} \frac{\log_k \zeta_f(r)}{(\log_k r)^{\beta_2}} = d_2 < \infty, \end{aligned}$$

for any two positive numbers $\beta_1, \beta_2 (> 1)$ and k a positive integer.

In view of Theorem 3.24, one can easily prove the following theorem.

Theorem 3.26 Let $f(z)$ be an entire function of nonzero finite lower order and $g(z)$ be an entire function of nonzero finite order. Then, $\lambda_{f \circ g} = \infty$ if one of the following conditions (I) and (II) is satisfied :

$$\begin{aligned} (I) \quad &\liminf_{r \rightarrow \infty} \frac{\log_{k-1} \zeta_g(r)}{(\log_k r)^{\beta_1}} = b_1 < \infty; \\ (II) \quad &\liminf_{r \rightarrow \infty} \frac{\log_k \zeta_g(r)}{(\log_k r)^{\beta_2}} = b_2 < \infty, \end{aligned}$$

for any two positive numbers $\beta_1, \beta_2 (> 1)$ and k a positive integer.

Next, two theorems are about ratios $\frac{v_{f \circ g}(r)}{v_f(r)}$ and $\frac{v_{f \circ g}(r)}{v_g(r)}$.

Theorem 3.27 Let $f(z)$ and $g(z)$ be two entire functions of nonzero finite order along with the condition $\lambda_g \geq \rho_f$. Then

$$\lim_{r \rightarrow \infty} \frac{v_{f \circ g}(r)}{v_f(r)} = \infty.$$

Proof By Definition 1.2, for any $\varepsilon > 0$ and for sufficiently large r , we get

$$\log v_f(r) < (\rho_f + \varepsilon) \log r,$$

that is

$$v_f(r) < r^{(\rho_f + \varepsilon)}. \quad (3.26)$$

Again, if there exists an α such that $0 < \alpha < \lambda_g$, then by Lemma 2.5, we have

$$v_{f \circ g}(r) > v_g(\exp(r^\alpha)).$$

By Definition 1.2

$$v_{f \circ g}(r) > (\exp(r^\alpha))^{(\lambda_g - \varepsilon)}. \quad (3.27)$$

In view of (3.26) and (3.27), we obtain

$$\frac{v_{f \circ g}(r)}{v_f(r)} > \frac{(\exp(r^\alpha))^{(\lambda_g - \varepsilon)}}{r^{(\rho_f + \varepsilon)}}.$$

Since $\lambda_g \geq \rho_f$, so for some chosen $\varepsilon > 0$, $\lambda_g - \varepsilon \geq \rho_f + \varepsilon$ and $\exp(r^\alpha) > r$ for given α .

Hence, the ratio $\frac{(\exp(r^\alpha))^{(\lambda_g - \varepsilon)}}{r^{(\rho_f + \varepsilon)}} \rightarrow \infty$ as $r \rightarrow \infty$ and the theorem is proved. $\square$

Example 3.28 If we consider two nonzero entire functions $f(z) = \cos(b\sqrt{z})$ ($b \neq 0$) and $g(z) = \exp(az)$ ($a \neq 0$), then $\rho_f = \frac{1}{2}$ and $\lambda_g = 1$ and therefore $\lambda_g \geq \rho_f$.

Since, $0 < \alpha < \lambda_g$, take $\alpha = \frac{1}{2}$, so that for $f \circ g(z) = \cos(b\sqrt{\exp(az)})$ ($a, b \neq 0$) using Lemma 2.5, we get $v_{f \circ g}(r) > \exp(\sqrt{r})$.

Therefore, along with Definition 1.2, we obtain $\frac{v_{f \circ g}(r)}{v_f(r)} > \frac{\exp(\sqrt{r})}{\sqrt{r}}$ and the right-hand side tends to ∞ as $r \rightarrow \infty$. Hence

$$\lim_{r \rightarrow \infty} \frac{v_{f \circ g}(r)}{v_f(r)} = \infty.$$

Theorem 3.29 Let $f(z)$ and $g(z)$ be two entire functions of nonzero finite order along with the condition $\lambda_f \geq \rho_g$. Then

$$\lim_{r \rightarrow \infty} \frac{v_{f \circ g}(r)}{v_g(r)} = \infty.$$

Proof By Definition 1.2, for any $\varepsilon > 0$ and for sufficiently large r , we get

$$\log v_g(r) < (\rho_g + \varepsilon) \log r,$$

that is

$$v_g(r) < r^{(\rho_g + \varepsilon)}. \quad (3.28)$$

Again, if there exists an α , such that $0 < \alpha < \lambda_g$, then by Lemma 2.4, we have

$$v_{f \circ g}(r) > v_f(\exp(r^\alpha)).$$

By Definition 1.2

$$v_{f \circ g}(r) > (\exp(r^\alpha))^{(\lambda_f - \varepsilon)}. \quad (3.29)$$

In view of (3.28) and (3.29), we obtain

$$\frac{v_{f \circ g}(r)}{v_g(r)} > \frac{(\exp(r^\alpha))^{(\lambda_f - \varepsilon)}}{r^{(\rho_g + \varepsilon)}}.$$

Since $\lambda_f \geq \rho_g$, so for some chosen $\varepsilon > 0$, $\lambda_f - \varepsilon \geq \rho_g + \varepsilon$ and $\exp(r^\alpha) > r$ for given α .

Hence, the ratio $\frac{(\exp(r^\alpha))^{(\lambda_f - \varepsilon)}}{r^{(\rho_g + \varepsilon)}} \rightarrow \infty$ as $r \rightarrow \infty$ and the theorem is proved. $\square$

Example 3.30 Consider two nonzero entire functions $f(z) = \cosh(z)$ and $g(z) = \cos(a\sqrt{z})$ ($a \neq 0$), then $\lambda_f = 1$ and $\rho_g = \lambda_g = \frac{1}{2}$, and therefore, $\lambda_f \geq \rho_g$.

Since, $0 < \alpha < \lambda_g$, take $\alpha = \frac{1}{4}$, so that for $f \circ g(z) = \cosh(\cos(a\sqrt{z}))$ ($a \neq 0$) using Lemma 2.4, we get $v_{f \circ g}(r) > \exp(\sqrt[4]{r})$.

Therefore, along with Definition 1.2, we obtain the ratio $\frac{v_{f \circ g}(r)}{v_g(r)} > \frac{\exp(\sqrt[4]{r})}{\sqrt{r}}$ and the right-hand side tends to ∞ as $r \rightarrow \infty$. Hence

$$\lim_{r \rightarrow \infty} \frac{v_{f \circ g}(r)}{v_g(r)} = \infty.$$

In the next theorem, we are discussing about the behaviour of the ratio $\frac{\log v_{f \circ g}(r)}{v_f(r)}$.

Theorem 3.31 Let $f(z)$ be an entire function of nonzero finite order and $g(z)$ be an entire function of nonzero finite lower order along with the condition $0 < \rho_f < \alpha < \lambda_g < \infty$. Then

$$\lim_{r \rightarrow \infty} \frac{\log v_{f \circ g}(r)}{v_f(r)} = \infty.$$

Proof By (3.26), for any $\varepsilon > 0$ and for sufficiently large r , we get

$$v_f(r) < r^{(\rho_f + \varepsilon)}. \quad (3.30)$$

Again, if there exists an α , such that $0 < \alpha < \lambda_g$, then by Lemma 2.4, we have

$$v_{f \circ g}(r) > v_f(\exp(r^\alpha)).$$

By Definition 1.2

$$\begin{aligned} \log v_{f \circ g}(r) &> (\lambda_f - \varepsilon) \log(\exp(r^\alpha)) \\ &> (\lambda_f - \varepsilon) r^\alpha. \end{aligned} \quad (3.31)$$

In view of (3.30) and (3.31), we obtain

$$\frac{\log v_{f \circ g}(r)}{v_f(r)} > \frac{(\lambda_f - \varepsilon) r^\alpha}{r^{(\rho_f + \varepsilon)}}.$$

Since for some $\varepsilon > 0$, $\alpha > \rho_f$ implies $\alpha > \rho_f + \varepsilon$, the ratio $\frac{r^\alpha}{r^{(\rho_f + \varepsilon)}} \rightarrow \infty$ as $r \rightarrow \infty$.

Hence, the theorem follows. $\square$

Example 3.32 Consider two nonzero entire functions $f(z) = \exp(az)$ ($a \neq 0$) and $g(z) = \exp(bz^2)$ ($b \neq 0$), then $\rho_f = \lambda_f = 1$ and $\lambda_g = 2$.

Since, $0 < \rho_f < \alpha < \lambda_g < \infty$, take $\alpha = \frac{3}{2}$, so that for $f \circ g(z) = \exp(a \exp(bz^2))$ ($a, b \neq 0$) using Lemma 2.4, we get $\log v_{f \circ g}(r) > r^{\frac{3}{2}}$.

Therefore, along with Definition 1.2, we obtain the ratio $\frac{\log v_{f \circ g}(r)}{v_f(r)} > \frac{r^{\frac{3}{2}}}{r} = r^{\frac{1}{2}}$ and the right-hand side tends to ∞ as $r \rightarrow \infty$. Hence

$$\lim_{r \rightarrow \infty} \frac{\log v_{f \circ g}(r)}{v_f(r)} = \infty.$$

Theorem 3.33 Let $f(z)$ be an entire function of nonzero finite order and $g(z)$ be an entire function of nonzero finite lower order. Then

$$\lim_{r \rightarrow \infty} \frac{\log v_{f \circ g}(r)}{\log v_f(r)} = \infty.$$

Proof By Definition 1.2, for any $\varepsilon > 0$ and for sufficiently large r , we get

$$\log v_f(r) < (\rho_f + \varepsilon) \log r. \quad (3.32)$$

Again, if there exists an α , such that $0 < \alpha < \lambda_g$, then by (3.31), we have

$$\log v_{f \circ g} > (\lambda_f - \varepsilon) r^\alpha. \quad (3.33)$$

In view of (3.32) and (3.33), we obtain

$$\frac{\log v_{f \circ g}(r)}{\log v_f(r)} > \frac{(\lambda_f - \varepsilon) r^\alpha}{(\rho_f + \varepsilon) \log r}.$$

Since $\alpha > 0$, the ratio $\frac{r^\alpha}{\log r} \rightarrow \infty$ as $r \rightarrow \infty$.

Hence, the theorem follows. $\square$

Example 3.34 For two nonzero entire functions $f(z) = \cosh(z)$ and $g(z) = \exp(az)$ ($a \neq 0$), we have $\rho_f = \lambda_f = 1$ and $\lambda_g = 1$.

Since, $0 < \alpha < \lambda_g$, take $\alpha = \frac{1}{2}$, so that for $f \circ g(z) = \cosh(\exp(az))$ ($a \neq 0$) using Lemma 2.4, we get $\log v_{f \circ g}(r) > r^{\frac{1}{2}}$.

Therefore, using Definition 1.2, we obtain the ratio $\frac{\log v_{f \circ g}(r)}{\log v_f(r)} > \frac{r^{\frac{1}{2}}}{\log r}$ and the right-hand side tends to ∞ as $r \rightarrow \infty$. Hence

$$\lim_{r \rightarrow \infty} \frac{\log v_{f \circ g}(r)}{\log v_f(r)} = \infty.$$

Theorem 3.35 Let $f(z)$ be an entire function of nonzero finite lower order and $g(z)$ be an entire function of nonzero finite order. Then

$$\lim_{r \rightarrow \infty} \frac{\log v_{f \circ g}(r)}{\log v_g(r)} = \infty.$$

Proof By Definition 1.2, for any $\varepsilon > 0$ and for sufficiently large r , we get

$$\log v_g(r) < (\rho_g + \varepsilon) \log r. \quad (3.34)$$

Again, if there exists an α , such that $0 < \alpha < \lambda_g$, then by Lemma 2.5, we have

$$v_{f \circ g} > v_g(\exp(r^\alpha)).$$

By Definition 1.2

$$\begin{aligned} \log v_{f \circ g} &> (\lambda_g - \varepsilon) \log(\exp(r^\alpha)) \\ &> (\lambda_g - \varepsilon) r^\alpha. \end{aligned} \quad (3.35)$$

In view of (3.34) and (3.35), we obtain

$$\frac{\log v_{f \circ g}(r)}{\log v_g(r)} > \frac{(\lambda_g - \varepsilon) r^\alpha}{(\rho_g + \varepsilon) \log r}.$$

Since $\alpha > 0$, the ratio $\frac{r^\alpha}{\log r} \rightarrow \infty$ as $r \rightarrow \infty$.

Hence, the theorem is proved. $\square$

Example 3.36 For two nonzero entire functions $f(z) = \cos(a\sqrt{z})$ ($a \neq 0$) and $g(z) = \sin z$, we have $\rho_f = \lambda_f = \frac{1}{2}$ and $\rho_g = \lambda_g = 1$.

Since, $0 < \alpha < \lambda_g$, take $\alpha = \frac{1}{3}$, so that for $f \circ g(z) = \cos(a\sqrt{\sin z})$ using Lemma 2.5, we get $\log v_{f \circ g}(r) > r^{\frac{1}{3}}$.

Therefore, using Definition 1.2, we obtain the ratio $\frac{\log v_{f \circ g}(r)}{\log v_g(r)} > \frac{r^{\frac{1}{3}}}{\log r}$ and the right-hand side tends to ∞ as $r \rightarrow \infty$. Hence

$$\lim_{r \rightarrow \infty} \frac{\log v_{f \circ g}(r)}{\log v_g(r)} = \infty.$$

Remark 3.37 The conclusions of the Theorems 3.27, 3.29, 3.31, 3.33, 3.35 remain valid with “limit superior” in place of “limit”.

4 Conclusion

We have proved several results of growth of composite entire functions in terms of their central indexes. There is huge scope of further study on the growth of entire functions by different growth measures in terms of their central indexes. The growth results of entire functions with respect to their generalized order may be investigated using their central indexes. Also, the relative order of an entire function with respect to another entire function may be defined alternatively in terms of their central indexes and the corresponding results may be investigated. Once we get these, we may implement them to the composite entire functions. Furthermore, some of these results may be applied to complex linear differential equations.

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