Arab. J. Math. (2022) 11:241–259
https://doi.org/10.1007/s40065-021-00356-8

Arabian Journal of Mathematics

György Gát · Ushangi Goginava

Almost everywhere convergence and divergence of Cesàro means with varying parameters of Walsh–Fourier series

Received: 22 May 2021 / Accepted: 7 December 2021 / Published online: 29 December 2021

Abstract In the present paper, we prove the almost everywhere convergence and divergence of subsequences of Cesàro means with zero tending parameters of Walsh–Fourier series.

Mathematics Subject Classification 42C10

1 Introduction

We denote the set of non-negative integers by $\mathbb{N}$. By a dyadic interval in $\mathbb{I} := [0, 1)$, we mean one of the form $I(l,k) := \left[\frac{l}{2^k}, \frac{l+1}{2^k}\right)$ for some $k \in \mathbb{N}$, $0 \le l < 2^k$. Given $k \in \mathbb{N}$ and $x \in [0,1)$, let $I_k(x)$ denote the dyadic interval of length 2^{-k} which contains the point x. Also, use the notation $I_n := I_n(0)\,(n \in \mathbb{N})$, $\overline{I}_k(x) := \mathbb{I} \backslash I_k(x)$. Let

$$x = \sum_{n=0}^{\infty} x_n 2^{-(n+1)}$$

be the dyadic expansion of $x \in \mathbb{I}$, where $x_n = 0$ or 1, and if x is a dyadic rational number, we choose the expansion which terminate in $0'$s. We also use the following notation:

$$I_k(x) = I_k(x_0, x_1, ..., x_{k-1}).$$

For any given $n \in \mathbb{N}$, it is possible to write n uniquely as

$$n = \sum_{k=0}^{\infty} \varepsilon_k(n) 2^k,$$

where $\varepsilon_k(n) = 0$ or 1 for $k \in \mathbb{N}$. This expression will be called the binary expansion of n and the numbers $\varepsilon_k(n)$ will be called the binary coefficients of n. Denote for $1 \le n \in \mathbb{N}$, $|n| := \max\{j \in \mathbb{N} : \varepsilon_j(n) \neq 0\}$, that is $2^{|n|} \le n < 2^{|n|+1}$.

G. Gát
Institute of Mathematics, University of Debrecen, Pf. 400, 4002 Debrecen, Hungary
E-mail: gat.gyorgy@science.unideb.hu

U. Goginava (✉)
Department of Mathematical Sciences, United Arab Emirates University, P.O. Box No. 15551, Al Ain, Abu Dhabi, UAE
E-mail: zazagoginava@gmail.com; ugoginava@uaeu.ac.ae

Set the definition of the nth ($n \in \mathbb{N}$) Walsh–Paley function at point $x \in \mathbb{I}$ as

$$w_n(x) = (-1)^{\sum\limits_{j=0}^{\infty} \varepsilon_j(n) x_j}.$$

Denote by $\dotplus$ the logical addition on $\mathbb{I}$. That is, for any $x, y \in \mathbb{I}$ and $k, n \in \mathbb{N}$

$$x \dotplus y := \sum_{n=0}^{\infty} |x_n - y_n| \, 2^{-(n+1)}.$$

Define the binary operator $\oplus : \mathbb{N}\times\mathbb{N}\to\mathbb{N}$ by

$$k \oplus n = \sum_{i=0}^{\infty} |\varepsilon_i(k) - \varepsilon_i(n)| \, 2^i. \tag{1}$$

It is well known (see, e.g., [13], p. 5) that

$$w_{m\oplus n}(x) = w_m(x)\, w_n(x), \; x \in [0, 1), n, m \in \mathbb{N}. \tag{2}$$

The Walsh–Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Set

$$D_n^* := w_n D_n.$$

Recall that [9,13]

$$D_{2^n}(x) = 2^n \chi_{I_n}(x), \tag{3}$$

where χ_E is the characteristic function of the set E.

Dyadic shift transformations of a function on the unit interval $\mathbb{I}$ will be denoted by $\tau_y f$ and it will be defined as

$$\left(\tau_y f\right)(x) := f(x \dotplus y) \quad (x \in \mathbb{I}).$$

The Fejér kernel of Walsh–Fourier series defined by

$$K_n(x) = \frac{1}{n} \sum_{j=0}^{n-1} D_j(x).$$

The partial sums of the Walsh–Fourier series are defined as follows:

$$S_m(f, x) = \sum_{j=0}^{m-1} \widehat{f}(j)\, w_j(x),$$

where the number

$$\widehat{f}(j) = \int_{\mathbb{I}} f w_j$$

is said to be the jth Walsh–Fourier coefficient of the function f.

The space $L_1(\mathbb{I})$ is defined by $\{f : \mathbb{I} \to \mathbb{R} : \|f\|_1 < \infty\}$, where

$$\|f\|_1 := \int_{\mathbb{I}} |f(x)|\, dx .$$

The space weak-$L_1(\mathbb{I})$ consists of all (Lebesgue) measurable functions f for which

$$\|f\|_{\text{weak}-L_1(\mathbb{I})} := \sup_{\lambda>0} \lambda \text{mes}\,(|f| > \lambda) < +\infty.$$

Let $f \in L_1(\mathbb{I})$. Then, the maximal function given by

$$E^*(f, x) = \sup_{n\in\mathbb{N}} \frac{1}{|I_n(x)|} \left| \int_{I_n(x)} f(u)\, du \right|, \ x \in \mathbb{I}.$$

For each $n \in \mathbb{N}$, let $\mathcal{A}_n$ represent the σ -algebra generated by the collection of dyadic intervals $\{I(k, n) : k = 0, 1, ..., 2^n - 1\}$. Thus, every element of $\mathcal{A}_n$ is a finite union of intervals of the form $\left[k2^{-n}, (k+1)\,2^{-n}\right)$ or an empty set.

Let $L(\mathcal{A}_n)$ represent the collection of $\mathcal{A}_n$-measurable functions on $\mathbb{I}$. By the Paley Lemma (see [13], Ch. 1, p. 12), $L(\mathcal{A}_n)$ coincides with the collection of Walsh polynomials of order less than 2^n.

A sequence of functions $(f_n : n \in \mathbb{N})$ is called a dyadic martingale if each f_n belongs to $L(\mathcal{A}_n)$ and

$$\int_E f_{n+1} = \int_E f_n \,(E \in \mathcal{A}_n, n \in \mathbb{N}) .$$

It is clear that the 2^nth partial sums of any Walsh series is a dyadic martingale. Conversely, it is easy to see that every dyadic martingale can be obtained in this way. Thus investigation of 2^nth partial sums of Walsh series leads to the study of dyadic martingales. It is well known that $(f_n : n \in \mathbb{N})$ is dyadic martingale if and only if $f_n \in L(\mathcal{A}_n)$ and

$$S_{2^n}(f_{n+1}) = f_n \quad (n \in \mathbb{N}) .$$

A martingale $(f_n : n \in \mathbb{N})$ will be called regular if there is an integrable function f, such that $f_n = S_{2^n}(f)$ for all $n \in \mathbb{N}$.

Let $\mathbf{A}$ denote the collection of sequences $\beta := \{\beta_n : n \in \mathbb{N}\}$ which satisfy $\beta_n \in L(\mathcal{A}_n)$ for $n \in \mathbb{N}$ and

$$\|\beta\| := \sup_{n\in\mathbb{N}} \|\beta_n\|_\infty < \infty.$$

For a given $\beta \in \mathbf{A}$ and $f \in L_1(\mathbb{I})$, the martingale transform of f is defined by

$$\mathbf{T}(\beta) f := \sum_{n=0}^{\infty} \beta_n \Delta_n f,$$

where $\Delta_n f := S_{2^{n+1}}(f) - S_{2^n}(f)$ for $n \in \mathbb{N}$. The maximal martingale transform is defined by

$$\mathbf{T}^*(\beta) f := \sup_{N\in\mathbb{N}} \left| \sum_{n=0}^{N} \beta_n \Delta_n f \right|.$$

In fact, we will use the following theorem (see [13], Ch. 3, Theorem 4; see more details in [16]).

Theorem MT *There exists an absolute constant c, such that*

$$\lambda mes\left(\left\{\mathbf{T}^*(\beta) f > \lambda\right\}\right) \le c\, \|\beta\|\, \|f\|_1$$

for all $f \in L_1(\mathbb{I})$, $\lambda > 0$, *and* $\beta \in \mathbf{A}$.

The (C, α_n) means of the Walsh–Fourier series of the function f is given by

$$\sigma_n^{\alpha_n}(f, x) = \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=1}^{n} A_{n-j}^{\alpha_n - 1} S_j(f, x) = \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=0}^{n-1} A_{n-1-j}^{\alpha_n} \widehat{f}(j)\, w_j(x),$$

where

$$A_n^{\alpha_n} := \frac{(1+\alpha_n)\ldots(n+\alpha_n)}{n!}$$

for any $n \in \mathbb{N}$, $\alpha_n \neq -1, -2, \ldots$. It is known that [20]

$$A_n^{\alpha_n} = \sum_{k=0}^{n} A_k^{\alpha_n - 1},\ A_n^{\alpha_n - 1} = \frac{\alpha_n}{\alpha_n + n} A_n^{\alpha_n}. \tag{4}$$

The (C, α_n) kernel is defined by

$$K_n^{\alpha_n} = \frac{1}{A_{n-1}^{\alpha}} \sum_{j=1}^{n} A_{n-j}^{\alpha_n - 1} D_j = \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=0}^{n-1} A_{n-j-1}^{\alpha_n} w_j.$$

The idea of Cesàro means with variable parameters of numerical sequences is due to Kaplan [11] and the introduction of these (C, α_n) means of Fourier series is due to Akhobadze ([1], [2]) who investigated the behavior of the L_1-norm convergence of $\sigma_n^{\alpha_n}(f) \to f$ for the trigonometric system.

The first result with respect to the a.e. convergence of the Walsh–Fejér means $\sigma_n^{\alpha_n}(f)$ for all integrable function f with constant sequence $\alpha_n = \alpha > 0$ is due to Fine [4] (see also Weisz [17]). On the rate of convergence of Cesà ro means in this constant case, see the paper of Yano [19], Fridli [5]. Approximation properties of Cesàro means of negative order with constant sequence were investigated by the second author [8].

For $n := \sum_{i=0}^{\infty} \varepsilon_i(n) 2^i$ ($\varepsilon_i(n) = 0, 1, i \in \mathbb{N}$), set the two variable function

$$P(n, \alpha) := \sum\nolimits_{i=0}^{\infty} \varepsilon_i(n) 2^{i\alpha_n} \ \ (n \in \mathbb{N}),\ \alpha := \{\alpha_n : n \in \mathbb{N}\}.$$

The function $P(n, \alpha)$ was introduced by Abu Joudeh and Gát in [10]. Also, set for sequence $\alpha := \{\alpha_n : n \in \mathbb{N}\}$ and positive reals K the subset of natural numbers

$$P_K(\alpha) := \left\{ n \in \mathbb{N} : \frac{P(n, \alpha)}{n^{\alpha_n}} \leq K \right\}.$$

Under some conditions on $\{\alpha_n : n \in \mathbb{N}\}$, Abu Joudeh and Gàt in [10] proved the almost everywhere convergence of the Cesàro (C, α_n) means of integrable functions. In particular, the following is proved.

Theorem JG *Suppose that $\alpha_n \in (0, 1)$. Let $f \in L_1(\mathbb{I})$. Then, we have the almost everywhere convergence $\sigma_n^{\alpha_n}(f) \to f$ provided that $P_K(\alpha) \ni n \to \infty$.*

The definition of the variation of an $n \in \mathbb{N}$ with binary coefficients

$$(\varepsilon_k(n) : k \in \mathbb{N})$$

was introduced in [13] by

$$V(n) := \sum_{i=0}^{\infty} |\varepsilon_i(n) - \varepsilon_{i+1}(n)|.$$

In this paper, we define the weighted version of variation of an $n \in \mathbb{N}$ with binary coefficients $(\varepsilon_k(n) : k \in \mathbb{N})$ by

$$V(n, \alpha) := \sum_{i=0}^{\infty} |\varepsilon_i(n) - \varepsilon_{i+1}(n)| \, 2^{i\alpha_n} \ \ (n \in \mathbb{N}).$$

Set for sequence $\alpha := \{\alpha_n : n \in \mathbb{N}\}$ and positive reals K the subset of natural numbers

$$V_K(\alpha) := \left\{ n \in \mathbb{N} : \frac{V(n,\alpha)}{n^{\alpha_n}} \leq K < \infty \right\}.$$

It is easy to see that $P_K(\alpha) \subsetneq V_{2K}(\alpha)$. On the other hand, if $\alpha_n \to 0$, then there exists K, such that $2^n - 1 \in V_K(\alpha)$ for all n, but there does not exist K, such that $2^n - 1 \in P_K(\alpha)$ for all n. In this paper, we are going to improve Theorem JG and to replace the condition $P_K(\alpha) \ni n \to \infty$ by the condition $V_K(\alpha) \ni n \to \infty$. In particular, the following will be proved.

Theorem 1.1 *Suppose that $\alpha_n \in (0,1)$. Let $f \in L_1(\mathbb{I})$. Then, we have the almost everywhere convergence $\sigma_n^{\alpha_n}(f) \to f$ provided that $V_K(\alpha) \ni n \to \infty$.*

From the proof of Theorem 1.1, we can obtain pointwise growth of Ces àro means with varying parameters of Walsh–Fourier series. The following is true.

Theorem 1.2 *Let $f \in L_1(\mathbb{I})$ and*

$$\lim_{n\to\infty} \frac{V(n,\alpha)}{n^{\alpha_n}} = \infty.$$

Then, we have the almost everywhere convergence

$$\lim_{n\to\infty} \frac{n^{\alpha_n}\sigma_n^{\alpha_n}(f,x)}{V(n,\alpha)} = 0.$$

Let $\lim_{n\to\infty} \alpha_n = 0$. We investigate two cases:
a) $\lim_{n\to\infty} (\alpha_n \log n) > 0$ and b) $\lim_{n\to\infty} (\alpha_n \log n) = 0$. For case a), we have

$$\frac{V(n,\alpha)}{n^{\alpha_n}} \leq \frac{c}{2^{|n|\alpha_n}} \sum_{i=0}^{|n|} 2^{i\alpha_n} \leq c\alpha_n^{-1},$$

and for case b), we obtain

$$\frac{V(n,\alpha)}{n^{\alpha_n}} \leq \frac{c}{2^{|n|\alpha_n}} \sum_{i=0}^{|n|} 2^{i\alpha_n} \leq \frac{c|n|2^{|n|\alpha_n}}{2^{|n|\alpha_n}} \leq c|n|.$$

Hence, from Theorem 1.2, we get the following.

Corollary 1.3 *Let $f \in L_1(\mathbb{I})$ and*

$$\lim_{n\to\infty} \alpha_n = 0.$$

Then, we have the almost everywhere convergence:
(a) If $\lim_{n\to\infty} (\alpha_n \log n) > 0$, then $\lim_{n\to\infty} \left(\alpha_n \sigma_n^{\alpha_n}(f,x)\right) = 0$;
(b) If $\lim_{n\to\infty} (\alpha_n \log n) = 0$, then $\lim_{n\to\infty} \frac{\sigma_n^{\alpha_n}(f,x)}{\log n} = 0$.

Theorem 1.4 *Let $f \in L_1(\mathbb{I})$ and $\alpha_n \in (0,1)$. Then, the operator $\sigma_n^{\alpha_n}(f)$ is of weak type (L_1, L_1).*

Theorem 1.4 imply

Corollary 1.5 *Let $f \in L_1(\mathbb{I})$ and $\alpha_n \in (0,1)$. Then, $\sigma_n^{\alpha_n}(f) \to f$ in measure as $n \to \infty$.*

Theorem 1.6 *Let $f \in L_1(\mathbb{I})$. Then, there exists a sequence $\mu_j(f)$, such that for each subsequence of natural numbers with $n_j \geq \mu_j(f)$, we have the a. e. relation*

$$\sigma_{n_j}^{\alpha_{n_j}}(f) \to f.$$

For the subsequence of the partial sums, we are going to prove the following.

Theorem 1.7 *For each sequence of natural numbers $\nu_j \uparrow \infty$, there exists a function $f \in L_1(\mathbb{I})$ and an another sequence of natural numbers with $N_j \geq \nu_j$ for which we have the everywhere divergence of $S_{N_j}(f)$.*

The a. e. divergence of Cesàro means with varying parameters of Walsh–Fourier series was investigated by Tetunashvili [14]. In particular, the following is proved: Assume that $\{\alpha_n\}$ is such that for a positive number n_0, we have

$$\alpha_n \leq \frac{c}{\log_2 n}, 0 \leq c < 1, n > n_0. \tag{5}$$

Then, there exists such a function f that the sequence $\sigma_n^{\alpha_n}(f)$ diverges everywhere unboundedly.

In this paper, we improve this theorem of Tetunashvili (5) in a way that we enlarge the set of sequences (α_n) for which we have divergence results of the Cesàro means with variable parameters. In particular, the following is true.

Theorem 1.8 *Assume that $\{\alpha_n\}$ is such that for some positive integer n_0, we have*

$$\frac{c_1}{\log_2 n} \leq \alpha_n \leq \frac{c_0 \log_2 \log_2 n}{\log_2 n}, 0 \leq c_0 < \frac{1}{2}, n > n_0.$$

Then, there exists a integrable function f that the sequence $\sigma_n^{\alpha_n}(f)$ diverges almost everywhere unboundedly.

The boundedness of maximal operators of subsequences of $(C, \alpha_n)-$ means of partial sums of Walsh–Fourier series from the Hardy space H_p into the space L_p is studied in [7]. In particular, the following is proved.

Theorem GG *Let $p > 0$. Then, there exists a positive constant c_p, such that*

$$\left\| \sup_{N\in\mathbb{N}} \left| f * \left| K_{2^N}^{\alpha_N} \right| \right| \right\|_p \leq c_p \|f\|_{H_p} \quad \left(f \in H_p\right).$$

Weisz [18] generalized Theorem GG for both the Cesàro and Riesz means by taking the supremum over all indices $n \in \mathbb{N}_v$. Here, $\mathbb{N}_v$ denotes the set of all $n = 2^{n_1} + \cdots + 2^{n_v}$ with a fixed parameter v. In particular, the following is proved.

Theorem W (Weisz [18]) *Let $p > 0$. Then, there exists a positive constant c_p, such that*

$$\left\| \sup_{n\in P_K(\alpha)} \left| f * K_n^{\alpha_n} \right| \right\|_p \leq c_p \, \||f|\|_{H_p} \quad \left(|f| \in H_p\right).$$

2 Auxiliary results

We shall need the following.

Lemma 2.1 *Let $k, n \in \mathbb{N}$. Then*

$$c_1(1+\alpha_n)(2+\alpha_n)k^{\alpha_n} < A_k^{\alpha_n} < c_2(1+\alpha_n)(2+\alpha_n)k^{\alpha_n}, -2 < \alpha_n < -1;$$
$$c_1(1+\alpha_n)k^{\alpha_n} < A_k^{\alpha_n} < c_2(1+\alpha_n)k^{\alpha_n}, -1 < \alpha_n < 0;$$
$$c_1(d)k^{\alpha_n} < A_k^{\alpha_n} < c_2(d)k^{\alpha_n}, 0 < \alpha_n \leq d.$$

The proof can be found in the paper of Akhobadze [1].

Set

$$n^{(s)} := \sum_{j=s}^{\infty} \varepsilon_j(n) 2^j, n_{(s)} = n - n^{(s+1)} = \sum_{j=0}^{s} \varepsilon_j(n) 2^j.$$

Lemma 2.2 *Let* $\alpha_n \in (0,1)$, $1 \le n \in \mathbb{N}$. *Then, we have*

$$\begin{aligned} K_n^{\alpha_n} &= \frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=0}^{|n|} \varepsilon_s(n) w_{n^{(s)}-1} \sum_{j=1}^{2^s-1} A_{n_{(s-1)}+j}^{\alpha_n-2} j K_j \\ &\quad -\frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=0}^{|n|} \varepsilon_s(n) w_{n^{(s)}-1} A_{n_{(s)}-1}^{\alpha_n-1} 2^s K_{2^s} \\ &\quad +\frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=0}^{|n|} \varepsilon_s(n) w_{n^{(s)}-1} A_{n_{(s)}-1}^{\alpha_n} D_{2^s} \\ &=: T_n^{(1)} + T_n^{(2)} + T_n^{(3)}. \end{aligned} \tag{6}$$

Proof of Lemma 2.2 We can write

$$\begin{aligned} A_{n-1}^{\alpha_n} K_n^{\alpha_n} &= \sum_{j=0}^{n-1} A_{n-j-1}^{\alpha_n} w_j = \sum_{s=0}^{|n|} \varepsilon_s(n) \sum_{j=n^{(s+1)}}^{n^{(s)}-1} A_{n-j-1}^{\alpha_n} w_j \\ &= \sum_{s=0}^{|n|} \varepsilon_s(n) \sum_{j=0}^{2^s-1} A_{n_{(s)}-j-1}^{\alpha_n} w_{j+n^{(s+1)}} \\ &= \sum_{s=0}^{|n|} \varepsilon_s(n) w_{n^{(s+1)}} \sum_{j=0}^{2^s-1} A_{n_{(s)}-j-1}^{\alpha_n} w_j. \end{aligned}$$

Since

$$n_{(s)} - j - 1 = n_{(s-1)} + 2^s - 1 - j, \varepsilon_s(n) = 1,$$

(otherwise nothing to be investigated here) and

$$2^s - 1 - j = \left(2^s - 1\right) \oplus j$$

from (2), we obtain

$$\begin{aligned} A_{n-1}^{\alpha_n} K_n^{\alpha_n} &= \sum_{s=0}^{|n|} \varepsilon_s(n) w_{n^{(s+1)}} \sum_{j=0}^{2^s-1} A_{n_{(s-1)}+j}^{\alpha_n} w_{(2^s-1)\oplus j} \\ &= \sum_{s=0}^{|n|} \varepsilon_s(n) w_{n^{(s)}-1} \sum_{j=0}^{2^s-1} A_{n_{(s-1)}+j}^{\alpha_n} w_j. \end{aligned} \tag{7}$$

Applying Abel's transformation (twice), we get

$$\sum_{j=0}^{2^s-1} A_{n_{(s-1)}+j}^{\alpha_n} w_j = \sum_{j=1}^{2^s-1} A_{n_{(s-1)}+j}^{\alpha_n-2} j K_j - A_{n_{(s)}-1}^{\alpha_n-1} 2^s K_{2^s} + A_{n_{(s)}-1}^{\alpha_n} D_{2^s}.$$

Hence, from (7), we conclude (6). □

From (4), we can write

$$\left|T_n^{(1)}\right| \le \frac{2}{A_{n-1}^{\alpha_n}} \sum_{s=0}^{|n|} \varepsilon_s(n) \sum_{j=1}^{2^s-1} A_{n_{(s-1)}+j}^{\alpha_n-1} \left|K_j\right| := \widetilde{T}_n^{(1)}.$$

Lemma 2.3 *Let $\alpha_n \in (0,1)$, $n \in \mathbb{N}$ and $f \in L_1(\mathbb{I})$, such that $supp(f) \subset I_N(u')$, $\int_{I_N(u')} f = 0$ for some dyadic interval $I_N(u')$. Then, we have*

$$\int_{\overline{I}_N(u')} \sup_{n\in\mathbb{N}} \left| f * \widetilde{T}_n^{(1)} \right| \leq c \left\| f \right\|_1 .$$

Proof of Lemma 2.3 Let $n \leq 2^N$. From the condition of the lemma, it is easy to see that $f * \widetilde{T}_n^{(1)} = 0$. Hence, we can suppose that $n > 2^N$. Without lost of generality, we may assume that $u' = 0$. It is easy to see that

$$\begin{aligned}
f * \left(\frac{\widetilde{T}_n^{(1)}}{2}\right) &= f * \left(\frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=0}^{|n|} \varepsilon_s(n) \sum_{j=1}^{2^s-1} A_{n_{(s-1)}+j}^{\alpha_n-1} \left|K_j\right|\right) \\
&= \int_{I_N} f(u) \frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=0}^{|n|} \varepsilon_s(n) \sum_{j=1}^{2^s-1} A_{n_{(s-1)}+j}^{\alpha_n-1} \left|K_j(x \dotplus u)\right| du \\
&= \int_{I_N} f(u) \frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=0}^{N} \varepsilon_s(n) \sum_{j=1}^{2^s-1} A_{n_{(s-1)}+j}^{\alpha_n-1} \left|K_j(x \dotplus u)\right| du \\
&\quad + \int_{I_N} f(u) \frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=N+1}^{|n|} \varepsilon_s(n) \sum_{j=1}^{2^N-1} A_{n_{(s-1)}+j}^{\alpha_n-1} \left|K_j(x \dotplus u)\right| du \\
&\quad + \int_{I_N} f(u) \frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=N+1}^{|n|} \varepsilon_s(n) \sum_{j=2^N}^{2^s-1} A_{n_{(s-1)}+j}^{\alpha_n-1} \left|K_j(x \dotplus u)\right| du \\
&= \frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=0}^{N} \varepsilon_s(n) \sum_{j=1}^{2^s-1} A_{n_{(s-1)}+j}^{\alpha_n-1} \left|K_j(x)\right| \int_{I_N} f(u)\, du \\
&\quad + \frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=N+1}^{|n|} \varepsilon_s(n) \sum_{j=1}^{2^N-1} A_{n_{(s-1)}+j}^{\alpha_n-1} \left|K_j(x)\right| \int_{I_N} f(u)\, du \\
&\quad + \int_{I_N} f(u) \frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=N+1}^{|n|} \varepsilon_s(n) \sum_{j=2^N}^{2^s-1} A_{n_{(s-1)}+j}^{\alpha_n-1} \left|K_j(x \dotplus u)\right| du \\
&= \int_{I_N} f(u) \frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=N+1}^{|n|} \varepsilon_s(n) \sum_{j=2^N}^{2^s-1} A_{n_{(s-1)}+j}^{\alpha_n-1} \left|K_j(x \dotplus u)\right| du.
\end{aligned}$$

It is easy to see from (4) and Lemma 2.1 that

$$\begin{aligned}
&\frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=1}^{|n|} \varepsilon_s(n) \sum_{j=1}^{2^s-1} A_{n_{(s-1)}+j}^{\alpha_n-1} \\
&= \frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=1}^{|n|} \varepsilon_s(n) \sum_{j=n_{(s-1)}+1}^{n_{(s)}-1} A_j^{\alpha_n-1} \\
&= \frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=1}^{|n|} \varepsilon_s(n) \left(A_{n_{(s)}-1}^{\alpha_n} - A_{n_{(s-1)}}^{\alpha_n}\right)
\end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=1}^{|n|} \left(A_{n_{(s)}}^{\alpha_n} - A_{n_{(s-1)}}^{\alpha_n} \right) \\ &< \frac{A_{n(|n|)}}{A_{n-1}^{\alpha_n}} \leq c. \end{aligned} \tag{8}$$

Set

$$K_{2^N}^* := \sup_{n \geq 2^N} |K_n| .$$

It is proved in [6] that

$$\int_{\overline{I}_N} K_{2^N}^* \leq c < \infty, N \in \mathbb{N}.$$

Then, from (8), we have

$$\begin{aligned} &\int_{\overline{I}_N} \sup_{n \geq 2^N} \frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=N+1}^{|n|} \sum_{j=2^N}^{2^s-1} A_{n_{(s-1)}+j}^{\alpha_n-1} \left| K_j(u) \right| du \\ &\leq \sup_{n \geq 2^N} \frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=N+1}^{|n|} \sum_{j=2^N}^{2^s-1} A_{n_{(s-1)}+j}^{\alpha_n-1} \int_{\overline{I}_N} K_{2^N}^* \\ &\leq c \sup_{n \geq 2^N} \frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=N+1}^{|n|} \sum_{j=2^N}^{2^s-1} A_{n_{(s-1)}+j}^{\alpha_n-1} \leq c < \infty. \end{aligned}$$

Consequently

$$\begin{aligned} &\int_{\overline{I}_N} \sup_{n \in \mathbb{N}} \left| f * \widetilde{T}_n^{(1)} \right| \\ &= \int_{\overline{I}_N} \sup_{n \geq 2^N} \left| \int_{I_N} f(u) \frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=N+1}^{|n|} \varepsilon_s(n) \sum_{j=2^N}^{2^s-1} A_{n_{(s-1)}+j}^{\alpha_n-1} \left| K_j(x \dotplus u) \right| du \right| \\ &\leq \int_{\overline{I}_N} \left(\int_{I_N} |f(u)| \sup_{n \geq 2^N} \frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=N+1}^{|n|} \sum_{j=2^N}^{2^s-1} A_{n_{(s-1)}+j}^{\alpha_n-1} \left| K_j(x \dotplus u) \right| du \right) dx \\ &= \int_{I_N} |f(u)| \left(\int_{\overline{I}_N} \sup_{n \geq 2^N} \frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=N+1}^{|n|} \sum_{j=2^N}^{2^s-1} A_{n_{(s-1)}+j}^{\alpha_n-1} \left| K_j(x \dotplus u) \right| dx \right) du \\ &\leq c \, \|f\|_1 . \end{aligned}$$

This completes the proof of Lemma 2.3. □

Lemma 2.4 *The operator* $\sup_{n \in \mathbb{N}} \left| f * \widetilde{T}_n^{(1)} \right|$ *is of type* (L_∞, L_∞).

Proof of Lemma 2.4 Since (see [13]) $\sup_n \|K_n\|_1 < 2$ from (8) (or even see [15] $\sup_n \|K_n\|_1 \leq 17/15$), we have

$$\sup_{n \in \mathbb{N}} \left\| \widetilde{T}_n^{(1)} \right\|_1 \leq \sup_{n \in \mathbb{N}} \frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=0}^{|n|} \varepsilon_s(n) \sum_{j=1}^{2^s-1} A_{n_{(s-1)}+j}^{\alpha_n-1} \leq c < \infty,$$

which implies the boundedness of operator $\sup_{n\in\mathbb{N}} \left| f * \widetilde{T}_n^{(1)} \right|$ from the space L_∞ to the space L_∞. □

Combine Lemmas 2.3 and 2.4 to have the following.

Lemma 2.5 *The operator* $\sup_{n\in\mathbb{N}} \left| f * \widetilde{T}_n^{(1)} \right|$ *is of weak type* (L_1, L_1).

Since

$$\left| f * T_n^{(1)} \right| \leq |f| * \widetilde{T}_n^{(1)},$$

from Lemma 2.5, we obtain

Lemma 2.6 *The operator* $\sup_{n\in\mathbb{N}} \left| f * T_n^{(1)} \right|$ *is of weak type* (L_1, L_1).

Analogously, we can prove

Lemma 2.7 *The operator* $\sup_{n\in\mathbb{N}} \left| f * T_n^{(2)} \right|$ *is of weak type* (L_1, L_1).

3 Proofs of main results

Proof of Theorem 1.1 We have

$$w_n T_n^{(3)} = \frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=0}^{|n|} \varepsilon_s(n)\, w_n w_{n^{(s)}-1} A_{n_{(s)}-1}^{\alpha_n} D_{2^s}.$$

From (2), we get ($\varepsilon_s(n) = 1$; otherwise, there is nothing to be discussed here)

$$\begin{aligned} w_n w_{n^{(s)}-1} &= w_n w_{n^{(s+1)}+2^s-1} = w_n w_{n^{(s+1)}} w_{2^s-1} \\ &= w_{n\oplus n^{(s+1)}} w_{2^s-1} = w_{n_{(s)}} w_{2^s-1} \\ &= w_{2^s} w_{n_{(s-1)}} w_{2^s-1} = w_{2^s} w_{n_{(s-1)\oplus(2^s-1)}}. \end{aligned}$$

Since $n_{(s-1)\oplus(2^s-1)} < 2^s$ from (3), we have

$$D_{2^s} w_{n_{(s-1)\oplus(2^s-1)}} = D_{2^s}.$$

Consequently

$$\begin{aligned} w_n T_n^{(3)} &= \frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=0}^{|n|} \varepsilon_s(n)\, A_{n_{(s)}-1}^{\alpha_n} w_{2^s} D_{2^s} \\ &= \frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=0}^{|n|} \varepsilon_s(n)\, A_{n_{(s)}-1}^{\alpha_n} \left(D_{2^{s+1}} - D_{2^s}\right) \\ &= \frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=1}^{|n|} \left(\varepsilon_{s-1}(n) - \varepsilon_s(n)\right) A_{n_{(s-1)}-1}^{\alpha_n} D_{2^s} \\ &\quad + \frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=1}^{|n|} \varepsilon_s(n) \left(A_{n_{(s-1)}-1}^{\alpha_n} - A_{n_{(s)}-1}^{\alpha_n}\right) D_{2^s} \\ &\quad + \frac{1}{A_{n-1}^{\alpha_n}} \varepsilon_{|n|}(n)\, A_{n_{(|n|)}-1}^{\alpha_n} D_{2^{|n|+1}} \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{A_{n-1}^{\alpha_n}}\varepsilon_0\left(n\right)A_{n_{(0)}-1}^{\alpha_n}D_1\\
&=: T_n^{(31)}+T_n^{(32)}+T_n^{(33)}+T_n^{(34)}.
\end{aligned}
\tag{9}
$$

From the condition of Theorem 1.1, we can write

$$
\begin{aligned}
&\sup_{n\in\mathbb{N}}\left(|f|*\left|T_n^{(3)}\right|\right)\\
&=\sup_{n\in\mathbb{N}}\left(|f|*\left|w_nT_n^{(3)}\right|\right)\\
&\leq\sup_{n\in\mathbb{N}}\left(|f|*\left|w_nT_n^{(31)}\right|\right)+\sup_{n\in\mathbb{N}}\left(|f|*\left|w_nT_n^{(32)}\right|\right)\\
&\quad+\sup_{n\in\mathbb{N}}\left(|f|*\left|w_nT_n^{(33)}\right|\right)+\sup_{n\in\mathbb{N}}\left(|f|*\left|w_nT_n^{(34)}\right|\right)\\
&\leq cE^*\left(x,|f|\right)\frac{1}{n^{\alpha_n}}\sum_{s=1}^{|n|}\left|\varepsilon_{s-1}\left(n\right)-\varepsilon_s\left(n\right)\right|2^{s\alpha_n}\\
&\quad+cE^*\left(x,|f|\right)\frac{1}{A_{n-1}^{\alpha_n}}\sum_{s=1}^{|n|}\left(A_{n_{(s-1)}-1}^{\alpha_n}-A_{n_{(s)}-1}^{\alpha_n}\right)\\
&\quad+cE^*\left(x,|f|\right)\\
&\leq c_KE^*\left(x,|f|\right).
\end{aligned}
\tag{10}
$$

Since the operator $E^*\left(x,|f|\right)$ is of weak type $\left(L_1,L_1\right)$, we obtain that

$$
\left\|\sup_{n\in V_K(\alpha)}\left(|f|*\left|T_n^{(3)}\right|\right)\right\|_{weak-L_1}\leq c_K\left\|f\right\|_1.
\tag{11}
$$

Combining Lemmas 2.6, 2.7, estimation (11) from (6) we conclude that

$$
\left\|\sup_{n\in V_K(\alpha)}\left|\sigma_n^{\alpha_n}f\right|\right\|_{weak-L_1}\leq c_K\left\|f\right\|_1.
\tag{12}
$$

Using the standard argument of Marcinkiewicz and Zygmund [12] from the estimation (12), we obtain the validity of Theorem 1.1. □

Proof of Theorem 1.2 From (6), we have

$$
\frac{n^{\alpha_n}\left(f*K_n^{\alpha_n}\right)}{V\left(n,\alpha\right)}=\frac{n^{\alpha_n}\left(f*T_n^{(1)}\right)}{V\left(n,\alpha\right)}+\frac{n^{\alpha_n}\left(f*T_n^{(2)}\right)}{V\left(n,\alpha\right)}+\frac{n^{\alpha_n}\left(f*T_n^{(3)}\right)}{V\left(n,\alpha\right)}.
\tag{13}
$$

Lemmas 2.6 and 2.7 imply that

$$
\sup_n\left|f*T_n^{(l)}\right|<\infty\ \text{ a. e. for } f\in L_1\left(\mathbb{I}\right),l=1,2.
$$

Hence

$$
\lim_{n\to\infty}\frac{n^{\alpha_n}\left(f*T_n^{(l)}\right)}{V\left(n,\alpha\right)}=0\text{ a. e. } l=1,2.
\tag{14}
$$

Using estimation (10), we have

$$
\sup_n\frac{n^{\alpha_n}\left(f*T_n^{(3)}\right)}{V\left(n,\alpha\right)}\leq cE^*\left(x,|f|\right).
$$

Since the operator $E^*(x,|f|)$ is of weak type (L_1, L_1), we obtain that the maximal operator

$$\sup_n \frac{n^{\alpha_n}\left(f * T_n^{(3)}\right)}{V(n,\alpha)}$$

is of weak type (L_1, L_1). It is clear that

$$\frac{n^{\alpha_n}\left(W * T_n^{(3)}\right)}{V(n,\alpha)} \to 0 \quad \text{as } n \to \infty$$

for every Walsh polynomial W. By the well-known density argument, we conclude that

$$\lim_{n\to\infty} \frac{n^{\alpha_n}\left(f * T_n^{(3)}\right)}{V(n,\alpha)} = 0 \quad \text{a. e.} \tag{15}$$

Combining (13)–(15), we conclude the proof of Theorem 1.2. □

Proof of Theorem 1.4 From (6), we have

$$\sigma_n^{\alpha_n}(f) = f * K_n^{\alpha_n} = f * T_n^{(1)} + f * T_n^{(2)} + f * T_n^{(3)}. \tag{16}$$

Applying Lemmas 2.6 and 2.7, we conclude that the operators $f * T_n^{(l)}, l = 1, 2$ are of weak type (L_1, L_1). Now, we consider the operator $f * T_n^{(3)}$. From (9), we have

$$\begin{aligned}
& f * T_n^{(3)} \\
&= w_n\left((f w_n) * w_n T_n^{(3)}\right) \\
&= \frac{w_n}{A_{n-1}^{\alpha_n}} \sum_{s=0}^{\infty} \varepsilon_s(n) A_{n_{(s)}-1}^{\alpha_n}\left((f w_n) * \left(D_{2^{s+1}} - D_{2^s}\right)\right) \\
&= w_n \sum_{s=0}^{\infty} \frac{\varepsilon_s(n) A_{n_{(s)}-1}^{\alpha_n}}{A_{n-1}^{\alpha_n}} \Delta_s(f w_n) \\
&= \mathbf{T}(\beta)(f w_n),
\end{aligned} \tag{17}$$

where

$$\beta := \left(\frac{\varepsilon_0(n) A_{n_{(0)}-1}^{\alpha_n}}{A_{n-1}^{\alpha_n}}, \ldots, \frac{\varepsilon_{|n|}(n) A_{n_{(|n|)}-1}^{\alpha_n}}{A_{n-1}^{\alpha_n}}, 0, \ldots\right).$$

Since $\|\beta\| \le 1$ from Theorem MT, we get that the operator $\left|(f w_n) * w_n T_n^{(3)}\right|$ is of weak type (L_1, L_1). Consequently

$$\left\| f * T_n^{(3)} \right\|_{weak-L_1(\mathbb{I})} \le c\|f\|_1. \tag{18}$$

From (16), we complete the proof of Theorem 1.4. □

Proof of Theorem 1.7 Basically, we use the method of Schipp (see [13], Ch. 4, Theorem 12) with some necessary modifications. For natural numbers n, k, set

$$i_n := \sum_{k=1}^{\lfloor n/2 \rfloor} 2^{2k-1} < 2^n, \quad g_n := sgn\, D_{i_n},$$

$$R_k^{(n)} := r_{2^n+k} \tau_{k/2^n} g_n, \quad Q_n := \prod_{k=0}^{2^n-1}\left(1 + R_k^{(n)}\right).$$

Then, in the sequel, we prove

$$S_{2^{2^n+k}+i_n}(Q_n,x)-S_{2^{2^n+k}}(Q_n,x)=r_{2^n+k}(x)g_n(x\dotplus k/2^n). \tag{19}$$

Since Q_n is the sum of the product of terms $R_k^{(n)}$, then we have to check $R:=R_{l_1}^{(n)}\dots R_{l_s}^{(n)}$ for $l_1<\cdots<l_s$ and let the empty product be 1. If the case is the latter, i.e., $R=1$, then the left-hand side of (19) is zero. Therefore, suppose that we are checking not the empty product. Then

$$R=R_{l_1}^{(n)}\dots R_{l_s}^{(n)}=r_{2^n+l_1}\dots r_{2^n+l_s}\left(\tau_{l_1 2^{-n}}g_n\dots\tau_{l_s 2^{-n}}g_n\right)=:r_{2^n+l_1}\dots r_{2^n+l_s}h,$$

where function h is $\mathcal{A}_n$ measurable. Therefore, in the case of $k<l_s$, we have

$$S_{2^{2^n+k}}(R)=0.$$

Besides, in the case of $k>l_s$, we have

$$S_{2^{2^n+k}}(R)=R.$$

That is, in both cases, the left-hand side of (19) is

$$S_{2^{2^n+k}+i_n}(R,x)-S_{2^{2^n+k}}(R,x)=r_{2^n+k}(x)S_{i_n}(Rr_{2^n+k},x), \tag{20}$$

which can be different from zero only in the case when $s=1$ and $l_s=k$. In this situation, it is exactly

$$r_{2^n+k}S_{i_n}(R_k^{(n)}r_{2^n+k})=r_{2^n+k}S_{i_n}(\tau_{k2^{-n}}g_n)=r_{2^n+k}\tau_{k2^{-n}}g_n=R_k^{(n)}. \tag{21}$$

Just add a few details to equality (21): Let $a=2\lfloor n/2\rfloor-1$. Then, $i_n=2^1+2^3+\dots 2^a$. It is easy to have that

$$g_n(x)=\begin{cases}1, & \text{if } x\in I_a,\\ r_a(x), & \text{if } x\in I_{a-2}\backslash I_a,\\ r_a(x)r_{a-2}(x), & \text{if } x\in I_{a-4}\backslash I_{a-2},\\ \dots, & \\ r_a(x)r_{a-2}(x)\cdots r_3(x), & \text{if } x\in I_1\backslash I_3,\\ 0, & \text{if } x\in \mathbb{I}\backslash I_1.\end{cases}$$

Let $e_i=1/2^{i+1}$. It gives that g_n is the sum of functions $g_{n,\epsilon}$

$$\begin{aligned}g_{n,\epsilon}(x):=&\frac{1}{2^a}D_{2^a}(x)+\frac{1}{2^a}D_{2^a}(x+\epsilon_{a-2}e_{a-2}+\epsilon_{a-1}e_{a-1})r_a(x)\\&+\frac{1}{2^{a-2}}D_{2^{a-2}}(x+\epsilon_{a-4}e_{a-4}+\epsilon_{a-3}e_{a-3})r_a(x)r_{a-2}(x)\\&+\dots+\frac{1}{2^3}D_{2^3}(x+\epsilon_1e_1+\epsilon_2e_2)r_a(x)r_{a-2}(x)\cdots r_3(x),\end{aligned}$$

where each ϵ_i is either 0 or 1, but $\epsilon_{a-2}+\epsilon_{a-1},\epsilon_{a-4}+\epsilon_{a-3},\dots,\epsilon_1+\epsilon_2\neq 0$ and we do the summing with respect to ϵ. That is, $g_n=\sum_\epsilon g_{n,\epsilon}$. Then, for any of the addends of type $g_{n,\epsilon}$, we have

$$\begin{aligned}&S_{i_n}\left(r_a\cdots r_{a-2i}D_{2^{a-2i}}(\cdot+\epsilon_{a-2i-2}e_{a-2i-2}+\epsilon_{a-2i-1}e_{a-2i-1})\right)\\&=r_a\cdots r_{a-2i}D_{2^{a-2i}}(\cdot+\epsilon_{a-2i-2}e_{a-2i-2}+\epsilon_{a-2i-1}e_{a-2i-1}),\end{aligned}$$

and consequently, $S_{i_n}g_n=g_n$. In other words, (19) is proved. Let $n_m\in\mathbb{N}$, $x\in\mathbb{I}$ be arbitrary and suppose that n_m is a cube and $n_m\geq\nu_{2m+1}$. Then, there exists one $k\in\{0,1,...,2^{n_m}-1\}$, such that

$$x\dotplus k2^{-n_m}\in I_{n_m}. \tag{22}$$

Set

$$N_{2m}:=2^{2^{n_m}+k},\ N_{2m+1}:=2^{2^{n_m}+k}+i_{n_m},\ m=1,2,....$$

It is easy to see that

$$N_{2m} \geq 2^{2^{n_m}} > n_m \geq \upsilon_{2m+1} > \upsilon_{2m}$$

and

$$N_{2m+1} \geq 2^{2^{n_m}} > n_m \geq \upsilon_{2m+1}.$$

Hence

$$N_j \geq \upsilon_j,\ j = 1, 2, ...$$

Let

$$f(x) := \sum_{m=1}^{\infty} \frac{Q_{n_m}(x)}{\sqrt[3]{n_m^2}}.$$

Since $\|Q_n\|_1 = 1$ (see [13, ch. 4, Theorem 12]), then $f \in L_1(\mathbb{I})$. From the definition of function Q_n, it follows for its spectrum:

$$\mathrm{sp}(Q_{n_j}) \subset \left[0, 2^{2^{n_j+1}}\right),$$

and since

$$N_{2m} \geq 2^{2^{n_m}} \geq 2^{2^{n_j+1}}\ (j < m),$$

we obtain

$$S_{N_{2m+1}}\left(Q_{n_j}, x\right) - S_{N_{2m}}\left(Q_{n_j}, x\right) = 0,\ j < m. \tag{23}$$

On the other hand, check the same difference of partial sums for Q_{n_j} $(j > m)$. Let again $R := R_{l_1}^{(n_j)} \dots R_{l_s}^{(n_j)}$ be different from the empty product. Then

$$\begin{aligned} &\left|S_{N_{2m+1}}\left(Q_{n_j}, x\right) - S_{N_{2m}}\left(Q_{n_j}, x\right)\right| \\ &= \left|S_{i_{n_m}}\left(R \cdot r_{2^{n_m}+k}\right)\right| \\ &= \left|S_{i_{n_m}}\left(r_{2^{n_j}+l_1} \cdots r_{2^{n_j}+l_s} \cdot r_{2^{n_m}+k} \cdot h\right)\right| = 0, \end{aligned} \tag{24}$$

because the function h is $\mathcal{A}_{n_j}$ measurable.

From (20), (21), (22), (23), and (24), we obtain

$$\begin{aligned} &\left|S_{N_{2m+1}}(f, x) - S_{N_{2m}}(f, x)\right| \\ &= \frac{1}{\sqrt[3]{n_m^2}}\left|S_{N_{2m+1}}\left(Q_{n_m}, x\right) - S_{N_{2m}}\left(Q_{n_m}, x\right)\right| \\ &= \frac{1}{\sqrt[3]{n_m^2}}\left|S_{i_{n_m}}(\tau_{k2^{-n_m}} g_{n_m}, x)\right| \\ &= \frac{1}{\sqrt[3]{n_m^2}}\left|\left(\tau_{k2^{-n_m}} g_{n_m}\right) * D_{i_{n_m}}(x)\right| \\ &= \frac{1}{\sqrt[3]{n_m^2}}\left|g_{n_m} * \left(\tau_{k2^{-n_m}} D_{i_{n_m}}(x)\right)\right| \\ &= \frac{1}{\sqrt[3]{n_m^2}}\left|g_{n_m} * D_{i_{n_m}}(0)\right| \\ &= \frac{1}{\sqrt[3]{n_m^2}}\left\|D_{i_{n_m}}\right\|_1 \geq c n_m^{1/3}. \end{aligned}$$

It means that for every $x \in \mathbb{I}$, we have

$$\sup_m \left|S_{N_{2m+1}}(f, x) - S_{N_{2m}}(f, x)\right| = \infty,$$

provided that $N_m \geq \nu_m$. This completes the proof of Theorem 1.7. □

Proof of Theorem 1.8 During the proof, we apply some idea of Bochkarev [3]. Consider the function $W_N(t)$ defined by

$$W_N(t) := \begin{cases} \frac{2^N}{\sqrt{N}} \sum_{j=2N}^{3N-1} w_{2^j}(t), t \in \bigcup_{y_0=0}^{1} \cdots \bigcup_{y_{3N-1}=0}^{1} I_{4N}(y_0, ..., y_{3N-1}, y_{2N}, ..., y_{3N-1}) \\ 0, \text{otherwise} \end{cases}.$$

Set

$$n(N, x) = \sum_{j=2N}^{3N-1} \varepsilon_j(x) 2^j + \sum_{j=3N}^{4N-1} \varepsilon_{j-N}(x) 2^j, \tag{25}$$

where $\varepsilon_j(x) = 0, 1$ which will be defined below. We suppose that

$$x \in I_{3N+1}(x_0, ..., x_{3N-1}, 1 - x_{2N}).$$

Denote

$$E'_N := \bigcup_{x_0=0}^{1} \cdots \bigcup_{x_{3N-1}=0}^{1} I_{3N+1}(x_0, ..., x_{3N-1}, 1 - x_{2N}),$$

$$E' := \bigcap_{k=1}^{\infty} \bigcup_{N=k}^{\infty} E'_N.$$

It is easy to see that

$$\text{mes}\left(E'\right) = 1.$$

and

$$I_{3N+1}(x_0, ..., x_{3N-1}, 1 - x_{2N}) \cap I_{4N}(y_0, ..., y_{3N-1}, y_{2N}, ..., y_{3N-1}) = \emptyset.$$

Let $\{N_v\}$ be a subsequence for which $x \in E'_{N_v}$, $v = 1, 2, ...$ Without lost of generality, we can suppose that $N'_v = N$. Since

$$(x \dotplus t)_{2N} \dotplus (x \dotplus t)_{3N} = 1, t \in supp(W_N), x \in E',$$

then from (3) and (9), we have (for the sake of brevity $A_{n(N,x)-1}^{\alpha_{n(N,x)}}$ will be denoted as $A_{n(N,x)-1}^{\alpha_n}$ which will not cause misunderstand)

$$W_N * T_{n(N,x)}^{(3)} = \frac{1}{A_{n(N,x)-1}^{\alpha_n}} \sum_{j=2N}^{3N} \varepsilon_j(x) A_{n_{(j)}(N,x)-1}^{\alpha_n} \times \int_{\mathbb{I}} W_N(t) w_{n(N,x)}(x \dotplus t) D^*_{2^j}(x \dotplus t) dt.$$

Set

$$q(N, x) := \sum_{j=3N}^{4N-1} \varepsilon_{j-N}(x) 2^j.$$

Then, we can write

$$w_{n(N,x)}(t) = w_{n(N,x)-q(Nx)}(t) w_{q(N,x)}(t) = 1, t \in supp(W_N).$$

Consequently

$$
\begin{aligned}
&W_N * T^{(3)}_{n(N,x)}\\
&= \frac{w_{n(N,x)}(x)}{A^{\alpha_n}_{n(N,x)-1}} \sum_{j=2N}^{3N} \varepsilon_j(x) A^{\alpha_n}_{n_{(j)}(N,x)-1} \frac{2^N}{\sqrt{N}}\\
&\quad \times \bigcup_{y_0=0}^{1} \cdots \bigcup_{y_{3N-1}=0}^{1} \int_{I_{4N}(y_0,\ldots,y_{3N-1},y_{2N},\ldots,y_{3N-1})} \left(\sum_{i=2N}^{3N-1} w_{2^i}(t)\right) D^*_{2^j}(x \dotplus t)\,dt\\
&= \frac{w_{n(N,x)}(x)}{\sqrt{N} A^{\alpha_n}_{n(N,x)-1}} \sum_{j=2N}^{3N} \varepsilon_j(x) A^{\alpha_n}_{n_{(j)}(N,x)-1}\\
&\quad \times \bigcup_{y_0=0}^{1} \cdots \bigcup_{y_{3N-1}=0}^{1} \int_{I_{3N}(y_0,\ldots,y_{3N-1})} \left(\sum_{i=2N}^{3N-1} w_{2^i}(t)\right) D^*_{2^j}(x \dotplus t)\,dt\\
&= \frac{w_{n(N,x)}(x)}{\sqrt{N} A^{\alpha_n}_{n(N,x)-1}} \sum_{j=2N}^{3N-1} \varepsilon_j(x) A^{\alpha_n}_{n_{(j)}(N,x)-1} \int_{\mathbb{I}} \left(\sum_{i=2N}^{3N-1} w_{2^i}(t)\right) D^*_{2^j}(x \dotplus t)\,dt\\
&= \frac{w_{n(N,x)}(x)}{\sqrt{N} A^{\alpha_n}_{n(N,x)-1}} \sum_{j=2N}^{3N-1} \varepsilon_j(x) A^{\alpha_n}_{n_{(j)}(N,x)-1} w_{2^j}(x).
\end{aligned}
$$

Two cases are possible:

(a)

$$\sum_{k=2N}^{3N-1} x_k < \frac{N}{3};$$

(b)

$$\sum_{k=2N}^{3N-1} x_k \geq \frac{N}{3}.$$

First, we consider the case a) and let us define digits $\varepsilon_k(x)$ by $\varepsilon_k(x) = 1 - x_k$. Then, we can write

$$\left|W_N * T^{(3)}_{n(N,x)}\right| \geq \frac{c}{\sqrt{N} 2^{4N\alpha_n}} \sum_{2N \leq j \leq 2N+(2N)/3} 2^{j\alpha_n} \geq \frac{c}{\sqrt{N} 2^{2N\alpha_n} \alpha_n}.$$

Since

$$\alpha_n \leq \frac{c_0 \log\log n(N,x)}{\log(N,x)} \leq \frac{c_0 \log(4N)}{2N} \quad (n > n_0),$$

we obtain

$$\left|W_N * T^{(3)}_{n(N,x)}\right| \geq \frac{cN^{1/2-c_0}}{\log(4N)}. \tag{26}$$

Now, we consider the case b). The digits $\varepsilon_k(x)$ define by $\varepsilon_k(x) = x_k$. Analogously, we can prove the validity of estimation (26).

Set

$$\gamma_N := \frac{N^{1/2-c_0}}{\log(4N)}.$$

Let $\{N_v : v \geq 1\}$ be a subsequence for which

$$x \in E_{N_v}, v = 1, 2, ...,$$

$$N_{v+1} \geq 2N_v, \tag{27}$$

$$\gamma_{N_v} \geq v^4, \tag{28}$$

$$\sum_{j=1}^{v-1} \frac{2^{N_j}\sqrt{N_j}}{\sqrt{\gamma_{N_j}}} < \frac{\sqrt{\gamma_{N_v}}}{v}. \tag{29}$$

Let

$$f(t) := \sum_{j=1}^{\infty} \frac{W_{N_j}(t)}{\sqrt{\gamma_{N_j}}}.$$

It is easy to show that

$$\begin{aligned}\|W_N\|_1 &= \frac{2^N}{\sqrt{N}} \bigcup_{x_0=0}^{1} \cdots \bigcup_{x_{3N-1}=0}^{1} \int_{I_{4N}(x_0,\ldots,x_{3N-1},x_{2N},\ldots,x_{3N-1})} \left| \sum_{j=2N}^{3N-1} w_{2^j}(t) \right| dt \\ &= \frac{1}{\sqrt{N}} \bigcup_{x_0=0}^{1} \cdots \bigcup_{x_{3N-1}=0}^{1} \int_{I_{3N}(x_0,\ldots,x_{3N-1})} \left| \sum_{j=2N}^{3N-1} w_{2^j}(t) \right| dt \\ &= \frac{1}{\sqrt{N}} \int_{\mathbb{I}} \left| \sum_{j=2N}^{3N-1} w_{2^j}(t) \right| dt \leq \frac{1}{\sqrt{N}} \left(\int_{I} \left| \sum_{j=2N}^{3N-1} w_{2^j}(t) \right|^2 dt \right)^{1/2} = 1.\end{aligned}$$

Then, from (28), we conclude that $f \in L_1(\mathbb{I})$.

It is easy to see that

$$f * T_{n(N_v,x)}^{(3)} = \sum_{j=1}^{v-1} \frac{1}{\sqrt{\gamma_{N_j}}} \left(W_{N_j} * T_{n(N_v,x)}^{(3)} \right) + \frac{1}{\sqrt{\gamma_{N_v}}} \left(W_{N_v} * T_{n(N_v,x)}^{(3)} \right). \tag{30}$$

We can write (see (6) and (25))

$$\begin{aligned}&W_{N_j} * T_{n(N_v,x)}^{(3)} \\ &= \frac{1}{A_{n(N_v,x)-1}^{\alpha_n}} \sum_{k=2N_v}^{3N_v-1} \varepsilon_k(x) A_{n_{(k)}(N_v,x)-1}^{\alpha_n} \left(W_{N_j} * \left(w_{n^{(k)}(N_v,x)-1} D_{2^k} \right) \right) \\ &\quad + \frac{1}{A_{n(N_v,x)-1}^{\alpha_n}} \sum_{k=3N_v}^{4N_v-1} \varepsilon_{k-N_v}(x) A_{n_{(k)}(N_v,x)-1}^{\alpha_n} \left(W_{N_j} * \left(w_{n^{(k)}(N_v,x)-1} D_{2^k} \right) \right).\end{aligned} \tag{31}$$

Let

$$n^{(k)}(N_v, x) - 1 = 2^k - 1 + n^{(k+1)}(N_v, x).$$

Suppose that $n^{(k+1)}(N_v, x) \neq 0$. Then, it is easy to see that

$$W_{N_j} * \left(w_{n^{(k)}(N_v,x)-1} D_{2^k} \right) = 0, \; j < v, 2N_v \leq k < 3N_v.$$

Hence, we can suppose that there exists $k_0 \in \{2N_v, ..., 3N_v - 1\}$, such that $n^{(k_0+1)}(N_v, x) = 0$ and $\varepsilon_{k_0}(x) = 1$. Since $n^{(k_0)}(N_v, x) \neq 0$, we conclude that

$$W_{N_j} * \left(w_{n^{(k)}(N_v,x)-1} D_{2^k}\right) = 0$$

when $k < k_0$. Consequently, we have $(w_{-1} = 0)$

$$\begin{aligned} &\frac{1}{A^{\alpha_n}_{n(N_v,x)-1}} \sum_{k=2N_v}^{3N_v-1} \varepsilon_k(x) A^{\alpha_n}_{n_{(k)}(N_v,x)-1} \left(W_{N_j} * \left(w_{n^{(k)}(N_v,x)-1} D_{2^k}\right)\right) \\ &= \frac{A^{\alpha_n}_{n_{(k_0)}(N_v,x)-1}}{A^{\alpha_n}_{n(N_v,x)-1}} \left(W_{N_j} * \left(w_{2^{k_0}-1} D_{2^{k_0}}\right)\right) \\ &= \frac{A^{\alpha_n}_{n_{(k_0)}(N_v,x)-1}}{A^{\alpha_n}_{n(N_v,x)-1}} \left(W_{N_j} * D_{2^{k_0}}\right) \\ &= \frac{A^{\alpha_n}_{n_{(k_0)}(N_v,x)-1}}{A^{\alpha_n}_{n(N_v,x)-1}} S_{2^{k_0}}\left(W_{N_j}\right) \\ &= \frac{A^{\alpha_n}_{n_{(k_0)}(N_v,x)-1}}{A^{\alpha_n}_{n(N_v,x)-1}} W_{N_j}. \end{aligned} \tag{32}$$

Analogously, we can prove that

$$\begin{aligned} &\frac{1}{A^{\alpha_n}_{n(N_v,x)}} \sum_{k=3N_v}^{4N_v-1} \varepsilon_{k-N_v}(x) A^{\alpha_n}_{n_{(k)}(N_v,x)-1} \left(W_{N_j} * \left(w_{n^{(k)}(N_v,x)-1} D_{2^k}\right)\right) \\ &= \frac{A^{\alpha_n}_{n_{(k_0)}(N_v,x)-1}}{A^{\alpha_n}_{n(N_v,x)-1}} W_{N_j}. \end{aligned} \tag{33}$$

Combining (31)–(33) from (29), we get

$$\begin{aligned} &\left|\sum_{j=1}^{v-1} \frac{1}{\sqrt{\gamma_{N_j}}} \left(W_{N_j} * T^{(3)}_{n(N_v,x)}\right)\right| \\ &\le \sum_{j=1}^{v-1} \frac{\left|W_{N_j}\right|}{\sqrt{\gamma_{N_j}}} \le \sum_{j=1}^{v-1} \frac{2^{N_j}\sqrt{N_j}}{\sqrt{\gamma_{N_j}}} < \frac{\sqrt{\gamma_{N_v}}}{v}. \end{aligned} \tag{34}$$

From (26), (30), and (34), we conclude that $\left(x \in E'\right)$

$$\left|f * T^{(3)}_{n(N_v,x)}\right| \ge c\sqrt{\gamma_{N_v}} \to \infty \text{ as } v \to \infty. \tag{35}$$

From (6), we can write

$$f * K^{\alpha_n}_{n(N_v,x)} = f * T^{(1)}_{n(N_v,x)} + f * T^{(2)}_{n(N_v,x)} + f * T^{(3)}_{n(N_v,x)}. \tag{36}$$

Lemmas 2.6 and 2.7 imply that

$$\sup_n \left|f * T^{(l)}_n\right| < \infty \text{ a. e. for } f \in L_1(\mathbb{I}), l = 1, 2. \tag{37}$$

Let E_0 be the set for which (37) does not hold. Denote $E := E' \backslash E_0$. Then, it is evident that $\text{mes}(E) = 1$. Let $x \in E$. Then, (35)–(37) imply that

$$\sup_n \left|\sigma^{\alpha_n}_n(f, x)\right| = \infty \quad (x \in E).$$

Theorem 1.8 is proved. □

Acknowledgements György Gát is supported by projects EFOP-3.6.2-16-2017-00015 and EFOP-3.6.1-16-2016-00022 supported by the European Union, cofinanced by the European Social Fund. Ushangi Goginava is supported by UAEU through the Start-up Grant "Fourier Series with respect to locally constant orthonormal system."

References

1. Akhobadze, T.: On the convergence of generalized Cesàro means of trigonometric Fourier series. II. Acta Math. Hungar. **115**(1–2), 79–100 (2007)
2. Akhobadze, T.: On the convergence of generalized Cesàro means of trigonometric Fourier series. I. Acta Math. Hungar. **115**(1–2), 59–78 (2007)
3. S.V. Bochkarev, Everywhere divergent Fourier series in the Walsh system and in multiplicative systems. (Russian, English) Russ. Math. Surv. 59, No.1, 103-124 (2004); translation from Usp. Mat. Nauk 59, No.1, 103-124 (2003).
4. Fine, J.: Cesàro summability of Walsh–Fourier series. Proc. Nat. Acad. Sci. USA **41**, 558–591 (1955)
5. Fridli, S.: On the rate of convergence of Cesàro means of Walsh–Fourier series. J. Approx. Theory **76**(1), 31–53 (1994)
6. Gát, G.: On $(C, 1)$ summability for Vilenkin-like systems. Studia Math. **144**(2), 101–120 (2001)
7. Gát, G.; Goginava, U.: Maximal operators of Cesàro means with varying parameters of Walsh–Fourier series. Acta Math. Hungar. **159**(2), 653–668 (2019)
8. Goginava, U.: On the approximation properties of Cesàro means of negative order of Walsh–Fourier series. J. Approx. Theory **115**(1), 9–20 (2002)
9. Golubov, B.I.; Efimov, A.V.; Skvortsov, V.A.: Series and transformations of Walsh, Moscow, 1987 (Russian). English translation, Kluwer Academic, Dordrecht (1991)
10. A. A. Abu Joudeh and G. Gát, Convergence of Cesàro means with varying parameters of Walsh–Fourier series, *Miskolc Mathematical Notes*, 19 (2018), no. 1, 303-317.
11. Kaplan, I.B.: Cesàro means of variable order. Izv. Vyssh. Uchebn. Zaved. Mat. **18**(5), 62–73 (1960)
12. Marcinkiewicz, I.; Zygmund, A.: On the summability of double Fourier series. Fund. Math. **32**, 112–132 (1939)
13. Schipp, F.; Wade, W.R.; Simon, P.; Pál, J.: Walsh Series. An Introduction to Dyadic Harmonic Analysis, Adam Hilger (Bristol-New York (1990)
14. Sh. Tetunashvili, On divergence of Fourier series by some methods of summability. J. Funct. Spaces Appl. 2012, Art. ID 542607, 9 pp.
15. Toledo, R.: On the boundedness of the L^1-norm of Walsh-Fejér kernels. J. Math. Anal. Appl. **457**(1), 153–178 (2018)
16. Weisz, F.: Martingale Hardy spaces and their applications in Fourier analysis, vol. 1568. Lecture Notes in Mathematics. Springer-Verlag, Berlin (1994)
17. Weisz, F.: (C, α) summability of Walsh–Fourier series. Anal. Math. **27**, 141–156 (2001)
18. Weisz, F.: Cesàro and Riesz summability with varying parameters of multi-dimensional Walsh–Fourier series. Acta Math. Hungar. **161**(1), 292–312 (2020)
19. Yano, S.: On approximation by Walsh functions. Proc. Amer. Math. Soc. **2**, 962–967 (1951)
20. Zygmund, A.: Trigonometric series, vol. 1. Cambridge Univ Press, Cambridge (1959)