K. Pushpa · K. R. Vasuki

On Eisenstein series, color partition and divisor function

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Abstract Ramanujan recorded Eisenstein series identities of level 5 of weight 2. The objective of this article is to prove these identities by classical method and also we find some new Eisenstein series identities for level 7. Finally we make use of these results to obtain convolution sums of divisor functions. We will also be introducing certain restricted color partition functions which helps us to represent the convolution sums in a much elegant form than the existing ones.

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1 Introduction

The smallest and the last chapter of the organized portion of the Ramanujan's notebooks is dedicated to Eisenstein series and deducing identities of the form $P_1 - nP_n =$ linear combinations of ratios of theta functions. Ramanujan documented identities of level 3, 5, 7, 9, 11, 17, 19, 23, 25, 31 and 35. The Chapter 21 of the Ramanujan's notebooks Part III edited by Berndt focuses on proofs of the equalities connecting a certain linear combinations of Eisenstein series with theta functions. Cooper and Toh [9] have proved results for $n = 5, 7$. Cooper [7] proved certain Eisenstein series identities for level 5 using k -parametrization. Cooper and Ye [11] proved some Eisenstein series identities for level 7 using the theory of modular forms. In Sect. 4 of this article, we establish certain new Eisenstein series identities for level 5 of the above type by making use of Bailey's formula which can be deduced from his very-well known ${}_6\psi_6$ summation formula:

$$\sum_{n=-\infty}^{\infty} \left[\frac{aq^{10n}}{(1-aq^{10n})^2} - \frac{bq^{10n}}{(1-bq^{10n})^2} \right] = af_6^6 \frac{f\left(-ab, \frac{-q^{10}}{ab}\right) f\left(\frac{-b}{a}, \frac{-aq^{10}}{b}\right)}{f^2\left(-a, \frac{-q^{10}}{a}\right) f^2\left(-b, \frac{-q^{10}}{b}\right)}, \quad (1.1)$$

where $f(a, b)$ and f_n are as defined below in Sect. 2.

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Let $\mathbb{N}$ denote the set of all natural numbers. As usual let

$$\sigma_k(n) := \sum_{d|n} d^k, \quad n, d, k \in \mathbb{N} \quad \text{and} \quad \sigma_k(n) = 0 \quad \forall \quad n \notin \mathbb{N}.$$

For convenience, we set $\sigma_1(n) = \sigma(n)$. We define the convolution sum $W_k(n)$ by

$$W_k(n) := \sum_{m < \frac{n}{k}} \sigma(m) \sigma(n - km) \quad \text{and} \quad W_{(a,b)}(n) := \sum_{\substack{l, m \\ al + bm = n}} \sigma(l) \sigma(m),$$

where $a, b, k, n \in \mathbb{N}$. Note that $W_{(1,k)}(n) = W_{(k,1)}(n) = W_k(n)$. The sum $W_k(n)$ have been evaluated for some k 's ≤ 64 , for all $n \in \mathbb{N}$. For a brief history and development of the this field, one may refer [17]. Most of the convolution sums are evaluated using arithmetic or analytic methods and some use theory of quasi-modular forms. The convolution sums for level 5, 7, 10 and 14 have a constant term. In Sect. 5 of this paper, our object is to deduce convolution sums for level 5, 7, 10 and 14 entirely in terms of divisor functions and partition functions by using the identities obtained in the previous sections and to prove certain fascinating congruence properties of these partition functions.

2 Preliminary results

As usual for any complex number a and q with $|q| < 1$, we define

$$(a; q)_{\infty} = \prod_{n=0}^{\infty} (1 - aq^n)$$

and the q shifted factorial

$$(a; q)_n = \frac{(a; q)_{\infty}}{(aq^n; q)_{\infty}},$$

where n is any integer.

On page 197 of his second notebook [19], Ramanujan defined his general theta-function as

$$f(a, b) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} = (-a; ab)_{\infty} (-b; ab)_{\infty} (ab; ab)_{\infty}, \quad |ab| < 1.$$

Further Ramanujan also defined three special cases of $f(a, b)$ namely,

$$\varphi(q) = f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q; q^2)_{\infty}^2 (q^2; q^2)_{\infty}, \quad \psi(q) = f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}},$$

and

$$f(-q) = f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_{\infty}.$$

He also defined

$$\chi(-q) = (q; q^2)_{\infty}.$$

For $q = e^{2\pi i \tau}$, $\text{Im}(\tau) > 0$, the classical Dedekind eta function $\eta(\tau)$ is defined by

$$\eta(\tau) = q^{\frac{1}{24}} (q; q)_{\infty} = q^{\frac{1}{24}} f(-q).$$

For any positive integer n , throughout this article we set

$$f_n = f(-q^n), \quad A_n = \frac{f_n}{q^{\frac{n}{24}} f_{2n}}, \quad B_n = \frac{f_n}{q^{\frac{n}{6}} f_{5n}}, \quad \text{and} \quad C_n = \frac{f_n}{q^{\frac{n}{4}} f_{7n}}.$$

Further for convenience, we set

$$\begin{aligned} u &= \frac{1}{B_1 B_2}, \quad v = \frac{1}{A_1 A_5}, \quad w = \frac{B_1}{B_2}, \quad x = \frac{1}{u} + 5u, \\ U &= \frac{1}{C_1 C_2}, \quad V = \frac{1}{A_1 A_7}, \quad W = \frac{C_1}{C_2} \quad \text{and} \quad y = \frac{1}{V} + 2V. \end{aligned} \quad (2.1)$$

The following u (resp. U), v (resp. V) and w (resp. W) relations are the important tools to prove our main results:

$$\frac{1}{u} + 5u = \frac{1}{w^3} + w^3, \quad (2.2)$$

$$\frac{1}{u} - 5u = \frac{1}{v^2} - 4v^2, \quad (2.3)$$

$$\frac{1}{v^2} + 4v^2 = \frac{1}{w^3} - w^3, \quad (2.4)$$

$$\frac{1}{U^2} + 49U^2 = \frac{1}{W^6} - \frac{8}{W^2} - 8W^2 + W^6, \quad (2.5)$$

$$\frac{1}{W^4} + W^4 = \frac{1}{V^3} + 8V^3 + 7, \quad (2.6)$$

and

$$\frac{1}{U^4} + 7^4 U^4 = \frac{1}{V^9} + (2V)^9 + \frac{5}{V^6} + 5(2V)^6 - \frac{8}{V^3} - 8(2V)^3 + 18. \quad (2.7)$$

Ramanujan recorded (2.2), (2.3) and (2.5) in the unorganized portion of his second notebook [19, p. 325 and 327] and popularly known as the P – Q theta function identities. He recorded (2.4) and (2.6) in the modular equation form [19, p. 237 and 240]. The identities (2.2), (2.3) and (2.5) are proved by Berndt and Zhang [4] using Ramanujan’s modular equations. Proof of (2.4) and (2.6) (modular equation form) can be found in [3, Entry 13(xiv) and Entry 19(ix)]. The identity (2.7) is due to Vasuki and Sreeramamurthy [25]. Recently Bhargava et al. [5] have found a very simple proof of (2.2), (2.3) and (2.4) using Ramanujan’s ${}_1\psi_1$ summation formula. Elementary proofs of (2.5), (2.6) and (2.7) by using only theta functions results of Ramanujan can be found in [26].

We make use of the famous Ramanujan’s ${}_1\psi_1$ Summation formula:

$$\sum_{n=-\infty}^{\infty} \frac{(a; q)_n}{(b; q)_n} z^n = \frac{(az; q)_{\infty} (q/az; q)_{\infty} (q; q)_{\infty} (b/a; q)_{\infty}}{(z; q)_{\infty} (b/az; q)_{\infty} (b; q)_{\infty} (q/a; q)_{\infty}}, \quad \left| \frac{b}{a} \right| < |z| < 1, \quad |q| < 1. \quad (2.8)$$

We also determine some Eisenstein series identities for level 7 in addition to that of Cooper and Ye [11] by using (2.5)–(2.7).

Ramanujan’s Eisenstein series $P(q)$ and $Q(q)$ are defined by

$$P(q) = 1 - 24 \sum_{j=1}^{\infty} \frac{j q^j}{1 - q^j} \quad \text{and} \quad Q(q) = 1 + 240 \sum_{j=1}^{\infty} \frac{j^3 q^j}{1 - q^j}.$$

Theorem 2.1 *For any positive integer n , we have*

$$P_n = 1 - 24 \sum_{j=1}^{\infty} \sigma(j) q^{jn}, \quad (2.9)$$

$$Q_n = 1 + 240 \sum_{j=1}^{\infty} \sigma_3(j) q^{jn}, \quad (2.10)$$

and

$$(P(q))^2 = 1 + \sum_{j=1}^{\infty} [240\sigma_3(j) - 288j\sigma(j)] q^j. \quad (2.11)$$

For a proof of (2.9) and (2.10), see [2, p. 318] and a proof (2.11), see [12] and [22, Eq (30)], [20, p. 142].

Theorem 2.2 *We have*

$$\varphi^8(-q) = \frac{1}{15}(-Q_1 + 16Q_2) \quad \text{and} \quad q\psi^8(q) = \frac{1}{240}(Q_1 - Q_2).$$

Berndt [3, p. 139] proved the above theorem by using the modular equation identities. Cooper [8] too proved using the elliptic function identities.

From the above theorem, we have

$$Q_1 = q^{\frac{1}{2}} f_1^4 f_2^4 \left(A_1^{12} + \frac{256}{A_1^{12}} \right) \quad (2.12)$$

and

$$Q_2 = q^{\frac{1}{2}} f_1^4 f_2^4 \left(A_1^{12} + \frac{16}{A_1^{12}} \right). \quad (2.13)$$

For any complex number a and for each positive integer n , let

$$(a)_n = a(a+1)(a+2) \cdots (a+n-1)$$

and

$$(a)_0 = 1.$$

The Gauss ordinary hypergeometric series is defined by

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k, \quad |z| < 1.$$

A modular equation of degree n is a relation between α and β that is induced by the equation

$$n \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\beta\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)}.$$

In such relation, we say β has n^{th} degree in α . We also define the multiplier m by

$$m = \frac{z_1}{z_n}, \quad (2.14)$$

where

$$z_1 = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right) \quad \text{and} \quad z_n = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right). \quad (2.15)$$

The following theorem of Ramanujan's play a central role in transforming a theta function identity into a modular equation and vice-versa.

Theorem 2.3 *For $q = \exp\left(-\pi \frac{K'}{K}\right)$ and $z = K$, then*

$$\varphi(q) = \sqrt{z}, \quad (2.16)$$

$$\varphi(-q) = \sqrt{z}(1-\alpha)^{1/4}, \quad (2.17)$$

$$\varphi(-q^2) = \sqrt{z}(1-\alpha)^{1/8}, \quad (2.18)$$

$$q^{1/8}\psi(q) = \sqrt{\frac{z}{2}}(\alpha)^{1/8}, \quad (2.19)$$

$$q^{1/8}\psi(-q) = \sqrt{\frac{z}{2}}\{\alpha(1-\alpha)\}^{1/8}, \quad (2.20)$$

$$q^{1/4}\psi(q^2) = \frac{\sqrt{z}}{2}(\alpha)^{1/4}, \quad (2.21)$$

$$q^{1/24}f(-q) = \frac{\sqrt{z}}{2^{1/6}}\{\alpha(1-\alpha)^4\}^{1/24}, \quad (2.22)$$

$$q^{1/12}f(-q^2) = \frac{\sqrt{z}}{2^{1/3}}\{\alpha(1-\alpha)\}^{1/12}, \quad (2.23)$$

$$q^{-1/24}\chi(q) = 2^{1/6}\{\alpha(1-\alpha)\}^{-1/24}, \quad (2.24)$$

and

$$q^{-1/24}\chi(-q) = 2^{1/6}\{\alpha(1-\alpha)^{-2}\}^{-1/24}, \quad (2.25)$$

where

$$K = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right) \text{ and } K' = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha\right), \quad |\alpha| < 1.$$

3 Quintic and septic identities for Q_n 's

In this section, we deduce Q_n 's for $n = 1, 2, 5, 7, 10$ and 14 in terms of f_n for $n = 1, 2, 5, 7, 10$ and 14 .

Theorem 3.1 If $T = q^{\frac{3}{4}}f_1f_2f_5f_{10}$, then

$$Q_1 = \frac{T^2}{2} \left[\left(\frac{1}{v^2} + 4v^2 \right) \left(3125u^2 + \frac{5}{u^2} + 250 \right) + 9375u^3 + 1875u + \frac{225}{u} - \frac{3}{u^3} \right], \quad (3.1)$$

$$Q_2 = \frac{T^2}{8} \left[\left(\frac{1}{v^2} + 4v^2 \right) \left(3125u^2 + \frac{5}{u^2} + 250 \right) - 9375u^3 - 1875u - \frac{225}{u} + \frac{3}{u^3} \right], \quad (3.2)$$

$$Q_5 = \frac{T^2}{2} \left[\left(\frac{1}{v^2} + 4v^2 \right) \left(5u^2 + \frac{5}{u^2} + 10 \right) + 15u^3 - 45u - \frac{15}{u} - \frac{3}{u^3} \right], \quad (3.3)$$

and

$$Q_{10} = \frac{T^2}{8} \left[\left(\frac{1}{v^2} + 4v^2 \right) \left(5u^2 + \frac{5}{u^2} + 10 \right) - 15u^3 + 45u + \frac{15}{u} + \frac{3}{u^3} \right]. \quad (3.4)$$

Proof The identity (2.4) is equivalent to

$$(A_1A_5)^2 + \frac{4}{(A_1A_5)^2} = \left(\frac{A_5}{A_1} \right)^3 - \left(\frac{A_1}{A_5} \right)^3. \quad (3.5)$$

From the above identity, one may deduce the following:

$$A_1^{12} + \frac{64}{A_1^{12}} = \frac{\sqrt{x^2-4}}{2} \left[(x^2-2)(x^2-16) - x(x^2-8)\sqrt{x^2-20} \right], \quad (3.6)$$

$$A_1^{12} - \frac{64}{A_1^{12}} = \frac{1}{2} \left[(x^2-2)(x^2-8)\sqrt{x^2-20} - x(x^2-16)(x^2-4) \right], \quad (3.7)$$

$$A_5^{12} + \frac{64}{A_5^{12}} = \frac{\sqrt{x^2-4}}{2} \left[(x^2-2)(x^2-16) + x(x^2-8)\sqrt{x^2-20} \right], \quad (3.8)$$

and

$$A_5^{12} - \frac{64}{A_5^{12}} = \frac{1}{2} \left[(x^2-2)(x^2-8)\sqrt{x^2-20} + x(x^2-16)(x^2-4) \right]. \quad (3.9)$$

From (2.3), (2.4) and the definition of x , we find that

$$\frac{1}{u} - 5u = \sqrt{x^2 - 20} \quad \text{and} \quad \frac{1}{v^2} + 4v^2 = \sqrt{x^2 - 4}. \quad (3.10)$$

Substituting the identity obtained by subtracting 3 times of (3.7) from 5 times of (3.6) in (2.12) and then using (3.10) in the resulting identity, we obtain (3.1).

Substituting the identity obtained by adding 5 times of (3.6) with 3 times of (3.7) in (2.13) and then employing (3.10) in the resulting identity, we obtain (3.2).

Replacing q by q^5 in (2.12) and then employing the identity obtained by subtracting 3 times of (3.9) from 5 times of (3.8) in the resulting identity and then using (3.9), we obtain (3.3).

Replacing q by q^5 in (2.13) and then employing the identity obtained by adding 5 times of (3.8) and 3 times of (3.9) in the resulting identity and then using (3.10), we obtain (3.4). $\square$

Theorem 3.2 If $S = qf_1 f_2 f_7 f_{14}$, then

$$Q_1 = S^2 \left(\frac{1}{V^3} + 4 \right) \left[1600V^6 + \frac{25}{V^3} + 1200V^3 + 220 - 24 \left(\frac{1}{W^4} - W^4 \right) (1 - 8V^3) \right], \quad (3.11)$$

$$Q_2 = S^2 (1 + 2V^3) \left[\frac{25}{V^6} + \frac{150}{V^3} + 200V^3 + 220 - 24 \left(\frac{1}{W^4} - W^4 \right) \left(\frac{1}{V^3} - 1 \right) \right], \quad (3.12)$$

$$Q_7 = \frac{S^2}{49} \left(\frac{1}{V^3} + 4 \right) \left[1600V^6 + \frac{25}{V^3} + 1200V^3 + 220 + 24 \left(\frac{1}{W^4} - W^4 \right) (1 - 8V^3) \right], \quad (3.13)$$

and

$$Q_{14} = \frac{S^2}{49} (1 + 2V^3) \left[\frac{25}{V^6} + \frac{150}{V^3} + 200V^3 + 220 + 24 \left(\frac{1}{W^4} - W^4 \right) \left(\frac{1}{V^3} - 1 \right) \right]. \quad (3.14)$$

Proof From (2.6), we find that

$$\frac{1}{W^4} - W^4 = \sqrt{(y^3 - 6y + 5)(y^3 - 6y + 9)}. \quad (3.15)$$

From (2.5) and the definition of y , we have

$$\frac{1}{U^2} + 49U^2 = (y^3 - 6y - 2) \sqrt{y^3 - 6y + 9} \quad \text{and} \quad \frac{1}{U^2} - 49U^2 = (y^2 - 2) \sqrt{(y^2 - 8)(y^3 - 6y + 5)}.$$

From the above two equation, we have

$$\frac{1}{U^2} = \frac{1}{2} \left[(y^3 - 6y - 2) \sqrt{y^3 - 6y + 9} + (y^2 - 2) \sqrt{(y^2 - 8)(y^3 - 6y + 5)} \right] \quad (3.16)$$

and

$$U^2 = \frac{1}{98} \left[(y^3 - 6y - 2) \sqrt{y^3 - 6y + 9} - (y^2 - 2) \sqrt{(y^2 - 8)(y^3 - 6y + 5)} \right]. \quad (3.17)$$

The identity (2.6) is equivalent to

$$(A_1 A_7)^3 + \frac{8}{(A_1 A_7)^3} + 7 = \left(\frac{A_7}{A_1} \right)^4 + \left(\frac{A_1}{A_7} \right)^4. \quad (3.18)$$

From the above identity, one may deduce the following identities:

$$\begin{aligned} A_1^{12} + \frac{64}{A_1^{12}} &= \frac{1}{2} \left[(y^6 - 12y^4 + 36y^2 - 16) (y^3 - 6y + 6) \sqrt{y^3 - 6y + 9} \right. \\ &\quad \left. - (y^5 - 8y^3 + 12y) (y^3 - 6y + 8) \sqrt{(y^2 - 8)(y^3 - 6y + 5)} \right], \end{aligned} \quad (3.19)$$

$$A_1^{12} - \frac{64}{A_1^{12}} = \frac{1}{2} \left[(y^5 - 8y^3 + 12y) (y^3 - 6y + 6) \sqrt{(y^2 - 8) (y^3 - 6y + 9)} \right. \\ \left. - (y^6 - 12y^4 + 36y^2 - 16) (y^3 - 6y + 8) \sqrt{y^3 - 6y + 5} \right], \quad (3.20)$$

$$A_7^{12} + \frac{64}{A_7^{12}} = \frac{1}{2} \left[(y^6 - 12y^4 + 36y^2 - 16) (y^3 - 6y + 6) \sqrt{y^3 - 6y + 9} \right. \\ \left. + (y^5 - 8y^3 + 12y) (y^3 - 6y + 8) \sqrt{(y^2 - 8) (y^3 - 6y + 5)} \right], \quad (3.21)$$

and

$$A_7^{12} - \frac{64}{A_7^{12}} = \frac{1}{2} \left[(y^5 - 8y^3 + 12y) (y^3 - 6y + 6) \sqrt{(y^2 - 8) (y^3 - 6y + 9)} \right. \\ \left. + (y^6 - 12y^4 + 36y^2 - 16) (y^3 - 6y + 8) \sqrt{y^3 - 6y + 5} \right]. \quad (3.22)$$

Using (3.16) and the identity obtained by subtracting 3 times of (3.20) from 5 times of (3.19) in (2.12) and then using (3.15) to eliminate $\sqrt{(y^3 - 6y + 5) (y^3 - 6y + 9)}$ and then employing the definition of y in the resulting identity, we obtain (3.11).

Using (3.16) and the identity obtained by adding 5 times of (3.19) and 3 times of (3.20) in (2.13) and then using (3.15) and the definition of y in the resulting identity, we obtain (3.12).

Replacing q by q^7 in (2.12) and using (3.17) and the identity obtained by subtracting 3 times of (3.22) from 5 times of (3.21) in the resulting identity and then using (3.15) and the definition of y , we obtain (3.13).

Replacing q by q^7 in (2.13) and using (3.17) and the identity obtained by adding 5 times of (3.21) and 3 times of (3.22) in the resulting identity and then using (3.15) and the definition of y , we obtain (3.14). $\square$

4 Eisenstein series of level 10 and 14

This section acts as the bridge between Sects. 3 and 5. In this section, we determine certain new identities relating Eisenstein series of level 5 and 10. We also find certain Eisenstein series identities for level 7 and 14.

Theorem 4.1 *We have*

$$-P_1 + P_2 + 5P_5 - 5P_{10} = 24q^2 \frac{\chi^2(q) f_{10}^4}{\chi^2(q^5)} + 12\varphi^4(-q^{10}) \frac{\chi(q) \chi(-q^5)}{\chi(-q) \chi(q^5)} - 12\varphi^4(q^5).$$

Proof Setting $a = q$, $b = -1$ in (1.1), we find that

$$\sum_{n=-\infty}^{\infty} \left[\frac{q^{10n+1}}{(1 - q^{10n+1})^2} + \frac{q^{10n}}{(1 + q^{10n})^2} \right] = \frac{\varphi^4(-q^{10})}{4} \frac{f^2(q, q^9)}{f^2(-q, -q^9)}.$$

Setting $a = q^3$, $b = -1$ in (1.1), we find that

$$\sum_{n=-\infty}^{\infty} \left[\frac{q^{10n+3}}{(1 - q^{10n+3})^2} + \frac{q^{10n}}{(1 + q^{10n})^2} \right] = \frac{\varphi^4(-q^{10})}{4} \frac{f^2(q^3, q^7)}{f^2(-q^3, -q^7)}.$$

Adding the above two identity, we have

$$\sum_{n=0}^{\infty} \left[\frac{q^{2n+1}}{(1 - q^{2n+1})^2} - \frac{q^{10n+5}}{(1 - q^{10n+5})^2} \right] + 2 \sum_{n=-\infty}^{\infty} \frac{q^{10n}}{(1 + q^{10n})^2} \\ = \frac{\varphi^4(-q^{10})}{4} \left[\frac{f^2(q, q^9)}{f^2(-q, -q^9)} + \frac{f^2(q^3, q^7)}{f^2(-q^3, -q^7)} \right]. \quad (4.1)$$

Setting $a = q^5$, $b = -1$ in (1.1), we find that

$$2 \sum_{n=0}^{\infty} \frac{q^{10n+5}}{(1 - q^{10n+5})^2} + \sum_{n=-\infty}^{\infty} \frac{q^{10n}}{(1 + q^{10n})^2} = \frac{\varphi^4(q^5)}{4}.$$

Using the above in (4.1), we have

$$\sum_{n=0}^{\infty} \left[\frac{q^{2n+1}}{(1 - q^{2n+1})^2} - 5 \frac{q^{10n+5}}{(1 - q^{10n+5})^2} \right] = \frac{\varphi^4(-q^{10})}{4} \left[\frac{f^2(q, q^9)}{f^2(-q, -q^9)} + \frac{f^2(q^3, q^7)}{f^2(-q^3, -q^7)} \right] - \frac{\varphi^4(q^5)}{2}. \quad (4.2)$$

From [3, p. 258], we have

$$4q^2 f^2(q, q^9) f^2(q^3, q^7) = \varphi^4(-q^{10}) \left[\frac{f^2(q, q^9)}{f^2(-q, -q^9)} + \frac{f^2(q^3, q^7)}{f^2(-q^3, -q^7)} \right] - \frac{2\varphi^4(-q^{10}) f(q, q^9) f(q^3, q^7)}{f(-q, -q^9) f(-q^3, -q^7)}. \quad (4.3)$$

From [3, p. 251], we have

$$f(-q, -q^9) f(-q^3, -q^7) = \frac{\chi(-q) f_{10}^2}{\chi(-q^5)} \quad \text{and} \quad f(q, q^9) f(q^3, q^7) = \frac{\chi(q) f_{10}^2}{\chi(q^5)}. \quad (4.4)$$

Using the above in (4.3) and then using the resulting identity in (4.2), we find that

$$\sum_{n=0}^{\infty} \left[\frac{q^{2n+1}}{(1 - q^{2n+1})^2} - 5 \frac{q^{10n+5}}{(1 - q^{10n+5})^2} \right] = \frac{q^2 \chi^2(q) f_{10}^4}{\chi^2(q^5)} + \frac{\varphi^4(-q^{10}) \chi(q) \chi(-q^5)}{2 \chi(-q) \chi(q^5)} - \frac{\varphi^4(q^5)}{2}.$$

Expanding each of the summands into geometric series, interchanging the order of summation, summing the inner geometric series and then using definition of P_n , we obtain the required result. $\square$

Theorem 4.2

$$P_2 - 4P_4 - 5P_{10} + 20P_{20} = 12\varphi^4(-q^5) - 24 \frac{q^2 \chi^2(-q) f_{10}^4}{\chi^2(-q^5)} - 48 \frac{q^3 f_{10} \psi^4(-q^5) \chi(-q) \chi(-q^2)}{\chi(-q^5) \varphi(-q^{10})}.$$

Proof Setting $a = -q^2$, $b = -q^5$ in (1.1), we find that

$$\sum_{n=-\infty}^{\infty} \left[\frac{-q^{10n+2}}{(1 + q^{10n+2})^2} + \frac{q^{10n+5}}{(1 + q^{10n+5})^2} \right] = -q^2 \psi^4(-q^5) \frac{f^2(-q^3, -q^7)}{f^2(q^2, q^8)}.$$

Setting $a = -q^4$, $b = -q^5$ in (1.1), we find that

$$\sum_{n=-\infty}^{\infty} \left[\frac{-q^{10n+4}}{(1 + q^{10n+4})^2} + \frac{q^{10n+5}}{(1 + q^{10n+5})^2} \right] = -q^4 \psi^4(-q^5) \frac{f^2(-q, -q^9)}{f^2(q^4, q^6)}.$$

Adding the above two identity, we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \left[\frac{-q^{2n+2}}{(1 + q^{2n+2})^2} + \frac{q^{10n+10}}{(1 + q^{10n+10})^2} \right] + 2 \sum_{n=-\infty}^{\infty} \frac{q^{10n+5}}{(1 + q^{10n+5})^2} \\ &= -q^4 \psi^4(-q^5) \left[\frac{f^2(-q^3, -q^7)}{q^2 f^2(q^2, q^8)} + \frac{f^2(-q, -q^9)}{f^2(q^4, q^6)} \right]. \end{aligned} \quad (4.5)$$

Setting $a = -1$, $b = -q^5$ in (1.1), we find that

$$\frac{1}{4} + 2 \sum_{n=1}^{\infty} \frac{q^{10n}}{(1 + q^{10n})^2} - \sum_{n=-\infty}^{\infty} \frac{q^{10n+5}}{(1 + q^{10n+5})^2} = \frac{\varphi^4(-q^5)}{4}.$$

Using the above in (4.5), we have

$$\sum_{n=0}^{\infty} \frac{-q^{2n+2}}{(1+q^{2n+2})^2} + 5 \sum_{n=1}^{\infty} \frac{q^{10n}}{(1+q^{10n})^2} = \frac{\varphi^4(-q^5)}{2} - \frac{1}{2} - q^4 \psi^4(-q^5) \left[\frac{f^2(-q^3, -q^7)}{q^2 f^2(q^2, q^8)} + \frac{f^2(-q, -q^9)}{f^2(q^4, q^6)} \right]. \quad (4.6)$$

From [3, Entry 29, p. 45], we find that

$$f(-q, -q^9) f(-q^3, -q^7) = -q \psi^2(-q^5) \left[\frac{f(-q, -q^9)}{f(q^4, q^6)} - \frac{f(-q^3, -q^7)}{q f(q^2, q^8)} \right]. \quad (4.7)$$

From [3, p. 263], we have

$$f(q^2, q^8) f(q^4, q^6) = \frac{\varphi(-q^{10}) f_{10}}{\chi(-q^2)}.$$

Squaring (4.7) and using the resulting identity along with (4.4) and the above identity in (4.6), we find that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{-q^{2n+2}}{(1+q^{2n+2})^2} + 5 \sum_{n=1}^{\infty} \frac{q^{10n}}{(1+q^{10n})^2} &= \frac{1}{2} \varphi^4(-q^5) - \frac{1}{2} - \frac{q^2 \chi^2(-q) f_{10}^4}{\chi^2(-q^5)} \\ &\quad - 2 \frac{q^3 f_{10} \psi^4(-q^5) \chi(-q) \chi(-q^2)}{\chi(-q^5) \varphi(-q^{10})}. \end{aligned}$$

Expanding each of the summands into geometric series, interchanging the order of summation, summing the inner geometric series and then using definition of P_n , we obtain the required result. $\square$

Lemma 4.3 *We have*

$$\frac{\psi^2(q)}{q \psi^2(q^5)} + 5q \frac{\psi^2(q^5)}{\psi^2(q)} = \frac{1}{v^4} + 6. \quad (4.8)$$

Proof Using $\psi(q) = \frac{f_2^2}{f_1}$ and the definition of u and w from (2.1), we have

$$\frac{\psi^2(q)}{q \psi^2(q^5)} + 5q \frac{\psi^2(q^5)}{\psi^2(q)} = \frac{1}{w^3 u} + 5w^3 u.$$

Using (2.2), (2.3) and (2.4), we have the required result. $\square$

Theorem 4.4 *We have*

$$\begin{aligned} &24q^2 \frac{\chi^2(q) f_{10}^4}{\chi^2(q^5)} + 12\varphi^4(-q^{10}) \frac{\chi(q) \chi(-q^5)}{\chi(-q) \chi(q^5)} - 12\varphi^4(q^5) \\ &= 24q^{\frac{3}{2}} \frac{f_2^4 f_{10}^4}{f_1^2 f_5^2} \left(\frac{\psi^2(q)}{q \psi^2(q^5)} - 2 + 5q \frac{\psi^2(q^5)}{\psi^2(q)} \right)^{\frac{1}{2}} \end{aligned} \quad (4.9)$$

and

$$\begin{aligned} &12\varphi^4(-q^5) - 24 \frac{q^2 \chi^2(-q) f_{10}^4}{\chi^2(-q^5)} - 48 \frac{q^3 f_{10} \psi^4(-q^5) \chi(-q) \chi(-q^2)}{\chi(-q^5) \varphi(-q^{10})} \\ &= 12q f_2^2 f_{10}^2 \left(\frac{\psi^2(q^2)}{q^2 \psi^2(q^{10})} - 2 + 5q^2 \frac{\psi^2(q^{10})}{\psi^2(q^2)} \right)^{\frac{1}{2}}. \end{aligned} \quad (4.10)$$

Proof Let β have degree 5 over α , z_1, z_5 be as defined in (2.15) and m be the multiplier for degree 5 defined by (2.14). From (2.16), (2.18), (2.23), (2.24) and (2.25), we have

$$24q^2 \frac{\chi^2(q)f_{10}^4}{\chi^2(q^5)} + 12\varphi^4(-q^{10}) \frac{\chi(q)\chi(-q^5)}{\chi(-q)\chi(q^5)} - 12\varphi^4(q^5) = \frac{12z_5^2}{2^{\frac{1}{3}}} \left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)} \right)^{\frac{1}{12}} + 12z_5^2 \left(\frac{(1-\beta)^5}{1-\alpha} \right)^{\frac{1}{8}} - 12z_5^2. \quad (4.11)$$

From [3, p. 280, Entry 13(iv)] and [3, p. 284, Eqs. (13.3) and (13.5)], we have

$$m = 1 + 2^{\frac{4}{3}} \left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)} \right)^{\frac{1}{24}}, \quad \rho = (m^3 - 2m^2 + 5m)^{\frac{1}{2}},$$

$$\text{and} \quad \left(\frac{(1-\beta)^5}{1-\alpha} \right)^{\frac{1}{8}} = \frac{\rho + m + 1}{4}. \quad (4.12)$$

Using the above in (4.11), we find that

$$24q^2 \frac{\chi^2(q)f_{10}^4}{\chi^2(q^5)} + 12\varphi^4(-q^{10}) \frac{\chi(q)\chi(-q^5)}{\chi(-q)\chi(q^5)} - 12\varphi^4(q^5) = \frac{3}{2}z_5^2(m^2 + 2\rho - 5). \quad (4.13)$$

Similarly transforming the square of the right hand side of (4.9) by using (2.19), (2.22) and (2.23) into modular equations, we have

$$576q^3 \frac{f_2^8 f_{10}^8}{f_1^4 f_5^4} \left(\frac{\psi^2(q)}{q\psi^2(q^5)} - 2 + 5q \frac{\psi^2(q^5)}{\psi^2(q)} \right) = 36z_5^4 m^2 (\alpha\beta)^{\frac{1}{2}} \left(m \left(\frac{\alpha}{\beta} \right)^{\frac{1}{4}} - 2 + \frac{5}{m} \left(\frac{\beta}{\alpha} \right)^{\frac{1}{4}} \right). \quad (4.14)$$

From [3, p. 286, Eqs. (13.12), (13.13) and (13.14)], we have

$$\left(\frac{\alpha}{\beta} \right)^{\frac{1}{4}} = \frac{2m + \rho}{m(m-1)}, \quad \left(\frac{\beta}{\alpha} \right)^{\frac{1}{4}} = \frac{2m - \rho}{5 - m} \quad \text{and} \quad (\alpha\beta)^{\frac{1}{2}} = \frac{4m^3 - 16m^2 + 20m + \rho(m^2 - 5)}{16m^2}. \quad (4.15)$$

Using the above in (4.14) and then taking the positive square root, we have

$$24q^{\frac{3}{2}} \frac{f_2^4 f_{10}^4}{f_1^2 f_5^2} \left(\frac{\psi^2(q)}{q\psi^2(q^5)} - 2 + 5q \frac{\psi^2(q^5)}{\psi^2(q)} \right)^{\frac{1}{2}} = \frac{3}{2}z_5^2(m^2 + 2\rho - 5). \quad (4.16)$$

Guided by Lemma 4.3, we have considered the positive root. The identity (4.9) follows from (4.13) and (4.16). The proof of (4.10) is similar to that of (4.9). $\square$

Corollary 4.5 *We have*

$$-P_1 + P_2 + 5P_5 - 5P_{10} = 24T \left(\frac{1}{v^2} + 4v^2 \right)^{\frac{1}{2}} v^2 \quad (4.17)$$

and

$$P_1 - 4P_2 - 5P_5 + 20P_{10} = 12T \left(\frac{1}{v^2} + 4v^2 \right)^{\frac{1}{2}} \frac{1}{v^2}. \quad (4.18)$$

Proof Using (4.8), (4.9) and the definition of v in Theorem 4.1, we deduce (4.17). Changing q to $q^{\frac{1}{2}}$ in Theorem 4.2 and in (4.10) and then using (4.8) and the definition of v , we deduce (4.18). $\square$

Remark From the above corollary, Entry 4(i) of Chapter 21 of Ramanujan's second notebook [19] follows.

Theorem 4.6 *We have*

$$-3P_1 + 2P_2 - 5P_5 + 30P_{10} = 24T \left(\frac{1}{v^2} + 4v^2 \right)^{\frac{1}{2}} \frac{1}{w^3} \quad (4.19)$$

and

$$-P_1 + 6P_2 - 15P_5 + 10P_{10} = 24T \left(\frac{1}{v^2} + 4v^2 \right)^{\frac{1}{2}} w^3, \quad (4.20)$$

where T is as defined in the Theorem 3.1.

Proof From (4.17) and (4.18), we have

$$-P_2 + 5P_{10} = 4T \left(\frac{1}{v^2} + 4v^2 \right)^{\frac{1}{2}} \left(\frac{1}{v^2} + 2v^2 \right). \quad (4.21)$$

From [8, p. 196, Theorem 3.26], we have

$$-P_1 + 2P_2 = 16q\psi^4(q^2) + \varphi^4(q) = q^{\frac{1}{3}} f_2^4 \left[\frac{A_2^8}{A_1^8} + \frac{16}{A_2^8} \right]. \quad (4.22)$$

From [3, Entry 25(vii), p. 40], we have

$$\frac{A_2^8}{A_1^8} - \frac{16}{A_2^8} = A_1^8 \implies \frac{A_2^8}{A_1^8} + \frac{16}{A_2^8} = \sqrt{A_1^{16} + \frac{64}{A_1^8}}.$$

Using the above identity in (4.22), we have

$$-P_1 + 2P_2 = q^{\frac{1}{4}} f_1^2 f_2^2 \left[A_1^{12} + \frac{64}{A_1^{12}} \right]^{\frac{1}{2}}. \quad (4.23)$$

From (3.6), we deduce that

$$A_1^{12} + \frac{64}{A_1^{12}} = \frac{\sqrt{x^2 - 4}}{4} \left[x^2 - 8 - x\sqrt{x^2 - 20} \right]^2.$$

Employing the above identity in (4.23) and then using (3.10) to eliminate $\sqrt{x^2 - 20}$ and $\sqrt{x^2 - 4}$ in the resulting identity, we deduce that

$$-P_1 + 2P_2 = T \left(\frac{1}{v^2} + 4v^2 \right)^{\frac{1}{2}} \left(\frac{1}{u} + 25u \right). \quad (4.24)$$

From (3.8), we deduce that

$$A_5^{12} + \frac{64}{A_5^{12}} = \frac{\sqrt{x^2 - 4}}{4} \left[x^2 - 8 + x\sqrt{x^2 - 20} \right]^2. \quad (4.25)$$

Changing q to q^5 in (4.23) and then employing (4.25) and (3.10) in the resulting identity, we find that

$$-P_5 + 2P_{10} = T \left(\frac{1}{v^2} + 4v^2 \right)^{\frac{1}{2}} \left(\frac{1}{u} + u \right). \quad (4.26)$$

Consider

$$-3P_1 + 2P_2 - 5P_5 + 30P_{10} = 3(-P_1 + 2P_2) + 4(-P_2 + 5P_{10}) + 5(-P_5 + 2P_{10}).$$

Using (4.21), (4.24), (4.26) in the right hand side of the above identity and then employing (2.2) and (2.4), we obtain (4.19). Now consider

$$-P_1 + 6P_2 - 15P_5 + 10P_{10} = (-P_1 + 2P_2) - 4(-P_2 + 5P_{10}) + 15(-P_5 + 2P_{10}).$$

Employing (4.21), (4.24), (4.26) in the right hand side of the above identity and then employing (2.2) and (2.4), we obtain (4.20). $\square$

Theorem 4.7 Let S be as defined in Theorem 3.2. Then

$$-3P_1 + 2P_2 - 7P_7 + 42P_{14} = S \left(16 \left(\frac{1}{W^4} - W^4 \right) + \frac{18}{V^3} + 144V^3 + 96 \right) \quad (4.27)$$

and

$$-P_1 + 6P_2 - 21P_7 + 14P_{14} = S \left(16 \left(\frac{1}{W^4} - W^4 \right) - \frac{18}{V^3} - 144V^3 - 96 \right). \quad (4.28)$$

Proof From [26, p. 54, Theorem 3.2.9], we have

$$-P_1 + P_2 + 7P_7 - 7P_{14} = 72q^2\psi^2(q)\psi^2(q^7) + 24q\varphi(-q)\varphi(-q^7)\psi(q)\psi(q^7)$$

and

$$P_1 - 4P_2 - 7P_7 + 28P_{14} = 18\varphi^2(-q)\varphi^2(-q^7) + 48q\varphi(-q)\varphi(-q^7)\psi(q)\psi(q^7).$$

From the above two equations, we have

$$-P_1 + 7P_7 = 6 \left(\varphi(-q)\varphi(-q^7) + 4q\psi(q)\psi(q^7) \right)^2. \quad (4.29)$$

From [26, p. 57, Theorem 3.3.1], we have

$$\left(\varphi(-q)\varphi(-q^7) + 4q\psi(q)\psi(q^7) \right)^3 = \frac{f_1^8 + 13qf_1^4f_7^4 + 49q^2f_7^8}{f_1f_7}.$$

Employing the above equation in (4.29), we have

$$-P_1 + 7P_7 = 6q^{\frac{2}{3}}f_1^2f_7^2 \left[C_1^4 + \frac{49}{C_1^4} + 13 \right]^{\frac{2}{3}}. \quad (4.30)$$

The identity (2.5) is equivalent to

$$(C_1C_2)^2 + \frac{49}{(C_1C_2)^2} = \left(\frac{C_2}{C_1} \right)^6 - 8 \left(\frac{C_2}{C_1} \right)^2 - 8 \left(\frac{C_1}{C_2} \right)^2 + \left(\frac{C_1}{C_2} \right)^6.$$

From the above identity, we have

$$\left(C_1^4 + 13 + \frac{49}{C_1^4} \right)^{\frac{1}{3}} = \frac{1}{2} \left(y^2 + y - 4 - (y-1)\sqrt{y^2-8} \right). \quad (4.31)$$

and

$$\left(C_2^4 + 13 + \frac{49}{C_2^4} \right)^{\frac{1}{3}} = \frac{1}{2} \left(y^2 + y - 4 + (y-1)\sqrt{y^2-8} \right). \quad (4.32)$$

Changing q to q^2 in (4.30) and then using (4.32) and the definition of y in the resulting identity, we obtain

$$-P_2 + 7P_{14} = 6S \left(\frac{1}{V^3} + 4V^3 + 4 \right). \quad (4.33)$$

Using (3.16) and (3.19), we find that

$$\frac{1}{U^2} \left(A_1^{12} + \frac{64}{A_1^{12}} \right) = \left(4\sqrt{(y^3-6y+5)(y^3-6y+9)} - 3(y^2-2)\sqrt{y^2-8} \right)^2.$$

Using the above identity in (4.23) and then using (3.15) and the definition of y in the resulting identity, we obtain

$$-P_1 + 2P_2 = 4S\left(\frac{1}{W^4} - W^4\right) - 3S\left(\frac{1}{V^3} - 8V^3\right). \quad (4.34)$$

Using (3.17) and (3.21), we find that

$$U^2\left(A_7^{12} + \frac{64}{A_7^{12}}\right) = \left(4\sqrt{(y^3 - 6y + 5)(y^3 - 6y + 9)} + 3(y^2 - 2)\sqrt{y^2 - 8}\right)^2. \quad (4.35)$$

Changing q to q^7 in (4.23) and then using the (4.35) and (3.15) and the definition of y in the resulting identity, we obtain

$$7(-P_7 + 2P_{14}) = 4S\left(\frac{1}{W^4} - W^4\right) + 3S\left(\frac{1}{V^3} - 8V^3\right). \quad (4.36)$$

Consider

$$-3P_1 + 2P_2 - 7P_7 + 42P_{14} = 3(-P_1 + 2P_2) + 4(-P_2 + 7P_{14}) + 7(-P_7 + 2P_{14}).$$

Using (4.33), (4.34) and (4.36) in the above equation, we obtain (4.27). Now consider

$$-P_1 + 6P_2 - 21P_7 + 14P_{14} = (-P_1 + 2P_2) - 4(-P_2 + 7P_{14}) + 21(-P_7 + 2P_{14}).$$

Using (4.33), (4.34) and (4.36) in the above equation, we obtain (4.28). $\square$

5 Applications to convolution sums of divisor function

In this section, we use some of the results of preceeding sections to evaluate convolution sums. We recall the definition of color partition and also we introduce certain restricted color partition functions which will be used in the evaluation of the convolution sums to avoid the coefficient functions of some infinite series existing in the convolution sums of [9, 10, 15, 16, 18, 24]. For convenience, we adopt the standard notation

$$(a_1, a_2, \dots, a_n; q)_\infty = \prod_{i=1}^n (a_i; q)_\infty$$

and define

$$(q^{m\pm}; q^n) = (q^m, q^{n-m}; q^n), \quad (m < n); \quad n, m \in \mathbb{N}.$$

Definition 5.1 [13] A positive integer n has l colors if there are l copies of n available colors and all of them are viewed as distinct objects. Partitions of a positive integer into parts with colors are called “colored partitions”.

For example, if 1 is allowed to have 2 colors, then all the colored partitions of 2 are 2, $1_r + 1_r$, $1_g + 1_g$ and $1_r + 1_g$. Where we use the indices r (red) and g (grey) to distinguish the two colors of 1. Also

$$\frac{1}{(q^u; q^v)_\infty^k}$$

is the generating function for the number of partitions of n where all the parts are congruent to $u \pmod{v}$ and have k colors.

We now introduce the following seven restricted color partition funtions for lateral use in this section:

Definition 5.2 Let $p_{(d,e)}(n)$ (resp. $p_{(d,o)}(n)$) denote the number of distinct partitions of n with even (resp. odd) number of parts and parts congruent to 0 (mod 5) with 8 colors, parts not congruent to 0 (mod 5) with 4 colors each.

From the above definition, it is clear that

$$\frac{T^2}{v^2} = qf_1^4 f_5^4 = \sum_{n=0}^{\infty} P^*(n)q^{n+1}, \quad (5.1)$$

where $P^*(n) = p_{(d,e)}(n) - p_{(d,o)}(n)$, with $P^*(0) = 1$. Let r —red, b —blue, y —yellow, g —green, O —orange, p —pink, br —brown and bl —black. Then we have

$$\begin{aligned} \sum_{n=0}^{\infty} P^*(n)q^n &= \prod_{j=1}^{\infty} (1 - q^{jr}) (1 - q^{jb}) (1 - q^{jy}) (1 - q^{jg}) \\ &\quad \times (1 - q^{5jO}) (1 - q^{5jp}) (1 - q^{5jbr}) (1 - q^{5jbl}). \end{aligned}$$

n	$p_{(d,e)}(n)$	$p_{(d,o)}(n)$	$P^*(n)$
1	0	$4(1r, 1b, 1y, 1g)$	−4
2	$6(1r + 1b, 1r + 1y, 1r + 1g, 1b + 1y, 1b + 1g, 1y + 1g)$	$4(2r, 2b, 2y, 2g)$	2
3	$16(1r + 2r, 1r + 2b, 1r + 2y, 1r + 2g, 1b + 2r, 1b + 2b, 1b + 2y, 1b + 2g, 1y + 2r, 1y + 2b, 1y + 2y, 1y + 2g, 1g + 2r, 1g + 2b, 1g + 2y, 1g + 2g)$	$8(1r + 1b + 1y, 1r + 1b + 1g, 1r + 1y + 1g, 1b + 1y + 1g, 3r, 3b, 3y, 3g)$	8
4	$23(2r + 2b, 2r + 2y, 2r + 2g, 2b + 2y, 2b + 2g, 2y + 2g, 1r + 3r, 1r + 3b, 1r + 3y, 1r + 3g, 1b + 3r, 1b + 3b, 1b + 3y, 1b + 3g, 1y + 3r, 1y + 3b, 1y + 3y, 1y + 3g, 1g + 2r, 1g + 2b, 1g + 2y, 1g + 2g, 1r + 1b + 1y + 1g)$	$28(4r, 4b, 4y, 4g, 1r + 1b + 2r, 1r + 1b + 2b, 1r + 1b + 2y, 1r + 1b + 2g, 1r + 1y + 2b, 1r + 1y + 2y, 1r + 1y + 2g, 1r + 1g + 2r, 1r + 1g + 2b, 1r + 1g + 2y, 1r + 1g + 2g, 1b + 1y + 2r, 1b + 1y + 2b, 1b + 1y + 2y, 1b + 1y + 2g, 1b + 1g + 2r, 1b + 1g + 2b, 1b + 1g + 2y, 1b + 1g + 2g, 1b + 1g + 2r, 1y + 1g + 2b, 1y + 1g + 2y, 1b + 1g + 2g)$	−5

Similarly for $n = 5$, one can easily find that $p_{(d,e)}(n) = 48$ and $p_{(d,o)}(n) = 56$.

Definition 5.3 Let $m_{(d,e)}(n)$ (resp. $m_{(d,o)}(n)$) denote the number of partitions of n such that

- parts which are congruent to $\pm 1, \pm 3 \pmod{10}$ have 1 color.
- parts which are not congruent to $0, \pm 1, \pm 3 \pmod{10}$ are distinct and have 4 colors each and parts congruent to $0 \pmod{10}$ are distinct and have 8 colors.
- number of parts which are not congruent to $\pm 1, \pm 3 \pmod{10}$ being even (resp. odd).

Using the above definition, we find that

$$\frac{T^2}{w^3} = \frac{qf_2^4 f_5^4}{(q^{1\pm}, q^{3\pm}; q^{10})_{\infty}} = \sum_{n=0}^{\infty} M(n)q^{n+1}, \quad (5.2)$$

where $M(n) = m_{(d,e)}(n) - m_{(d,o)}(n)$, with $M(0) = 1$. Let r —red, b —blue, y —yellow, g —green, O —orange, p —pink, br —brown and bl —black. Then we have

$$\begin{aligned} \sum_{n=0}^{\infty} M(n)q^n &= \prod_{j=1}^{\infty} \left[(1 - q^{2jr}) (1 - q^{2jb}) (1 - q^{2jy}) (1 - q^{2jg}) (1 - q^{5jO}) (1 - q^{5jp}) (1 - q^{5jbr}) \right. \\ &\quad \times (1 - q^{5jbl}) (1 + q^{1r} + q^{1r+1r} + q^{1r+1r+1r} + \dots) (1 + q^{3r} + q^{3r+3r} + q^{3r+3r+3r} + \dots) \\ &\quad \left. \times (1 + q^{7r} + q^{7r+7r} + q^{7r+7r+7r} + \dots) (1 + q^{9r} + q^{9r+9r} + q^{9r+9r+9r} + \dots) \times \dots \right]. \end{aligned}$$

Similarly for $n = 5$, one can easily find that $m_{(d,e)}(n) = 8$ and $m_{(d,o)}(n) = 16$.

n	$m_{(d,e)}(n)$	$m_{(d,o)}(n)$	$M(n)$
1	1 (1r)	0	1
2	1 (1r + 1r)	4 (2r, 2b, 2y, 2g)	-3
3	2 (1r + 1r + 1r, 3r)	4 (1r + 2r, 1r + 2b, 1r + 2y, 1r + 2g)	-2
4	8(2r + 2b, 2r + 2y, 2r + 2g, 2b + 2y, 2b + 2g, 2y + 2g, 1r + 1r + 1r + 1r, 1r + 3r)	8(4r, 4b, 4y, 4g, 1r + 1r + 2r, 1r + 1r + 2b, 1r + 1r + 2y, 1r + 1r + 2g)	0

One can also find the values of coefficients involved in following definitions by using the same technique used in the above two tables.

Definition 5.4 Let $t_{(d,e)}(n)$ (resp. $t_{(d,o)}(n)$) denote the number of distinct partitions of n with even (resp. odd) number of parts and parts congruent to 0 (mod 10) with 8 colors, parts congruent to $\pm 2, \pm 4, 5$ (mod 10) with 4 colors each, parts congruent to $\pm 1, \pm 3$ (mod 10) with 5 colors each.

The above definition yields

$$T^2 w^3 = q^2 f_1^4 f_{10}^4 (q^{1\pm}, q^{3\pm}; q^{10})_{\infty} = \sum_{n=0}^{\infty} T^*(n) q^{n+2}, \quad (5.3)$$

where $T^*(n) = t_{(d,e)}(n) - t_{(d,o)}(n)$, with $T^*(0) = 1$.

Definition 5.5 Let $l_{(d,e)}(n)$ (resp. $l_{(d,o)}(n)$) denote the number of distinct partitions of n with even (resp. odd) number of parts and parts congruent to 0 (mod 14) with 8 colors, parts congruent to $\pm 2, \pm 4, \pm 6$ (mod 14) with 4 colors each, parts congruent to $\pm 1, \pm 3, \pm 5$ (mod 14) with 5 colors each, parts congruent to 7 (mod 14) with 10 colors.

From the above definition, one can easily see that

$$\frac{S^2}{V^3} = q f_1^4 f_7^4 (q^{1\pm}, q^{3\pm}, q^{5\pm}, q^7, q^7; q^{14})_{\infty} = \sum_{n=0}^{\infty} L(n) q^{n+1}, \quad (5.4)$$

where, $L(n) = l_{(d,e)}(n) - l_{(d,o)}(n)$, with $L(0) = 1$.

Definition 5.6 Let $k_{(d,e)}(n)$ (resp. $k_{(d,o)}(n)$) denote the number of distinct partitions of n with even (resp. odd) number of parts and parts congruent to 0 (mod 14) with 8 colors, parts congruent to $\pm 2, \pm 4, \pm 6, 7$ (mod 14) with 4 colors each, parts congruent to $\pm 1, \pm 3, \pm 5$ (mod 14) with 2 colors each.

From the above definition, we see that

$$S^2 = q^2 f_1^2 f_2^2 f_7^2 f_{14}^2 = \sum_{n=0}^{\infty} K(n) q^{n+2}, \quad (5.5)$$

where $K(n) = k_{(d,e)}(n) - k_{(d,o)}(n)$, with $K(0) = 1$.

Definition 5.7 Let $a_{(d,e)}(n)$ (resp. $a_{(d,o)}(n)$) denote the number of partitions of n such that

- parts which are congruent to $\pm 1, \pm 3, \pm 5$ (mod 14) have 2 colors.
- parts which are not congruent to 0, $\pm 1, \pm 3, \pm 5$ (mod 14) are distinct and have 4 colors each and parts congruent to 0 (mod 14) are distinct and have 8 colors.
- number of parts which are not congruent to $\pm 1, \pm 3, \pm 5$ (mod 14) being even (resp. odd).

From the above definition, we have

$$\frac{S^2}{W^4} = \frac{q f_2^4 f_7^4}{(q^{1\pm}, q^{3\pm}, q^{5\pm}; q^{14})_{\infty}^2} = \sum_{n=0}^{\infty} A(n) q^{n+1}, \quad (5.6)$$

where $A(n) = a_{(d,e)}(n) - a_{(d,o)}(n)$, with $A(0) = 1$.

Definition 5.8 Let $b_{(d,e)}(n)$ (resp. $b_{(d,o)}(n)$) denote the number of distinct partitions of n with even (resp. odd) number of parts and parts congruent to 0 (mod 14) with 8 colors, parts congruent to $\pm 2, \pm 4, \pm 6, 7$ (mod 14) with 4 colors each, parts congruent to $\pm 1, \pm 3, \pm 5$ (mod 14) with 6 colors each.

From the above definition, one may find that

$$S^2 W^4 = q^3 f_1^4 f_{14}^4 \left(q^{1\pm}, q^{3\pm}, q^{5\pm}; q^{14} \right)_{\infty}^2 = \sum_{n=0}^{\infty} B(n) q^{n+3}, \quad (5.7)$$

where $B(n) = b_{(d,e)}(n) - b_{(d,o)}(n)$, with $B(0) = 1$.

Note: We set $P^*(n) = M(n) = T^*(n) = L(n) = K(n) = A(n) = B(n) = 0 \quad \forall \quad n \notin \mathbb{N} \cup \{0\}$.

As stated in the beginning of this section, the identity (5.8) and (5.17) are same as that deduced by M. Lemire and K. S. Williams [15] and also Cooper and Toh [9] have deduced (5.8) and (5.17), apart from the constant term, which we have represented using partition function. Lemire and Williams have deduced these identities by finding the direct relations between Eisenstein series of weight 2 and 4 from existing identities found in [21]. Cooper and Toh have deduced by employing modular equations and differentiation. Royer [24] too have deduced (5.8), (5.10), (5.17) and (5.19) by the method based on quasimodular forms. Similarly the identity (5.9) is same as that deduced by Cooper and Ye [10] by quasi-modular. Ntjenjem [16] have deduced (5.18) and (5.19) by employing modular groups and modular forms. Ramakrishnan, Sahu and Singh [18] have also deduced (5.17), (5.18) and (5.19). Our method is very elementary as we used certain Eisenstein series identities of weight 2 (also deduced by elementary approach) to find the required convolution sums.

Theorem 5.9 We have

$$W_5(n) = \frac{5}{312} \sigma_3(n) + \frac{125}{312} \sigma_3\left(\frac{n}{5}\right) + \left(\frac{1}{24} - \frac{n}{20}\right) \sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right) \sigma\left(\frac{n}{5}\right) - \frac{1}{130} P^*(n-1), \quad (5.8)$$

$$W_{(2,5)}(n) = \frac{1}{312} \sigma_3(n) + \frac{1}{78} \sigma_3\left(\frac{n}{2}\right) + \frac{25}{312} \sigma_3\left(\frac{n}{5}\right) + \frac{25}{78} \sigma_3\left(\frac{n}{10}\right) + \left(\frac{1}{24} - \frac{n}{20}\right) \sigma\left(\frac{n}{2}\right) + \left(\frac{1}{24} - \frac{n}{8}\right) \sigma\left(\frac{n}{5}\right) - \frac{1}{312} M(n-1) + \frac{31}{1560} T^*(n-2), \quad (5.9)$$

and

$$W_{10}(n) = \frac{1}{312} \sigma_3(n) + \frac{1}{78} \sigma_3\left(\frac{n}{2}\right) + \frac{25}{312} \sigma_3\left(\frac{n}{5}\right) + \frac{25}{78} \sigma_3\left(\frac{n}{10}\right) + \left(\frac{1}{24} - \frac{n}{40}\right) \sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right) \sigma\left(\frac{n}{10}\right) - \frac{31}{1560} M(n-1) + \frac{1}{312} T^*(n-2). \quad (5.10)$$

Proof Adding 4 times of (4.17) to (4.18), we have

$$-P_1 + 5P_5 = 4T \left(\frac{1}{v^2} + 4v^2 \right)^{\frac{1}{2}} \left(\frac{1}{v^2} + 8v^2 \right).$$

Squaring the above identity on both sides and then using (2.2), (2.3) and (2.4), we obtain

$$P_1^2 + 25P_5^2 - 10P_1P_5 = X_1, \quad (5.11)$$

where

$$X_1 = T^2 \left(3000u^3 + 120u - \frac{24}{u} - \frac{24}{u^3} + \left(\frac{1}{v^2} + 4v^2 \right) \left(\frac{40}{u^2} + 1000u^2 + 176 \right) \right).$$

Adding 8 times of (3.1) and 5 times of (3.3) and then using (2.3), we have

$$\frac{1}{13} (8Q_1 + 200Q_5) = X_1 - \frac{576T^2}{13v^2}. \quad (5.12)$$

Replacing q by q^5 in (2.9) and in (2.11) and substituting the resulting identities in the left hand side of (5.11) and using (2.9) and (2.11), we obtain

$$5760 \sum_{n=1}^{\infty} \left(\sum_{m < \frac{n}{5}} \sigma(m) \sigma(n-5m) \right) q^n = 16 + 25 \sum_{n=1}^{\infty} \left(240\sigma_3\left(\frac{n}{5}\right) - \frac{288}{5}n\sigma\left(\frac{n}{5}\right) \right) q^n + 240 \sum_{n=1}^{\infty} \sigma(n) q^n \\ + 240 \sum_{n=1}^{\infty} \sigma\left(\frac{n}{5}\right) q^n + \sum_{n=1}^{\infty} (240\sigma_3(n) - 288n\sigma(n)) q^n - X_1.$$

Using (5.12) in the above to eliminate X_1 and then using (2.10) and (5.1) and then equating the coefficients of q^n on both sides of the resulting identity, we obtain (5.8).

Subtracting 3 times of (4.20) from (4.19), we have

$$-2P_2 + 5P_5 = 3T \left(\frac{1}{v^2} + 4v^2 \right)^{\frac{1}{2}} \left(\frac{1}{w^3} - 3w^3 \right).$$

Squaring the above on both sides and then using (2.2) and (2.4), we obtain

$$4P_2^2 + 25P_5^2 - 20P_2P_5 = X_2, \quad (5.13)$$

where

$$X_2 = T^2 \left(\left(\frac{1}{v^2} + 4v^2 \right) \left(\frac{45}{u^2} + 1125u^2 + 306 \right) - 4500u^3 - \frac{36}{u^3} - 1980u - \frac{396}{u} \right).$$

From Theorem 3.1, (2.2) and (2.4), we have

$$\frac{1}{13} (-2Q_1 + 44Q_2 + 275Q_5 - 200Q_{10}) = X_2 - \frac{96T^2}{13} \left(\frac{5}{w^3} - 31w^3 \right). \quad (5.14)$$

Replacing q by q^2 in (2.9) and in (2.11) and q by q^5 in (2.9) and in (2.11) and substituting the resulting identities in the left hand side of (5.13), we find that

$$11520 \sum_{n=1}^{\infty} \left(\sum_{2l+5m=n} \sigma(l) \sigma(m) \right) q^n = 9 + 25 \sum_{n=1}^{\infty} \left(240\sigma_3\left(\frac{n}{5}\right) - \frac{288}{5}n\sigma\left(\frac{n}{5}\right) \right) q^n + 480 \sum_{n=1}^{\infty} \sigma\left(\frac{n}{2}\right) q^n \\ + 480 \sum_{n=1}^{\infty} \sigma\left(\frac{n}{5}\right) q^n + 4 \sum_{n=1}^{\infty} \left(240\sigma_3\left(\frac{n}{2}\right) - 144n\sigma\left(\frac{n}{2}\right) \right) q^n - X_2.$$

Using (5.14) in the above to eliminate X_2 and then using (2.10), (5.2) and (5.3) and then equating the coefficients of q^n on both sides of the resulting identity, we obtain (5.9).

Subtracting (4.20) from 3 times of (4.19), we have

$$-P_1 + 10P_{10} = 3T \left(\frac{1}{v^2} + 4v^2 \right)^{\frac{1}{2}} \left(\frac{3}{w^3} - w^3 \right).$$

Squaring the above on both sides and then using (2.2) and (2.4), we obtain

$$P_1^2 + 100P_{10}^2 - 20P_1P_{10} = X_3, \quad (5.15)$$

where

$$X_3 = T^2 \left(\left(\frac{1}{v^2} + 4v^2 \right) \left(\frac{45}{u^2} + 1125u^2 + 306 \right) + 4500u^3 + \frac{36}{u^3} + 1980u + \frac{396}{u} \right).$$

From Theorem 3.1, (2.2) and (2.4), we find that

$$\frac{1}{13} (11Q_1 - 8Q_2 - 50Q_5 + 1100Q_{10}) = X_3 - \frac{96T^2}{13} \left(\frac{31}{w^3} - 5w^3 \right). \quad (5.16)$$

Replacing q by q^{10} in (2.9) and in (2.11) and substituting the resulting identities in the left hand side of (5.4) and using (2.9) and (2.11), we obtain

$$\begin{aligned} 11520 \sum_{n=1}^{\infty} \left(\sum_{m < \frac{n}{10}} \sigma(m) \sigma(n-10m) \right) q^n &= 81 + 100 \sum_{n=1}^{\infty} \left(240\sigma_3 \left(\frac{n}{10} \right) - \frac{144}{5} n \sigma \left(\frac{n}{10} \right) \right) q^n \\ &\quad + 480 \sum_{n=1}^{\infty} \sigma(n) q^n + 480 \sum_{n=1}^{\infty} \sigma \left(\frac{n}{10} \right) q^n \\ &\quad + \sum_{n=1}^{\infty} (240\sigma_3(n) - 288n\sigma(n)) q^n - X_3. \end{aligned}$$

Using (5.16) in the above to eliminate X_3 and then using (2.10), (5.2) and (5.3) and then equating the coefficients of q^n on both sides of the resulting identity, we obtain (5.10). $\square$

Theorem 5.10 *We have*

$$\begin{aligned} W_7(n) &= \frac{1}{120} \sigma_3(n) + \frac{49}{120} \sigma_3 \left(\frac{n}{7} \right) + \left(\frac{1}{24} - \frac{n}{28} \right) \sigma(n) + \left(\frac{1}{24} - \frac{n}{4} \right) \sigma \left(\frac{n}{7} \right) \\ &\quad - \frac{1}{70} L(n-1) - \frac{2}{35} K(n-2), \end{aligned} \quad (5.17)$$

$$\begin{aligned} W_{(2,7)}(n) &= \frac{1}{600} \sigma_3(n) + \frac{1}{150} \sigma_3 \left(\frac{n}{2} \right) + \frac{49}{600} \sigma_3 \left(\frac{n}{7} \right) + \frac{49}{150} \sigma_3 \left(\frac{n}{14} \right) + \left(\frac{1}{24} - \frac{n}{28} \right) \sigma \left(\frac{n}{2} \right) \\ &\quad + \left(\frac{1}{24} - \frac{n}{8} \right) \sigma \left(\frac{n}{7} \right) + \frac{2}{175} K(n-2) - \frac{1}{600} A(n-1) - \frac{107}{4200} B(n-3), \end{aligned} \quad (5.18)$$

and

$$\begin{aligned} W_{14}(n) &= \frac{1}{600} \sigma_3(n) + \frac{1}{150} \sigma_3 \left(\frac{n}{2} \right) + \frac{49}{600} \sigma_3 \left(\frac{n}{7} \right) + \frac{49}{150} \sigma_3 \left(\frac{n}{14} \right) + \left(\frac{1}{24} - \frac{n}{56} \right) \sigma(n) \\ &\quad + \left(\frac{1}{24} - \frac{n}{4} \right) \sigma \left(\frac{n}{14} \right) + \frac{2}{175} K(n-2) - \frac{107}{4200} A(n-1) - \frac{1}{600} B(n-3). \end{aligned} \quad (5.19)$$

Proof Using (4.31) and the definition of y in (4.30), we obtain

$$-P_1 + 7P_7 = 6S \left(\frac{1}{V^3} + 16V^3 + 8 \right).$$

Squaring the above identity on both sides, we obtain

$$P_1^2 + 49P_7^2 - 14P_1P_7 = X_4, \quad (5.20)$$

where

$$X_4 = 36S^2 \left(\frac{1}{V^6} + 256V^6 + \frac{16}{V^3} + 256V^3 + 96 \right).$$

Adding (3.11) and 49 times of (3.13), we have

$$\frac{18}{25} (Q_1 + 49Q_7) = X_4 - \frac{576S^2}{5} \left(\frac{1}{V^3} + 4 \right). \quad (5.21)$$

Replacing q by q^7 in (2.9) and in (2.11) and substituting the resulting identities in the left hand side of (5.20) and using (2.9) and (2.11), we obtain

$$8064 \sum_{n=1}^{\infty} \left(\sum_{m < \frac{n}{7}} \sigma(m) \sigma(n-7m) \right) q^n = 36 + 49 \sum_{n=1}^{\infty} \left(240\sigma_3\left(\frac{n}{7}\right) - \frac{288}{7}n\sigma\left(\frac{n}{7}\right) \right) q^n + 336 \sum_{n=1}^{\infty} \sigma(n) q^n \\ + 336 \sum_{n=1}^{\infty} \sigma\left(\frac{n}{7}\right) q^n + \sum_{n=1}^{\infty} (240\sigma_3(n) - 288n\sigma(n)) q^n - X_4.$$

Using (5.21) in the above to eliminate X_4 and then using (2.10), (5.4) and (5.5) and then equating the coefficients of q^n on both sides of the resulting identity, we obtain (5.17).

Subtracting 3 times of (4.28) from (4.27), we have

$$-2P_2 + 7P_7 = S \left(\frac{9}{V^3} + 72V^3 + 48 - 4 \left(\frac{1}{W^4} - W^4 \right) \right).$$

Squaring the above on both sides and then using (2.6), we obtain

$$4P_2^2 + 49P_7^2 - 28P_2P_7 = X_5, \quad (5.22)$$

where

$$X_5 = S^2 \left(\frac{97}{V^6} + 6208V^6 + \frac{1088}{V^3} + 8704V^3 + 4576 \right) - S^2 \left(\frac{1}{W^4} - W^4 \right) \left(\frac{72}{V^3} + 576V^3 + 384 \right).$$

From Theorem 3.2 and (2.6), we have

$$\frac{1}{125} (-14Q_1 + 444Q_2 + 5439Q_7 - 2744Q_{14}) = X_5 - \frac{96S^2}{25} \left(\frac{7}{W^4} + 107W^4 - 48 \right). \quad (5.23)$$

Replacing q by q^2 in (2.9) and in (2.11) and q by q^7 in (2.9) and in (2.11) and substituting the resulting identities in the left hand side of (5.22), we find that

$$16128 \sum_{n=1}^{\infty} \left(\sum_{2l+7m=n} \sigma(l) \sigma(m) \right) q^n = 25 + 49 \sum_{n=1}^{\infty} \left(240\sigma_3\left(\frac{n}{7}\right) - \frac{288}{7}n\sigma\left(\frac{n}{7}\right) \right) q^n + 672 \sum_{n=1}^{\infty} \sigma\left(\frac{n}{2}\right) q^n \\ + 672 \sum_{n=1}^{\infty} \sigma\left(\frac{n}{7}\right) q^n + 4 \sum_{n=1}^{\infty} \left(240\sigma_3\left(\frac{n}{2}\right) - 144n\sigma\left(\frac{n}{2}\right) \right) q^n - X_5.$$

Using (5.23) in the above to eliminate X_5 and then using (2.10), (5.5), (5.6) and (5.7) and then equating the coefficients of q^n on both sides of the resulting identity, we obtain (5.18).

Subtracting (4.28) from 3 times of (4.27), we have

$$-P_1 + 14P_{14} = S \left(\frac{9}{V^3} + 72V^3 + 48 + 4 \left(\frac{1}{W^4} - W^4 \right) \right).$$

Squaring the above on both sides and then using (2.6), we obtain

$$P_1^2 + 196P_{14}^2 - 28P_1P_{14} = X_6, \quad (5.24)$$

where

$$X_6 = S^2 \left(\frac{97}{V^6} + 6208V^6 + \frac{1088}{V^3} + 8704V^3 + 4576 \right) + S^2 \left(\frac{1}{W^4} - W^4 \right) \left(\frac{72}{V^3} + 576V^3 + 384 \right).$$

From Theorem 3.2 and (2.6), we have

$$\frac{1}{125} (111Q_1 - 56Q_2 - 686Q_7 + 21756Q_{14}) = X_6 - \frac{96S^2}{25} \left(\frac{107}{W^4} + 7W^4 - 48 \right). \quad (5.25)$$

Replacing q by q^{14} in (2.9) and in (2.11) and substituting the resulting identities in the left hand side of (5.24) and using (2.9) and (2.11), we obtain

$$\begin{aligned} 16128 \sum_{n=1}^{\infty} \left(\sum_{m < \frac{n}{14}} \sigma(m) \sigma(n-14m) \right) q^n &= 169 + 196 \sum_{n=1}^{\infty} \left(240\sigma_3\left(\frac{n}{14}\right) - \frac{144}{7}n\sigma\left(\frac{n}{14}\right) \right) q^n \\ &\quad + 672 \sum_{n=1}^{\infty} \sigma(n) q^n + 672 \sum_{n=1}^{\infty} \sigma\left(\frac{n}{14}\right) q^n \\ &\quad + \sum_{n=1}^{\infty} (240\sigma_3(n) - 288n\sigma(n)) q^n - X_6. \end{aligned}$$

Using (5.25) in the above to eliminate X_6 and then using (2.10), (5.5), (5.6) and (5.7) and then equating the coefficients of q^n on both sides of the resulting identity, we obtain (5.19). $\square$

Let $\mathbb{Z}$ denote the set of integers and $x_i \in \mathbb{Z}$ for $1 \leq i \leq 8$. For $a, b, n \in \mathbb{N}$, let $T_{(a,b)}(n)$ be the number of representations of n by the form

$$(x_1^2 + x_1x_2 + ax_2^2 + x_3^2 + x_3x_4 + ax_4^2) + b(x_5^2 + x_5x_6 + ax_6^2 + x_7^2 + x_7x_8 + ax_8^2).$$

The identity and proof of the following theorem is similar to that of Ramakrishnan, Sahu and Singh [18], apart from the constant term, which we have represented using partition functions.

Theorem 5.11 *If $n \in \mathbb{N}$, then*

$$\begin{aligned} T_{(2,2)}(n) &= \frac{24}{25}\sigma_3(n) + \frac{96}{25}\sigma_3\left(\frac{n}{2}\right) + \frac{1176}{25}\sigma_3\left(\frac{n}{7}\right) + \frac{4705}{25}\sigma_3\left(\frac{n}{14}\right) \\ &\quad - \frac{64}{25}K(n-2) + \frac{76}{25}A(n-1) + \frac{76}{25}B(n-3). \end{aligned}$$

Proof For $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, let

$$s_{(a,4)}(k) = \text{card} \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid k = x_1^2 + x_1x_2 + ax_2^2 + x_3^2 + x_3x_4 + ax_4^2 \right\}.$$

Clearly $s_{(a,4)}(0) = 1$. From [6, Example 3, p.6], we have

$$1 + \sum_{j=1}^{\infty} s_{(2,4)}(j)q^j = \left(\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+mn+2n^2} \right)^2 = -\frac{1}{6}P_1 + \frac{7}{6}P_7.$$

(See [1, Entry 18.2.15, p.405], [3, Entry 5, p.467] for further details.) From the above identity, we have

$$s_{(2,4)}(k) = 4\sigma(k) - 28\sigma\left(\frac{k}{7}\right), \quad k \in \mathbb{N}. \quad (5.26)$$

We have

$$T_{(2,2)}(n) = \sum_{\substack{l, m \in \mathbb{Z} \\ l, m \geq 0 \\ l+2m=n}} s_{(2,4)}(l)s_{(2,4)}(m) = s_{(2,4)}(0)s_{(2,4)}\left(\frac{n}{2}\right) + s_{(2,4)}(n)s_{(2,4)}(0) + \sum_{\substack{l, m \in \mathbb{N} \\ l+2m=n}} s_{(2,4)}(l)s_{(2,4)}(m).$$

Using (5.26) in the above, we obtain

$$\begin{aligned} T_{(2,2)}(n) &= 4\sigma\left(\frac{n}{2}\right) - 28\sigma\left(\frac{n}{14}\right) + 4\sigma(n) - 28\sigma\left(\frac{n}{7}\right) + 16W_2(n) \\ &\quad - 112W_{(2,7)}(n) - 112W_{14}(n) + 784W_2\left(\frac{n}{7}\right). \end{aligned}$$

Using the convolution sum $W_2(n)$ from [14, Theorem 2] and employing (5.18) and (5.19) in the above, we complete the proof. $\square$

Ramanujan observed certain curious congruence properties of $p(n)$ (number of unrestricted partitions of n) as soon as he noticed the table calculated by Major MacMahon, of the values of $p(n)$, for all values of n from 1 to 200. He [23] found quite simple proofs of the following congruences of $p(n)$:

$$p(5n + 4) \equiv 0 \pmod{5}, \quad p(7n + 5) \equiv 0 \pmod{7} \quad \text{and} \quad p(35n + 19) \equiv 0 \pmod{35}.$$

Following the above many mathematicians proved interesting congruence properties of certain color and restricted color partition functions. We too have discovered some intriguing congruence properties of the seven restricted color partition functions introduced in this section. Theorems 5.12–5.18 seem to be new.

Theorem 5.12 *Let n be any non negative integer and let $P^*(n)$ be as in (5.1). Then*

$$P^*(4n + 1) \equiv 0 \pmod{2}, \tag{5.27}$$

$$P^*(4n + 2) \equiv 0 \pmod{2}, \tag{5.28}$$

$$P^*(4n + 3) \equiv 0 \pmod{2}, \tag{5.29}$$

$$P^*(5n + 1) \equiv 0 \pmod{2}, \tag{5.30}$$

$$P^*(5n + 2) \equiv 0 \pmod{2}, \tag{5.31}$$

$$P^*(2n + 1) \equiv 0 \pmod{4}, \tag{5.32}$$

and

$$P^*(5n + 4) \equiv 0 \pmod{5}. \tag{5.33}$$

Proof From (5.1), we have

$$\sum_{n=0}^{\infty} P^*(n)q^n = f_1^4 f_5^4. \tag{5.34}$$

From [3, p. 40, Entry 25, (v) and (vi)], we have

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - \frac{4q f_2^2 f_8^4}{f_4^2}. \tag{5.35}$$

Changing q to q^5 in the above and substituting the resulting identity along with (5.35) in the right hand side of (5.34), we find that

$$\sum_{n=0}^{\infty} P^*(n)q^n = \frac{f_4^{10} f_{20}^{10}}{f_2^2 f_8^4 f_{10}^2 f_{40}^4} - \frac{4q f_2^2 f_8^4 f_{20}^{10}}{f_4^2 f_{10}^2 f_{40}^4} - \frac{4q^5 f_4^{10} f_{10}^2 f_{40}^4}{f_2^2 f_8^4 f_{20}^2} + \frac{16q^6 f_2^2 f_8^4 f_{10}^2 f_{40}^4}{f_4^2 f_{20}^2}. \tag{5.36}$$

Extracting the terms involving q^{2n+1} from both sides of the above, we have

$$\sum_{n=0}^{\infty} P^*(2n + 1)q^{2n+1} = -4q \left(\frac{f_2^2 f_8^4 f_{20}^{10}}{f_4^2 f_{10}^2 f_{40}^4} + \frac{q^4 f_4^{10} f_{10}^2 f_{40}^4}{f_2^2 f_8^4 f_{20}^2} \right).$$

Dividing both sides of the above by q and then replacing q^2 by q , we deduce (5.32). For any positive integer k , it is easy to see that

$$f_k^2 \equiv f_{2k} \pmod{2}. \tag{5.37}$$

Using the above in (5.36), we have

$$\sum_{n=0}^{\infty} P^*(n)q^n \equiv \frac{f_4^9 f_{20}^9}{f_8^4 f_{40}^4} \pmod{2}.$$

Extracting the terms involving q^{4n+1} from both sides of the above congruence, dividing both sides by q and then replacing q^4 by q , we deduce (5.27). Similarly by extracting the terms involving q^{4n+2} and q^{4n+3} respectively

from both sides of the above congruence, dividing both sides by q^2 and q^3 respectively and then replacing q^4 by q , we deduce (5.28) and (5.29) respectively. Ramanujan [20] recorded the following identity:

$$f_1 = f_{25} \left(R(q^5) - q - \frac{q^2}{R(q^5)} \right), \quad \text{where } R(q) = \frac{(q^2; q^5)(q^3; q^5)}{(q; q^5)(q^4; q^5)}. \quad (5.38)$$

Using the above in (5.34), we find that

$$\sum_{n=0}^{\infty} P^*(n)q^n = f_5^4 f_{25}^4 \left(R^4(q^5) - 4qR^3(q^5) + 2q^2R^2(q^5) + 8q^3R(q^5) - 5q^4 \right. \\ \left. - \frac{8q^5}{R(q^5)} + \frac{2q^6}{R^2(q^5)} + \frac{4q^7}{R^3(q^5)} + \frac{q^8}{R^4(q^5)} \right).$$

Extracting the terms involving q^{5n+1} from both sides of the above and then dividing by q and replacing q^5 by q in the resulting identity, we obtain (5.30). Extracting the terms involving q^{5n+2} from both sides of the above and then dividing by q^2 and replacing q^5 by q in the resulting identity, we obtain (5.31). Extracting the terms involving q^{5n+4} from both sides of the above and then dividing by q^4 and replacing q^5 by q in the resulting identity, we obtain (5.33). $\square$

Ramanujan [20, p. 212, Identity (14)] also recorded the following identity:

$$\frac{1}{f_1} = \frac{f_{25}^5}{f_5^6} \left(R^4(q^5) + qR^3(q^5) + 2q^2R^2(q^5) + 3q^3R(q^5) + 5q^4 - \frac{3q^5}{R(q^5)} \right. \\ \left. + \frac{2q^6}{R^2(q^5)} - \frac{q^7}{R^3(q^5)} + \frac{q^8}{R^4(q^5)} \right). \quad (5.39)$$

From [3, p. 262, Entry 10, (i) and (ii)], we have

$$\frac{f_2^2}{f_1} = \psi(q) = f(q^{10}, q^{15}) + qf(q^5, q^{20}) + q^3 \frac{f_{50}^2}{f_{25}} \quad (5.40)$$

and

$$\frac{f_1^2}{f_2} = \varphi(-q) = -2qf(-q^{15}, -q^{35}) + 2q^4f(-q^5, -q^{45}) + \frac{f_{25}^2}{f_{50}}. \quad (5.41)$$

From [3, p. 303, Entry 17, (iii) and (iv)], we have

$$\frac{f_1^2}{f_2} = \varphi(-q) = -2qf(-q^{35}, -q^{63}) + 2q^4f(-q^{21}, -q^{77}) - 2q^9f(-q^7, -q^{91}) + \frac{f_{49}^2}{f_{98}} \quad (5.42)$$

and

$$\frac{f_2^2}{f_1} = \psi(q) = f(q^{21}, q^{28}) + qf(q^{14}, q^{35}) + q^3f(q^7, q^{42}) + q^6 \frac{f_{98}^2}{f_{49}}. \quad (5.43)$$

From [3, p. 40, Entry 25, (i) and (ii)], we have

$$f_1^2 = \frac{f_2 f_8^5}{f_4 f_{16}^2} - 2q \frac{f_2 f_{16}^2}{f_8} \quad \text{and} \quad \frac{1}{f_1} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q \frac{f_4 f_{16}^2}{f_2^5 f_8}. \quad (5.44)$$

Using appropriate identities from (5.35), (5.38) and from (5.39) - (5.44), one may easily prove the following theorems:

Theorem 5.13 For all integer $n \geq 0$, we have

$$M(5n+4) \equiv 0 \pmod{5},$$

where $M(n)$ is as in (5.2).

Theorem 5.14 *Let n be any non negative integer and let $T^*(n)$ be as in (5.3). Then*

$$T^*(5n + 2) \equiv 0 \pmod{2}, \quad T^*(5n + 4) \equiv 0 \pmod{2}, \quad T^*(2n + 1) \equiv 0 \pmod{5},$$

and

$$T^*(5n + 3) \equiv 0 \pmod{5}.$$

Theorem 5.15 *For all integer $n \geq 0$, we have*

$$L(7n + 2) \equiv 0 \pmod{2}, \quad L(7n + 4) \equiv 0 \pmod{2}, \quad \text{and} \quad L(7n + 5) \equiv 0 \pmod{2},$$

where $L(n)$ is as in (5.4).

Theorem 5.16 *Let n be any non negative integer and let $K(n)$ be as in (5.5). Then*

$$K(2n + 1) \equiv 0 \pmod{2}, \quad K(7n + 1) \equiv 0 \pmod{2}, \quad K(7n + 3) \equiv 0 \pmod{2},$$

and

$$K(7n + 4) \equiv 0 \pmod{2}.$$

Theorem 5.17 *For all integer $n \geq 0$, we have*

$$A(2n + 1) \equiv 0 \pmod{2},$$

where $A(n)$ is as in (5.6).

Theorem 5.18 *Let n be any non negative integer and let $B(n)$ be as in (5.7). Then*

$$B(2n + 1) \equiv 0 \pmod{2}.$$

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