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Uniform and weak stability of Bresse system with one infinite memory in the shear angle displacements

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Abstract In this paper, we consider a one-dimensional linear Bresse system in a bounded open interval with one infinite memory acting only on the shear angle equation. First, we establish the well posedness using the semigroup theory. Then, we prove two general (uniform and weak) decay estimates depending on the speeds of wave propagations and the arbitrary growth at infinity of the relaxation function.

Mathematics Subject Classification 35B40 · 35L45 · 74H40 · 93D20 · 93D15

1 Introduction

In this paper, we consider a Bresse system in one-dimensional open bounded interval subjected to homogeneous Dirichlet–Neumann–Neumann boundary conditions and with the presence of one infinite memory acting on the shear angle equation. Precisely, we are concerned with the following problem:

$$\begin{cases} \rho_1\varphi_{tt} - k_1(\varphi_x + \psi + lw)_x - lk_3(w_x - l\varphi) = 0, \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi + lw) + \displaystyle\int_0^{+\infty} g(s)\psi_{xx}(x, t-s)\,\mathrm{d}s = 0, \\ \rho_1 w_{tt} - k_3(w_x - l\varphi)_x + lk_1(\varphi_x + \psi + lw) = 0, \\ \varphi(0,t) = \psi_x(0,t) = w_x(0,t) = \varphi(L,t) = \psi_x(L,t) = w_x(L,t) = 0, \\ \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,-t) = \psi_0(x,t),\ \psi_t(x,0) = \psi_1(x), \\ w(x,0) = w_0(x),\ w_t(x,0) = w_1(x), \end{cases} \tag{1.1}$$

where $(x,t) \in]0, L[\times\mathbb{R}_+$, $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a given function, and $L,\ l,\ \rho_i,\ i = 1, 2$, and $k_j,\ j = 1, 2, 3$, are positive constants. The integral term in system (1.1) represents the infinite memory, and the state (unknown) is

$$(\varphi, \psi, w) :]0, L[\times]0, +\infty[\to \mathbb{R}^3.$$

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Our objective is to establish the well posedness and the asymptotic stability of this problem in terms of the growth of g at infinity and the speeds of wave propagations given by

$$s_1 = \sqrt{\frac{k_1}{\rho_1}}, \quad s_2 = \sqrt{\frac{k_2}{\rho_2}} \quad \text{and} \quad s_3 = \sqrt{\frac{k_3}{\rho_1}}. \tag{1.2}$$

The Bresse system is known as the circular arch problem and is given by the following equations:

$$\rho_1 \varphi_{tt} = Q_x + lN + F_1, \quad \rho_2 \psi_{tt} = M_x - Q + F_2, \quad \rho_1 w_{tt} = N_x - lQ + F_3,$$

with

$$N = k_0(w_x - l\varphi), \quad Q = k(\varphi_x + lw + \psi) \quad \text{and} \quad M = b\psi_x,$$

where $\rho_1,\ \rho_2,\ l,\ k,\ k_0$ and b are positive physical constants, $N,\ Q$ and M denote, respectively, the axial force, the shear force and the bending moment, and $w,\ \varphi$ and ψ represent, respectively, the longitudinal, vertical and shear angle displacements. Here,

$$\rho_1 = \rho A, \quad \rho_2 = \rho I, \quad k_0 = EA, \quad k = k'GA, \quad b = EI \quad \text{and} \quad l = R^{-1},$$

such that ρ, E, G, k', A, I and R are positive constants and denote, respectively, the density, the modulus of elasticity, the shear modulus, the shear factor, the cross-sectional area, the second moment of area of the cross-section and the radius of curvature. Finally, F_1, F_2 and F_3 are the external forces defined in $]0, L[\times]0, +\infty[$. For more reading about this matter, we refer to Lagnese et al. [18,19]. It is worth noting that the system considered by Bresse [3] is obtained by taking

$$(F_1, F_2, F_3) = (0, -\gamma\psi_t, 0), \tag{1.3}$$

with $\gamma > 0$.

To stabilize the Bresse system, various dampings have been employed and several decay results have been established. Alabau-Boussouira et al. [1] considered the case (1.3) and proved that the exponential stability is equivalent to

$$s_1 = s_2 = s_3. \tag{1.4}$$

When (1.4) is not satisfied, they showed that the norm of solutions decays polynomially to zero with rates depending on the regularity of the initial data. These latter results were extended and improved in [22] by considering a locally distributed dissipation (that is, γ in (1.3) is replaced by a non-negative function a : $]0, L[\to \mathbb{R}_+$ which is positive only on a part of $]0, L[$). In their work, the authors of [22] obtained a better decay rate when (1.4) does not hold. The exponential stability result of [1] was also established by Soriano et al. [29] for the case of indefinite damping. That is, when $\gamma = a(x)$, where $a :]0, L[\to \mathbb{R}$ is a function with a positive average on $]0, L[$ and such that

$$\left\| a - \int_0^L a(x)\,dx \right\|_{L^2(]0,L[)}$$

is small enough. In such a situation, a may change sign in $]0, L[$. Also, some optimal polynomial decay rates for Bresse systems for the case (1.3) were proved in [7] when (1.4) does not hold. Wehbe and Youcef [31] treated the case

$$(F_1, F_2, F_3) = (0, -a_1(x)\psi_t, -a_2(x)w_t),$$

where $a_i :]0, L[\to \mathbb{R}_+$ are non-negative functions which can vanish on some part of $]0, L[$, and proved that the exponential stability holds if and only if $s_1 = s_2$. When $s_1 \neq s_2$, a polynomial decay rate depending on the regularity of the initial data was obtained. This rate, in the case of classical solutions, is $t^{-\frac{1}{2}+\epsilon}$.

When only the first and second equations are controlled by means of linear frictional dampings; that is,

$$(F_1, F_2, F_3) = (-\gamma_1\varphi_t, -\gamma_2\psi_t, 0),$$

with $\gamma_i > 0$, the equivalence between the exponential stability and the equality $s_1 = s_3$ was established in [2]. In addition, a polynomial stability was also shown when $s_1 \neq s_3$, where the decay rate depends on the regularity of the initial data. In the particular case of classical solutions, the polynomial decay of [2] is of the rate $t^{-\frac{1}{2}}$ and it is optimal. Soufyane and Said-Houari [30] looked into the case of three frictional dampings in the whole space $\mathbb{R}$ (instead of $]0, L[$) and established some polynomial stability estimates. For stabilization via nonlinear frictional dampings, we refer the readers to [4,28].

Concerning the stabilization via heat effect, one of the earliest results concerning the asymptotic behavior of the Bresse system is due to Liu and Rao [20], where a Bresse system of the form

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - lk_0(w_x - l\varphi) + l\gamma\chi = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) + \gamma\theta_x = 0, \\ \rho_1 w_{tt} - k_0(w_x - l\varphi)_x + lk(\varphi_x + \psi + lw) + \gamma\chi_t = 0, \\ \rho_3\theta_t - \theta_{xx} + \gamma\psi_{xt} = 0, \\ \rho_3\chi_t - \chi_{xx} + \gamma(w_x - l\varphi)_t = 0, \end{cases} \tag{1.5}$$

in a bounded interval, together with initial and boundary conditions, has been considered. In that work, Liu and Rao [20] proved that the norm of solutions decays exponentially if and only if (1.4) holds. Otherwise, the solutions decay polynomially with rates depending on the regularity of the initial data. For the classical solutions, with boundary conditions of Dirichlet–Neumann–Neumann or Dirichlet–Dirichlet–Dirichlet type, these rates are of the form $t^{-\frac{1}{4}+\epsilon}$ or $t^{-\frac{1}{8}+\epsilon}$, respectively, where $\epsilon > 0$ is an arbitrary "small" constant. Other results similar to those of [20] were obtained in [8] for the Bresse system (1.5) without χ. The obtained decay for classical solutions when (1.4) is not satisfied is, in general, of the rate $t^{-\frac{1}{6}+\epsilon}$; whereas the rate is $t^{-\frac{1}{3}+\epsilon}$ when $s_1 \neq s_2$ and $s_1 = s_3$. Najdi and Wehbe [21] extended the results of [8] to the case where the thermal dissipation is locally distributed, and improved the polynomial stability estimate to $t^{-\frac{1}{2}}$ when (1.4) is not satisfied. Recently, Keddi et al. [16] studied a thermoelastic Bresse system with Cattaneo's thermal dissipation of the form

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - lk_0(w_x - l\varphi) = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) + \gamma\theta_x = 0, \\ \rho_1 w_{tt} - k_0(w_x - l\varphi)_x + lk(\varphi_x + \psi + lw) = 0, \\ \rho_3\theta_t + q_x + \gamma\psi_{xt} = 0, \\ \tau q_t + \beta q + \theta_x = 0, \end{cases}$$

in a bounded interval, where φ, ψ and w are, respectively, the vertical, shear angle and longitudinal displacements, θ and q denote the temperature difference and the heat flux, and $\rho_1,\ \rho_2,\ \rho_3,\ k,\ k_0,\ b,\ \beta,\ \gamma$ and τ are positive constants. Under suitable relations between the constants, the authors of [16] showed exponential and optimal polynomial decay rates. The same system was treated by Said-Houari and Hamadouche [25] in the whole space $\mathbb{R}$, where they showed that the coupling of the Bresse system with the heat conduction of the Cattaneo theory leads to a loss of regularity of the solution and they proved that the decay rate of the solution in the L^2-norm is of the rate $t^{-1/12}$. For more problems of thermoelastic Bresse systems, we refer the reader to [24], where a global existence was proved using two heat equations, and to [26,27], where Cauchy thermoelastic Bresse problems were treated.

Concerning the stability of Bresse systems via memories, there are only very few results. For instance, Guesmia and Kafini [10] discussed, without restrictions on the speeds, the stability issue for the case when the three equations are controlled via infinite memories of the form

$$F_1 = -\int_0^{+\infty} g_1(s)\varphi_{xx}(x, t-s)\,\mathrm{d}s, \quad F_2 = -\int_0^{+\infty} g_2(s)\psi_{xx}(x, t-s)\,\mathrm{d}s,$$
$$F_3 = -\int_0^{+\infty} g_3(s)w_{xx}(x, t-s)\,\mathrm{d}s,$$

where $g_i : \mathbb{R}_+ \to \mathbb{R}_+$ are differentiable, non-increasing and integrable functions on $\mathbb{R}_+$. Their decay estimate depends only on the growth of the relaxation functions g_i at infinity, which are allowed to have a decay rate at infinity arbitrary close to $\frac{1}{s}$. The same stability estimate of [10] was later established in [11] when only two infinite memories are considered, that is

$$(F_1, F_2, F_3) = \left(0, -\int_0^{+\infty} g_2(s)\psi_{xx}(x, t-s)\,\mathrm{d}s, -\int_0^{+\infty} g_3(s)w_{xx}(x, t-s)\,\mathrm{d}s\right), \tag{1.6}$$

$$(F_1, F_2, F_3) = \left(- \int_0^{+\infty} g_1(s)\varphi_{xx}(x, t-s)\,\mathrm{d}s, 0, - \int_0^{+\infty} g_3(s) w_{xx}(x, t-s)\,\mathrm{d}s \right) \tag{1.7}$$

or

$$(F_1, F_2, F_3) = \left(- \int_0^{+\infty} g_1(s)\varphi_{xx}(x, t-s)\,\mathrm{d}s, - \int_0^{+\infty} g_2(s)\psi_{xx}(x, t-s)\,\mathrm{d}s, 0 \right), \tag{1.8}$$

under the following conditions on the speeds of wave propagations:

$$s_1 = s_2 \text{ in cases (1.6) and (1.7)}, \quad s_1 = s_3 \text{ in case (1.8)}. \tag{1.9}$$

When (1.9) does not hold, a weak stability estimate was given in [11], where the decay rate depends also on the smoothness of the initial data. Similar results were obtained in [15] when the memory term acts on the longitudinal displacements. However, when the memory term acts on the vertical displacements, it was proved in [14] that the system can not be exponentially stable even if the speeds of wave propagations are equal, but it is still polynomially stable.

To the best of our knowledge, the only known stability results for Bresse systems with only one infinite memory acting on the shear angle displacements are the ones obtained in [6] in case

$$(F_1, F_2, F_3) = \left(0, - \int_0^{+\infty} g(s)\psi_{xx}(x, t-s)\,\mathrm{d}s, 0 \right), \tag{1.10}$$

where $g : \mathbb{R}_+ \to \mathbb{R}_+$ is differentiable, non-increasing and integrable function on $\mathbb{R}_+$. In [6], it was assumed that g satisfies, for $\alpha_1, \alpha_2 > 0$,

$$-\alpha_2 g(s) \leq g'(s) \leq -\alpha_1 g(s), \quad \forall s \in \mathbb{R}_+, \tag{1.11}$$

and was shown that the exponential stability holds if and only if (1.4) is satisfied. Otherwise, only the polynomial stability with a decay rate of type $t^{-\frac{1}{2}}$ and its optimality were obtained. Notice that the condition (1.11) implies that g converges exponentially to zero at infinity and satisfies

$$g(0)\mathrm{e}^{-\alpha_2 s} \leq g(s) \leq g(0)\mathrm{e}^{-\alpha_1 s}, \quad \forall s \in \mathbb{R}_+. \tag{1.12}$$

Our goal in this work is to study the well posedness and asymptotic stability of system (1.1) in terms of the arbitrary growth at infinity of the kernel g and the speeds of wave propagations (1.2). We prove that the systems is well posed and its energy converges to zero when time goes to infinity and provide two general decay estimates: a uniform stability estimate under (1.4), and another weak stability result in general. Our results generalize those of [6] and allow a wider class of relaxation functions. See Remark 3.3 below.

The proof of the well posedness is based on the semigroup theory. For the stability estimates, we use the energy method and an approach introduced by the present authors in [12,13].

The paper is organized as follows. In Sect. 2, we present our assumptions on the relaxation function g and state and prove the well posedness of (1.1). In Sect. 3, we present our stability results. The proof of our uniform and weak decay estimates are given, respectively, in Sects. 4 and 5.

2 Well posedness

In this section, we discuss the well posedness of (1.1) using the semigroup approach. Following the method of [5], we consider the functional

$$\eta(x, t, s) = \psi(x, t) - \psi(x, t-s) \quad \text{in }]0, L[\times \mathbb{R}_+ \times \mathbb{R}_+. \tag{2.1}$$

This functional satisfies

$$\begin{cases} \eta_t + \eta_s - \psi_t = 0 & \text{in }]0, L[\times \mathbb{R}_+ \times \mathbb{R}_+, \\ \eta_x(0, t, s) = \eta_x(L, t, s) = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}_+, \\ \eta(x, t, 0) = 0 & \text{in }]0, L[\times \mathbb{R}_+. \end{cases} \tag{2.2}$$

Let $\eta^0(x,s)=\eta(x,0,s)$,

$$U^0=\left(\varphi_0,\psi_0,w_0,\varphi_1,\psi_1,w_1,\eta^0\right)^T, \tag{2.3}$$
$$U=(\varphi,\psi,w,\varphi_t,\psi_t,w_t,\eta)^T \tag{2.4}$$

and

$$g^0=\int_0^{+\infty} g(s)\,\mathrm{d}s. \tag{2.5}$$

Then, the system (1.1) takes the following abstract form:

$$\begin{cases} U_t=\mathcal{A}U,\\ U(t=0)=U^0, \end{cases} \tag{2.6}$$

where $\mathcal{A}$ is the linear operator defined by

$$\mathcal{A}U=\begin{pmatrix} \varphi_t\\ \psi_t\\ w_t\\ k_1\bar\rho_1\varphi_{xx}-l^2k_3\bar\rho_1\varphi+k_1\bar\rho_1\psi_x+l\bar\rho_1(k_1+k_3)w_x\\ -k_1\bar\rho_2\varphi_x+1\bar\rho_2\left(k_2-g^0\right)\psi_{xx}-k_1\bar\rho_2\psi-lk_1\bar\rho_2 w+1\bar\rho_2\int_0^{+\infty} g\eta_{xx}\,\mathrm{d}s\\ -l\bar\rho_1(k_1+k_3)\varphi_x-lk_1\bar\rho_1\psi+k_3\bar\rho_1 w_{xx}-l^2k_1\bar\rho_1 w\\ \psi_t-\eta_s \end{pmatrix}.$$

Let

$$L_2=\left\{v:\mathbb{R}_+\to H_*^1(]0,L[),\ \int_0^L\int_0^{+\infty} g v_x^2\,\mathrm{d}s\,\mathrm{d}x<+\infty\right\} \tag{2.7}$$

and

$$\mathcal{H}=H_0^1(]0,L[)\times\left(H_*^1(]0,L[)\right)^2\times L^2(]0,L[)\times\left(L_*^2(]0,L[)\right)^2\times L_2, \tag{2.8}$$

where

$$L_*^2(]0,L[)=\left\{v\in L^2(]0,L[),\ \int_0^L v\,\mathrm{d}x=0\right\} \tag{2.9}$$

and

$$H_*^1(]0,L[)=\left\{v\in H^1(]0,L[),\ \int_0^L v\,\mathrm{d}x=0\right\}. \tag{2.10}$$

The domain $D(\mathcal{A})$ of $\mathcal{A}$ is defined by

$$\begin{aligned} D(\mathcal{A})=\Big\{&V=(v_1,\dots,v_7)^T\in\mathcal{H},\ \mathcal{A}V\in\mathcal{H},\ v_7(0)=0,\ \partial_x v_2(0)=\partial_x v_3(0)=0,\\ &\partial_x v_2(L)=\partial_x v_3(L)=0,\ \partial_x v_7(\cdot,0)=\partial_x v_7(\cdot,L)=0\Big\}; \end{aligned} \tag{2.11}$$

that is, according to the definition of $\mathcal{H}$ and $\mathcal{A}$,

$$\begin{aligned} D(\mathcal{A})=\Big\{&(v_1,\dots,v_7)^T\in\mathcal{H},\ (v_1,\dots,v_6)^T\in H_0^1(]0,L[)\times\left(H_*^1(]0,L[)\right)^2\times H_0^1(]0,L[)\times\left(H_*^1(]0,L[)\right)^2,\\ &v_1,\ v_3\in H^2(]0,L[),\ \left(k_2-g^0\right)\partial_{xx}v_2+\int_0^{+\infty} g\,\partial_{xx}v_7\,\mathrm{d}s\in L_*^2(]0,L[),\ \partial_s v_7\in L_2,\\ &v_7(0)=0,\ \partial_x v_2(0)=\partial_x v_3(0)=\partial_x v_2(L)=\partial_x v_3(L)=0,\ \partial_x v_7(\cdot,0)=\partial_x v_7(\cdot,L)=0\Big\}. \end{aligned}$$

More generally, for $n \in \mathbb{N}$,

$$D(\mathcal{A}^n) = \begin{cases} \mathcal{H} & \text{if} \quad n = 0, \\ D(\mathcal{A}) & \text{if} \quad n = 1, \\ \left\{ V \in D(\mathcal{A}^{n-1}),\ \mathcal{A}V \in D(\mathcal{A}^{n-1}) \right\} & \text{if} \quad n = 2, 3, \ldots. \end{cases}$$

Remark 2.1 As in [11], by integrating on $]0, L[$ the second and third equations in (1.1), and using the boundary conditions, we get

$$\partial_{tt}\left(\int_0^L \psi \, \mathrm{d}x\right) + \frac{k_1}{\rho_2}\int_0^L \psi \, \mathrm{d}x + \frac{lk_1}{\rho_2}\int_0^L w \, \mathrm{d}x = 0 \tag{2.12}$$

and

$$\partial_{tt}\left(\int_0^L w \, \mathrm{d}x\right) + \frac{l^2 k_1}{\rho_1}\int_0^L w \, \mathrm{d}x + \frac{lk_1}{\rho_1}\int_0^L \psi \, \mathrm{d}x = 0. \tag{2.13}$$

Therefore, (2.12) implies that

$$\int_0^L w \, \mathrm{d}x = -\frac{\rho_2}{lk_1}\partial_{tt}\left(\int_0^L \psi \, \mathrm{d}x\right) - \frac{1}{l}\int_0^L \psi \, \mathrm{d}x. \tag{2.14}$$

Substituting (2.14) into (2.13), we get

$$\partial_{tttt}\left(\int_0^L \psi \, \mathrm{d}x\right) + \left(\frac{k_1}{\rho_2} + \frac{l^2 k_1}{\rho_1}\right)\partial_{tt}\left(\int_0^L \psi \, \mathrm{d}x\right) = 0. \tag{2.15}$$

Let $l_0 = \sqrt{\frac{k_1}{\rho_2} + \frac{l^2 k_1}{\rho_1}}$. Then, solving (2.15), we find

$$\int_0^L \psi \, dx = \tilde{c}_1 \cos(l_0 t) + \tilde{c}_2 \sin(l_0 t) + \tilde{c}_3 t + \tilde{c}_4, \tag{2.16}$$

where $\tilde{c}_1, \ldots, \tilde{c}_4$ are real constants. By combining (2.14) and (2.16), we get

$$\int_0^L w \, dx = \tilde{c}_1\left(\frac{\rho_2 l_0^2}{lk_1} - \frac{1}{l}\right)\cos(l_0 t) + \tilde{c}_2\left(\frac{\rho_2 l_0^2}{lk_1} - \frac{1}{l}\right)\sin(l_0 t) - \frac{\tilde{c}_3}{l}t - \frac{\tilde{c}_4}{l}. \tag{2.17}$$

Let

$$(\tilde{\psi}_0(x), \tilde{w}_0(x)) = (\psi_0(x, 0), w_0(x)).$$

Using the initial data of ψ and w in (1.1), we see that

$$\begin{cases} \tilde{c}_1 = \frac{k_1}{\rho_2 l_0^2}\displaystyle\int_0^L \tilde{\psi}_0 \, dx + \frac{lk_1}{\rho_2 l_0^2}\int_0^L \tilde{w}_0 \, dx, \\ \tilde{c}_2 = \frac{k_1}{\rho_2 l_0^3}\displaystyle\int_0^L \psi_1 \, dx + \frac{lk_1}{\rho_2 l_0^3}\int_0^L w_1 \, dx, \\ \tilde{c}_3 = \left(1 - \frac{k_1}{\rho_2 l_0^2}\right)\displaystyle\int_0^L \psi_1 \, dx - \frac{lk_1}{\rho_2 l_0^2}\int_0^L w_1 \, dx, \\ \tilde{c}_4 = \left(1 - \frac{k_1}{\rho_2 l_0^2}\right)\displaystyle\int_0^L \tilde{\psi}_0 \, dx - \frac{lk_1}{\rho_2 l_0^2}\int_0^L \tilde{w}_0 \, dx. \end{cases}$$

Let

$$\tilde{\psi} = \psi - \frac{1}{L}\left(\tilde{c}_1 \cos(l_0 t) + \tilde{c}_2 \sin(l_0 t) + \tilde{c}_3 t + \tilde{c}_4\right) \tag{2.18}$$

and

$$\tilde{w} = w - \frac{1}{L}\left(\tilde{c}_1\left(\frac{\rho_2 l_0^2}{lk_1} - \frac{1}{l}\right)\cos(l_0 t) + \tilde{c}_2\left(\frac{\rho_2 l_0^2}{lk_1} - \frac{1}{l}\right)\sin(l_0 t) - \frac{\tilde{c}_3}{l}t - \frac{\tilde{c}_4}{l}\right). \tag{2.19}$$

Then, from (2.16) and (2.17), one can check that

$$\int_0^L \tilde{\psi}\,\mathrm{d}x = \int_0^L \tilde{w}\,\mathrm{d}x = 0, \tag{2.20}$$

and, hence,

$$\int_0^L \tilde{\eta}\,\mathrm{d}x = 0, \tag{2.21}$$

where

$$\tilde{\eta}(x,t,s) = \tilde{\psi}(x,t) - \tilde{\psi}(x,t-s) \quad \text{in }]0,L[\times\mathbb{R}_+\times\mathbb{R}_+.$$

Therefore, Poincaré's inequality

$$\exists\, c_0 > 0 : \int_0^L v^2\,\mathrm{d}x \le c_0 \int_0^L v_x^2\,\mathrm{d}x, \quad \forall v \in H_*^1(]0,L[) \tag{2.22}$$

is applicable for $\tilde{\psi}$, $\tilde{w}$ and $\tilde{\eta}$, provided that $\tilde{\psi},\ \tilde{w} \in H^1(]0,L[)$. In addition, $(\varphi, \tilde{\psi}, \tilde{w})$ satisfies the boundary conditions and the first three equations in (1.1) with initial data

$$\begin{aligned}
&\psi_0 - \frac{1}{L}(\tilde{c}_1 + \tilde{c}_4), \quad \psi_1 - \frac{1}{L}(l_0\tilde{c}_2 + \tilde{c}_3),\\
&w_0 - \frac{1}{L}\left(\tilde{c}_1\left(\frac{\rho_2 l_0^2}{lk_1} - \frac{1}{l}\right) - \frac{\tilde{c}_4}{l}\right) \quad \text{and} \quad w_1 - \frac{1}{L}\left(\tilde{c}_2 l_0\left(\frac{\rho_2 l_0^2}{lk_1} - \frac{1}{l}\right) - \frac{\tilde{c}_3}{l}\right)
\end{aligned}$$

instead of ψ_0, ψ_1, w_0 and w_1, respectively. In the sequel, we work with $\tilde{\psi}$, $\tilde{w}$ and $\tilde{\eta}$ instead of ψ, w and η, but, for simplicity of notation, we use ψ, w and η instead of $\tilde{\psi}$, $\tilde{w}$ and $\tilde{\eta}$, respectively.

Now, to prove the well posedness of (2.6), we make the following hypothesis:
(H1) The function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is differentiable, non-increasing and integrable on $\mathbb{R}_+$ such that there exists a positive constant k_0 such that, for any

$$(\varphi, \psi, w)^T \in H_0^1(]0,L[) \times \left(H_*^1(]0,L[)\right)^2,$$

we have

$$k_0 \int_0^L \left(\varphi_x^2 + \psi_x^2 + w_x^2\right)\mathrm{d}x \le \int_0^L \left(\left(k_2 - g^0\right)\psi_x^2 + k_1(\varphi_x + \psi + lw)^2 + k_3(w_x - l\varphi)^2\right)\mathrm{d}x. \tag{2.23}$$

Moreover, there exists a positive constant β such that

$$-\beta g(s) \le g'(s), \quad \forall s \in \mathbb{R}_+. \tag{2.24}$$

Remark 2.2 1. It is evident that (2.23) implies that

$$k_0 \int_0^L \left(\varphi_x^2 + \psi_x^2 + w_x^2\right)\mathrm{d}x \le \int_0^L \left(k_2\psi_x^2 + k_1(\varphi_x + \psi + lw)^2 + k_3(w_x - l\varphi)^2\right)\mathrm{d}x. \tag{2.25}$$

On the other hand, thanks to (2.22) applied for ψ and w, and Poincaré's inequality

$$\exists\, \tilde{c}_0 > 0 : \int_0^L v^2\,dx \le \tilde{c}_0 \int_0^L v_x^2\,\mathrm{d}x, \quad \forall v \in H_0^1(]0,L[) \tag{2.26}$$

applied for φ, there exists a positive constant $\tilde{k}_0$ such that, for any

$$(\varphi, \psi, w)^T \in H_0^1(]0, L[) \times \left(H_*^1(]0, L[)\right)^2,$$

we have

$$\int_0^L \left(k_2\psi_x^2 + k_1(\varphi_x + \psi + lw)^2 + k_3(w_x - l\varphi)^2\right) \mathrm{d}x \le \tilde{k}_0 \int_0^L \left(\varphi_x^2 + \psi_x^2 + w_x^2\right) \mathrm{d}x. \tag{2.27}$$

Thus, from (2.25) and (2.27), we deduce that the right hand side of the inequality (2.25) defines a norm on $H_0^1(]0, L[) \times \left(H_*^1(]0, L[)\right)^2$ equivalent to the natural norm of $\left(H^1(]0, L[)\right)^3$.

2. As in [11], we conclude from (2.23) that

$$k_0 + g^0 - k_2 \le 0. \tag{2.28}$$

Indeed, for the choice $\varphi = w = 0$, (2.23) gives

$$\left(k_0 + g^0 - k_2\right) \int_0^L \psi_x^2 \,\mathrm{d}x \le k_1 \int_0^L \psi^2 \,\mathrm{d}x, \quad \forall \psi \in H_*^1(]0, L[).$$

This inequality implies, for $\psi(x) = \cos(\lambda x) - \frac{1}{\lambda L}\sin(\lambda L)$ and $\lambda \in]0, +\infty[$ (notice that $\psi \in H_*^1(]0, L[)$),

$$\left(k_0 + g^0 - k_2\right)\left(L - \frac{1}{2\lambda}\sin(2\lambda L)\right) \le \frac{k_1}{\lambda^2}\left(L + \frac{1}{2\lambda}\sin(2\lambda L) - \frac{2}{\lambda^2 L}\sin^2(\lambda L)\right), \quad \forall \lambda > 0.$$

By letting λ go to $+\infty$, we deduce (2.28).

According to Remark 2.2, we notice that, under the hypothesis **(H1)**, the sets L_2 and $\mathcal{H}$ are Hilbert spaces equipped, respectively, with the inner products that generate the norms, for $v \in L_2$ and $V = (v_1, \ldots, v_7)^T \in \mathcal{H}$,

$$\|v\|_{L_2}^2 = \int_0^L \int_0^{+\infty} g v_x^2 \,\mathrm{d}s\,\mathrm{d}x \tag{2.29}$$

and

$$\begin{aligned}\|V\|_{\mathcal{H}}^2 = &\int_0^L \left(\left(k_2 - g^0\right)(\partial_x v_2)^2 + k_1(\partial_x v_1 + v_2 + lv_3)^2 + k_3(\partial_x v_3 - lv_1)^2\right) \mathrm{d}x \\ &+ \int_0^L \left(\rho_1 v_4^2 + \rho_2 v_5^2 + \rho_1 v_6^2\right) \mathrm{d}x + \|v_7\|_{L_2}^2.\end{aligned} \tag{2.30}$$

Now, a simple computation implies that, for any $V = (v_1, \ldots, v_7)^T \in D(\mathcal{A})$,

$$\langle \mathcal{A}V, V\rangle_{\mathcal{H}} = \frac{1}{2}\int_0^L \int_0^{+\infty} g'(\partial_x v_7)^2 \,\mathrm{d}s\,\mathrm{d}x. \tag{2.31}$$

Since g is non-increasing, we deduce from (2.31) that

$$\langle \mathcal{A}V, V\rangle_{\mathcal{H}} \le 0. \tag{2.32}$$

This implies that A is dissipative. Notice that, according to (2.24) and the fact that g is non-increasing, we see that, for $v \in L_2$,

$$\begin{aligned}\left|\int_0^L \int_0^{+\infty} g' v_x^2 \,\mathrm{d}s\,\mathrm{d}x\right| &= -\int_0^L \int_0^{+\infty} g' v_x^2 \,\mathrm{d}s\,\mathrm{d}x \\ &\le \beta \int_0^L \int_0^{+\infty} g v_x^2 \,\mathrm{d}s\,\mathrm{d}x \\ &\le \beta \|v\|_{L_2}^2 \\ &< +\infty,\end{aligned}$$

so the integral in the right hand side of (2.31) is well defined.

Next, we follow the proof given in [11] to prove that $Id - \mathcal{A}$ is surjective, where Id is the identity operator. Let $F = (f_1, \dots, f_7)^T \in \mathcal{H}$. We seek the existence of $V = (v_1, \dots, v_7)^T \in D(\mathcal{A})$, a solution of the equation

$$(Id - \mathcal{A})V = F. \tag{2.33}$$

The first three equations in (2.33) take the form

$$\begin{cases} v_4 = v_1 - f_1, \\ v_5 = v_2 - f_2, \\ v_6 = v_3 - f_3. \end{cases} \tag{2.34}$$

Using (2.34), the last equation in (2.33) is equivalent to

$$\partial_s v_7 + v_7 = v_2 + f_7 - f_2. \tag{2.35}$$

By integrating (2.35) and using the fact that $v_7(0) = 0$ (from (2.11)), we get

$$v_7(s) = (1 - \mathrm{e}^{-s})(v_2 - f_2) + \mathrm{e}^{-s} \int_0^s \mathrm{e}^{\tau} f_7(\tau)\, \mathrm{d}\tau, \tag{2.36}$$

We see that, from (2.34), if $(v_1, v_2, v_3) \in H_0^1(]0, L[) \times \left(H_*^1(]0, L[)\right)^2$, then $(v_4, v_5, v_6) \in H_0^1(]0, L[) \times \left(H_*^1(]0, L[)\right)^2$. On the other hand, using Fubini theorem, Hölder's inequality and noting that $f_7 \in L_2$, we get

$$\begin{aligned}
\int_0^L \int_0^{+\infty} g(s) \left(\mathrm{e}^{-s} \int_0^s \mathrm{e}^{\tau} \partial_x f_7(\tau)\, \mathrm{d}\tau\right)^2 ds\, \mathrm{d}x \\
&\le \int_0^{+\infty} \mathrm{e}^{-2s} g(s) \left(\int_0^s \mathrm{e}^{\tau}\, \mathrm{d}\tau\right) \int_0^s \mathrm{e}^{\tau} (\partial_x f_7(\tau))^2\, \mathrm{d}\tau\, \mathrm{d}s\, \mathrm{d}x \\
&\le \int_0^L \int_0^{+\infty} \mathrm{e}^{-s}(1 - \mathrm{e}^{-s}) g(s) \int_0^s \mathrm{e}^{\tau} (\partial_x f_7(\tau))^2\, \mathrm{d}\tau\, \mathrm{d}s\, \mathrm{d}x \\
&\le \int_0^L \int_0^{+\infty} \mathrm{e}^{-s} g(s) \int_0^s \mathrm{e}^{\tau} (\partial_x f_7(\tau))^2\, \mathrm{d}\tau\, \mathrm{d}s\, \mathrm{d}x \\
&\le \int_0^L \int_0^{+\infty} \mathrm{e}^{\tau} (\partial_x f_7(\tau))^2 \int_{\tau}^{+\infty} \mathrm{e}^{-s} g(s)\, \mathrm{d}s\, \mathrm{d}\tau\, \mathrm{d}x \\
&\le \int_0^L \int_0^{+\infty} \mathrm{e}^{\tau} g(\tau) (\partial_x f_7(\tau))^2 \int_{\tau}^{+\infty} \mathrm{e}^{-s}\, \mathrm{d}s\, \mathrm{d}\tau\, \mathrm{d}x \\
&\le \int_0^L \int_0^{+\infty} g(\tau) (\partial_x f_7(\tau))^2\, \mathrm{d}\tau\, \mathrm{d}x \\
&\le \|f_7\|_{L_2}^2 \\
&< +\infty,
\end{aligned}$$

$$\begin{aligned}
\int_0^L \int_0^{+\infty} g(s) \left(\mathrm{e}^{-s} \int_0^s \mathrm{e}^{\tau} \partial_x f_7(\tau)\, \mathrm{d}\tau\right)^2 ds\, \mathrm{d}x &\le \int_0^L \int_0^{+\infty} \mathrm{e}^{-2s} g(s) \left(\int_0^s \mathrm{e}^{\tau}\, \mathrm{d}\tau\right) \int_0^s \mathrm{e}^{\tau} (\partial_x f_7(\tau))^2\, \mathrm{d}\tau\, \mathrm{d}s\, \mathrm{d}x \\
&\le \int_0^L \int_0^{+\infty} \mathrm{e}^{-s}(1 - \mathrm{e}^{-s}) g(s) \int_0^s \mathrm{e}^{\tau} (\partial_x f_7(\tau))^2\, \mathrm{d}\tau\, \mathrm{d}s\, \mathrm{d}x \\
&\le \int_0^L \int_0^{+\infty} \mathrm{e}^{-s} g(s) \int_0^s \mathrm{e}^{\tau} (\partial_x f_7(\tau))^2\, \mathrm{d}\tau\, \mathrm{d}s\, \mathrm{d}x \\
&\le \int_0^L \int_0^{+\infty} \mathrm{e}^{\tau} (\partial_x f_7(\tau))^2 \int_{\tau}^{+\infty} \mathrm{e}^{-s} g(s)\, \mathrm{d}s\, \mathrm{d}\tau\, \mathrm{d}x \\
&\le \int_0^L \int_0^{+\infty} \mathrm{e}^{\tau} g(\tau) (\partial_x f_7(\tau))^2 \int_{\tau}^{+\infty} \mathrm{e}^{-s}\, \mathrm{d}s\, \mathrm{d}\tau\, \mathrm{d}x \\
&\le \int_0^L \int_0^{+\infty} g(\tau) (\partial_x f_7(\tau))^2\, \mathrm{d}\tau\, \mathrm{d}x \\
&\le \|f_7\|_{L_2}^2 \\
&< +\infty,
\end{aligned}$$

then

$$s \mapsto \mathrm{e}^{-s} \int_0^s \mathrm{e}^{\tau} f_7(\tau)\, \mathrm{d}\tau \in L_2,$$

and therefore, (2.36) implies that $v_7 \in L_2$. Moreover, $\partial_s v_7 \in L_2$ by (2.35). Therefore, to prove that (2.33) admits a solution $V \in D(\mathcal{A})$, it is enough to show that

$$\partial_x v_7(\cdot, 0) = \partial_x v_7(\cdot, L) = 0 \tag{2.37}$$

and (v_1, v_2, v_3) exists and satisfies the required regularity and boundary conditions in $D(\mathcal{A})$, that is

$$(v_1, v_2, v_3)^T \in \left(H^2(]0, L[) \cap H_0^1(]0, L[)\right) \times H_*^1(]0, L[) \times \left(H^2(]0, L[) \cap H_*^1(]0, L[)\right)^2, \tag{2.38}$$

$$\left(k_2 - g^0\right) \partial_{xx} v_2 + \int_0^{+\infty} g \partial_{xx} v_7 \,\mathrm{d}s \in L_*^2(]0, L[) \tag{2.39}$$

and

$$\partial_x v_2(0) = \partial_x v_3(0) = \partial_x v_2(L) = \partial_x v_3(L) = 0. \tag{2.40}$$

Let us assume that (2.37)–(2.40) hold. Multiplying the fourth, fifth and sixth equations in (2.33) by $\rho_1 \tilde{v}_1$, $\rho_2 \tilde{v}_2$ and $\rho_1 \tilde{v}_3$, respectively, integrating their sum over $]0, L[$, using the boundary conditions (2.37) and (2.40), and inserting (2.34) and (2.36), we get that (v_1, v_2, v_3) solves the variational problem

$$a_1\left((v_1, v_2, v_3)^T, (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)^T\right) = \tilde{a}_1\left((\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)^T\right), \tag{2.41}$$

for any $(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)^T \in H_0^1(]0, L[) \times \left(H_*^1(]0, L[)\right)^2$, where

$$\begin{aligned} & a_1\left((v_1, v_2, v_3)^T, (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)^T\right) \\ & \quad = \int_0^L (k_1(\partial_x v_1 + v_2 + l v_3)(\partial_x \tilde{v}_1 + \tilde{v}_2 + l\tilde{v}_3) + k_3(\partial_x v_3 - l v_1)(\partial_x \tilde{v}_3 - l\tilde{v}_1))\, \mathrm{d}x \\ & \qquad + \int_0^L \left(\rho_1 v_1 \tilde{v}_1 + \rho_2 v_2 \tilde{v}_2 + \rho_1 v_3 \tilde{v}_3 + (k_2 - \tilde{g}^0)\partial_x v_2 \partial_x \tilde{v}_2\right) \mathrm{d}x, \end{aligned} \tag{2.42}$$

$\tilde{g}^0 = \int_0^{+\infty} \mathrm{e}^{-s} g(s)\, \mathrm{d}s$ and

$$\begin{aligned} \tilde{a}_1\left((\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)^T\right) = & \int_0^L (\rho_1(f_1 + f_4)\tilde{v}_1 + \rho_2(f_2 + f_5)\tilde{v}_2 + \rho_1(f_3 + f_6)\tilde{v}_3)\, \mathrm{d}x \\ & + (g^0 - \tilde{g}^0) \int_0^L \partial_x f_2 \partial_x \tilde{v}_2 \,\mathrm{d}x \\ & - \int_0^L \left(\int_0^{+\infty} \mathrm{e}^{-s} g(s) \int_0^s \mathrm{e}^{\tau} \partial_x f_7(\tau)\, \mathrm{d}\tau\, ds\right) \partial_x \tilde{v}_2 \,\mathrm{d}x. \end{aligned} \tag{2.43}$$

We note that, as before, using again Fubini theorem, Hölder's inequality and the fact that $f_7 \in L_2$,

$$
\begin{aligned}
&\int_0^L \left(\int_0^{+\infty} \mathrm{e}^{-s} g(s) \int_0^s \mathrm{e}^{\tau} \partial_x f_7(\tau)\, \mathrm{d}\tau\, ds \right)^2 \mathrm{d}x \\
&\qquad \le \int_0^L \left(\int_0^{+\infty} \mathrm{e}^{-s} g(s) \int_0^s \mathrm{e}^{\tau} |\partial_x f_7(\tau)|\, \mathrm{d}\tau\, \mathrm{d}s \right)^2 \mathrm{d}x \\
&\qquad \le \int_0^L \left(\int_0^{+\infty} \mathrm{e}^{\tau} |\partial_x f_7(\tau)| \int_\tau^{+\infty} g(s) \mathrm{e}^{-s}\, \mathrm{d}s\, \mathrm{d}\tau \right)^2 \mathrm{d}x \\
&\qquad \le \int_0^L \left(\int_0^{+\infty} \mathrm{e}^{\tau} g(\tau) |\partial_x f_7(\tau)| \int_\tau^{+\infty} \mathrm{e}^{-s}\, \mathrm{d}s\, \mathrm{d}\tau \right)^2 \mathrm{d}x \\
&\qquad \le \int_0^L \left(\int_0^{+\infty} g(\tau) |\partial_x f_7(\tau)|\, \mathrm{d}\tau \right)^2 \mathrm{d}x \\
&\qquad \le \int_0^L \left(\int_0^{+\infty} g(\tau)\, \mathrm{d}\tau \right) \left(\int_0^{+\infty} g(\tau) (\partial_x f_7(\tau))^2\, \mathrm{d}\tau \right) \mathrm{d}x \\
&\qquad \le g^0 \| f_7 \|_{L_2}^2 \\
&\qquad < +\infty,
\end{aligned}
$$

$$
\begin{aligned}
\int_0^L \left(\int_0^{+\infty} \mathrm{e}^{-s} g(s) \int_0^s \mathrm{e}^{\tau} \partial_x f_7(\tau)\, \mathrm{d}\tau\, ds \right)^2 \mathrm{d}x &\le \int_0^L \left(\int_0^{+\infty} \mathrm{e}^{-s} g(s) \int_0^s \mathrm{e}^{\tau} |\partial_x f_7(\tau)|\, \mathrm{d}\tau\, \mathrm{d}s \right)^2 \mathrm{d}x \\
&\le \int_0^L \left(\int_0^{+\infty} \mathrm{e}^{\tau} |\partial_x f_7(\tau)| \int_\tau^{+\infty} g(s) \mathrm{e}^{-s}\, \mathrm{d}s\, \mathrm{d}\tau \right)^2 \mathrm{d}x \\
&\le \int_0^L \left(\int_0^{+\infty} \mathrm{e}^{\tau} g(\tau) |\partial_x f_7(\tau)| \int_\tau^{+\infty} \mathrm{e}^{-s}\, \mathrm{d}s\, \mathrm{d}\tau \right)^2 \mathrm{d}x \\
&\le \int_0^L \left(\int_0^{+\infty} g(\tau) |\partial_x f_7(\tau)|\, \mathrm{d}\tau \right)^2 \mathrm{d}x \\
&\le \int_0^L \left(\int_0^{+\infty} g(\tau)\, \mathrm{d}\tau \right) \left(\int_0^{+\infty} g(\tau) (\partial_x f_7(\tau))^2\, \mathrm{d}\tau \right) \mathrm{d}x \\
&\le g^0 \| f_7 \|_{L_2}^2 \\
&< +\infty,
\end{aligned}
$$

which implies that

$$
x \mapsto \int_0^{+\infty} \mathrm{e}^{-s} g(s) \int_0^s \mathrm{e}^{\tau} \partial_x f_7(\tau)\, \mathrm{d}\tau\, ds \in L^2(]0, L[).
$$

On the other hand, $\tilde{g}^0 \le g^0 < k_2$ (by (2.28)). Then, by virtue of (2.23) and (2.27), we have a_1 is a bilinear, continuous and coercive form on

$$
\left(H_0^1(]0, L[) \times \left(H_*^1(]0, L[) \right)^2 \right) \times \left(H_0^1(]0, L[) \times \left(H_*^1(]0, L[) \right)^2 \right),
$$

and $\tilde{a}_1$ is a linear and continuous form on $H_0^1(]0, L[) \times \left(H_*^1(]0, L[) \right)^2$. Consequently, using the Lax–Milgram theorem, we deduce that (2.41) has a unique solution

$$
(v_1, v_2, v_3)^T \in H_0^1(]0, L[) \times \left(H_*^1(]0, L[) \right)^2 .
$$

Therefore, using classical elliptic regularity arguments, we conclude that the forth, fifth and sixth equations in (2.33) are satisfied with $(v_1, v_2, v_3)^T$ satisfying (2.38) and (2.40), and, using (2.34) and (2.36), v_7 satisfies (2.37) and (2.39). Thus, we deduce that (2.33) admits a unique solution $V \in D(\mathcal{A})$, and then $Id - \mathcal{A}$ is surjective.

The operator $-\mathcal{A}$ is then linear maximal monotone, and $D(\mathcal{A})$ is dense in $\mathcal{H}$. Finally, thanks to the Hille–Yosida theorem (see [23]), we deduce from (2.32) and (2.33) that $\mathcal{A}$ generates a C_0-semigroup of contractions in $\mathcal{H}$. This gives the following well-posedness results of (2.6) (see [17,23]).

Theorem 2.3 *Assume that* **(H1)** *holds. For any* $n \in \mathbb{N}$ *and* $U^0 \in D(\mathcal{A}^n)$, (2.6) *has a unique solution*

$$U \in \bigcap_{k=0}^{n} C^{n-k}\left(\mathbb{R}_+; D\left(\mathcal{A}^k\right)\right). \tag{2.44}$$

3 Stability

In this section, we study the stability of (2.6), where the obtained two (uniform and weak) decay rates of solution depend on the speeds of wave propagations (1.2) and the growth of g at infinity characterized by the following additional hypothesis:

(H2) Assume that $g(0) > 0$ and there exists a non-increasing differentiable function $\xi : \mathbb{R}_+ \to \mathbb{R}_+^*$ such that

$$g'(s) \leq -\xi(s)g(s), \quad \forall s \in \mathbb{R}_+. \tag{3.1}$$

We start by considering the case where the speeds of wave propagations (1.2) satisfy (1.4).

Theorem 3.1 *Assume that* **(H1)**, **(H2)** *and* (1.4) *are satisfied such that*

$$l \text{ is small enough}. \tag{3.2}$$

Let $U^0 \in \mathcal{H}$ *be such that*

$$\xi \equiv \text{ constant } \text{ or } \sup_{s\in\mathbb{R}_+} \int_0^L \left(\eta_x^0(x,s)\right)^2 \mathrm{d}x < +\infty. \tag{3.3}$$

Then, there exist constants $\beta_0 \in]0,1]$ *and* $\alpha_1 > 0$ *such that, for all* $\alpha_0 \in]0, \beta_0[$, *the solution of* (2.6) *satisfies*

$$\|U(t)\|_{\mathcal{H}}^2 \leq \alpha_1 \left(1 + \int_0^t (g(s))^{1-\alpha_0}\,\mathrm{d}s\right) \mathrm{e}^{-\alpha_0 \int_0^t \xi(s)\,\mathrm{d}s} + \alpha_1 \int_t^{+\infty} g(s)\,\mathrm{d}s, \quad \forall t \in \mathbb{R}_+. \tag{3.4}$$

When (1.4) does not hold, we prove the following weaker stability result for (2.6).

Theorem 3.2 *Assume that* **(H1)**, **(H2)** *and* (3.2) *are satisfied. Let* $U^0 \in D(\mathcal{A})$ *be such that*

$$\xi \equiv \text{ constant } \text{ or } \sup_{s\in\mathbb{R}_+} \max_{k=0,1} \int_0^L \left(\partial_s^k \eta_x^0(x,s)\right)^2 \mathrm{d}x < +\infty \tag{3.5}$$

and

$$s_1 = s_3. \tag{3.6}$$

Then, there exists a positive constant α_1 *such that*

$$\|U(t)\|_{\mathcal{H}}^2 \leq \frac{\alpha_1\left(1 + \int_0^t \xi(s) \int_s^{+\infty} g(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\right)}{\int_0^t \xi(s)\,\mathrm{d}s}, \quad \forall t > 0. \tag{3.7}$$

Remark 3.3 1. If (3.1) holds with $\xi \equiv$ constant, then (3.4) and (3.7) give, respectively, for some positive constants d_1 and d_2,

$$\|U(t)\|_{\mathcal{H}}^2 \leq d_1 \mathrm{e}^{-d_2 t}, \quad \forall t \in \mathbb{R}_+ \tag{3.8}$$

and

$$\|U(t)\|_{\mathcal{H}}^2 \leq \frac{d_1}{t}, \quad \forall t > 0. \tag{3.9}$$

Therefore, this particular case includes the results of [6]. The estimates (3.8) and (3.9) give the best decay rates which can be obtained from (3.4) and (3.7), respectively.

2. When $\xi \equiv$ constant, condition (3.1) implies that g converges exponentially to zero at infinity. However, when $\xi \neq$ constant, condition (3.1) allows $s \mapsto g(s)$ to have a decay rate arbitrarily close to $\frac{1}{s}$ at infinity, which represents the critical limit, since g is integrable on $\mathbb{R}_+$. To illustrate our general stability estimates, we give here some particular examples of g satisfying (3.1), and show the specific corresponding decay rates given by (3.4) and (3.7).
   (i) Let $g(t) = d\mathrm{e}^{-(1+t)^q}$ with $0 < q < 1$ and $d > 0$ (g converges to zero at infinity faster than any polynomial). Then, (3.1) holds with $\xi(t) = q(1+t)^{q-1}$, and consequently, (3.4) and (3.7) give, respectively, for two positive constants c_1 and c_2,
$$E(t) \leq c_1 \mathrm{e}^{-c_2(1+t)^q}, \quad \forall t \in \mathbb{R}_+$$
and
$$E(t) \leq c_1(1+t)^{-q}, \quad \forall t \in \mathbb{R}_+.$$
   (ii) Let $g(t) = d(1+t)^{-q}$ with $q > 1$ and $d > 0$ (g has at most a polynomial decay at infinity). Assumption (3.1) holds with $\xi(t) = q(1+t)^{-1}$, and consequently, (3.4) and (3.7) give, respectively, for two positive constants c_1 and c_2,
$$E(t) \leq c_1(1+t)^{-c_2}, \quad \forall t \in \mathbb{R}_+$$
and
$$E(t) \leq c_1(\ln(1+t))^{-1}, \quad \forall t > 0.$$

To prove (3.4) and (3.7), we will consider suitable multipliers and construct appropriate Lyapunov functionals satisfying some differential inequalities, for any $U^0 \in D(\mathcal{A})$ and $t \in \mathbb{R}_+$; so all the calculations are justified. By integrating these differential inequalities, we get (3.4) and (3.7), for any $U^0 \in D(\mathcal{A})$. By simple density arguments ($D(\mathcal{A})$ is dense in $\mathcal{H}$), (3.4) remains valid, for any $U^0 \in \mathcal{H}$.

We will use c, throughout the rest of this paper, to denote a generic positive constant which depends continuously on the initial data U^0 and the fixed parameters in (1.1), (2.22) and (2.26), and can be different from step to step. When c depends on some new constants $y_1, y_2, \ldots$, introduced in the proof, the constant c is noted $c_{y_1}, c_{y_1,y_2}, \ldots$.

Let us consider the energy functional E associated to (2.6) defined by
$$E(t) = \frac{1}{2}\|U(t)\|_{\mathcal{H}}^2. \tag{3.10}$$
From (2.6) and (2.31), we see that
$$E_i'(t) = \frac{1}{2}\int_0^L \int_0^{+\infty} g' \eta_x^2 \, \mathrm{d}s \, \mathrm{d}x. \tag{3.11}$$
Recalling that g is non-increasing, (3.11) implies that E is non-increasing, and consequently, (2.6) is dissipative.

4 Proof of uniform decay (3.4)

First, we consider the following functional:
$$I(t) = -\rho_2 \int_0^L \psi_t \int_0^{+\infty} g(s)\eta \, \mathrm{d}s \, \mathrm{d}x. \tag{4.1}$$

Lemma 4.1 *For any $\delta_0 > 0$, there exists $c_{\delta_0} > 0$ such that*
$$\begin{aligned} I'(t) \leq & -\rho_2\left(g^0 - \delta_0\right)\int_0^L \psi_t^2 \, \mathrm{d}x + \delta_0 \int_0^L \left(\psi_x^2 + (\varphi_x + \psi + lw)^2\right) \mathrm{d}x \\ & + c_{\delta_0}\int_0^L \int_0^{+\infty} \left(g(s) - g'(s)\right)\eta_x^2 \, \mathrm{d}s \, \mathrm{d}x. \end{aligned} \tag{4.2}$$

Proof First, we note that

$$\begin{aligned}\partial_t \int_0^{+\infty} g(s)\eta \,\mathrm{d}s &= \partial_t \int_{-\infty}^{t} g(t-s)(\psi(t)-\psi(s))\,\mathrm{d}s\\ &= \int_{-\infty}^{t} g'(t-s)(\psi(t)-\psi(s))\,\mathrm{d}s + \left(\int_{-\infty}^{t} g(t-s)\,\mathrm{d}s\right)\psi_t;\end{aligned}$$

that is

$$\partial_t \int_0^{+\infty} g(s)\eta \,\mathrm{d}s = \int_0^{+\infty} g'(s)\eta\,\mathrm{d}s + g^0\psi_t. \tag{4.3}$$

Second, using Young's and Hölder's inequalities, we get the following inequality: for all $\lambda > 0$, there exists $c_\lambda > 0$ such that, for any $v \in L^2(]0, L[)$ and $\hat{\eta} \in \{\eta, \partial_x \eta\}$,

$$\left|\int_0^L v \int_0^{+\infty} g(s)\hat{\eta}\,\mathrm{d}s\,\mathrm{d}x\right| \le \lambda \int_0^L v^2\,\mathrm{d}x + c_\lambda \int_0^L \int_0^{+\infty} g(s)\hat{\eta}^2\,\mathrm{d}s\,\mathrm{d}x. \tag{4.4}$$

Similarly,

$$\left|\int_0^L v \int_0^{+\infty} g'(s)\hat{\eta}\,\mathrm{d}s\,\mathrm{d}x\right| \le \lambda \int_0^L v^2\,\mathrm{d}x - c_\lambda \int_0^L \int_0^{+\infty} g'(s)\hat{\eta}^2\,\mathrm{d}s\,\mathrm{d}x. \tag{4.5}$$

Now, direct computations, using the first equation in (1.1), integrating by parts and using the boundary conditions and (4.3), yield

$$\begin{aligned}I'(t) = &-\rho_2 g^0 \int_0^L \psi_t^2\,\mathrm{d}x + \int_0^L \left(\int_0^{+\infty} g(s)\eta_x\,\mathrm{d}s\right)^2 \mathrm{d}x\\ &+ \left(k_1 - g^0\right)\int_0^L \psi_x \int_0^{+\infty} g(s)\eta_x\,\mathrm{d}s\,\mathrm{d}x\\ &+k_1 \int_0^L (\varphi_x + \psi + lw) \int_0^{+\infty} g(s)\eta\, ds\, dx\\ &-\rho_2 \int_0^L \psi_t \int_0^{+\infty} g'(s)\eta\,\mathrm{d}s\,\mathrm{d}x.\end{aligned}$$

Using (4.4) and (4.5) for the last three terms of this equality, Poincaré's inequality (2.22) for η, and Hölder's inequality to estimate

$$\left(\int_0^{+\infty} g(s)\partial_x\eta\,\mathrm{d}s\right)^2,$$

we get (4.2). □

Lemma 4.2 *Let*

$$\begin{aligned}J(t) = \rho_2 \int_0^L (\varphi_x + \psi + lw)\psi_t\,\mathrm{d}x + \frac{k_2\rho_1}{k_1}\int_0^L \psi_x\varphi_t\,\mathrm{d}x\\ -\tfrac{\rho_1}{k_1}\int_0^L \varphi_t \int_0^{+\infty} g(s)\psi_x(t-s)\,\mathrm{d}s\ \mathrm{d}x.\end{aligned} \tag{4.6}$$

Then, for any $\delta_0, \epsilon_0, \epsilon_1, \epsilon_2 > 0$, *there exist* $c_{\delta_0}, c_{\epsilon_0} > 0$ *such that*

$$\begin{aligned}J'(t) \le &-k_1 \int_0^L (\varphi_x + \psi + lw)^2\,\mathrm{d}x + \left(\delta_0 + \frac{lk_2k_3\epsilon_1}{2k_1} + \frac{lk_3g^0\epsilon_2}{2k_1}\right)\int_0^L (w_x - l\varphi)^2\,\mathrm{d}x\\ &+\delta_0 \int_0^L \varphi_t^2\,\mathrm{d}x + \left(\frac{lk_2k_3}{2k_1\epsilon_1} + \frac{lk_3g^0}{2k_1\epsilon_2}\right)\int_0^L \psi_x^2\,\mathrm{d}x + \int_0^L \left(c_{\epsilon_0}\psi_t^2 + \epsilon_0 w_t^2\right)\mathrm{d}x\\ &+ \left(\tfrac{k_2\rho_1}{k_1} - \rho_2\right)\int_0^L \psi_{xt}\varphi_t\,\mathrm{d}x + c_{\delta_0}\int_0^L\int_0^{+\infty} (g(s) - g'(s))\eta_x^2\,\mathrm{d}s\,\mathrm{d}x.\end{aligned} \tag{4.7}$$

Proof First, notice that

$$\begin{aligned}\partial_t \int_0^{+\infty} g(s)\psi_x(t-s)\,\mathrm{d}s &= \partial_t \int_{-\infty}^{t} g(t-s)\psi_x(s)\,\mathrm{d}s\\ &= g(0)\psi_x(t) + \int_{-\infty}^{t} g'(t-s)\psi_x(s)\,\mathrm{d}s\\ &= -\int_0^{+\infty} g'(s)\psi_x(t)\,\mathrm{d}s + \int_0^{+\infty} g'(s)\psi_x(t-s)\,\mathrm{d}s;\end{aligned}$$

that is

$$\partial_t \int_0^{+\infty} g(s)\psi_x(t-s)\,\mathrm{d}s = -\int_0^{+\infty} g'(s)\eta_x\,\mathrm{d}s. \tag{4.8}$$

Now, by exploiting the first two equations in (1.1), integrating by parts, using (4.8) and the boundary conditions, we get

$$\begin{aligned}J'(t) = &-k_1\int_0^L (\varphi_x+\psi+lw)^2\,\mathrm{d}x + \left(\frac{k_2\rho_1}{k_1}-\rho_2\right)\int_0^L \psi_{xt}\varphi_t\,\mathrm{d}x + \rho_2\int_0^L \psi_t^2\,\mathrm{d}x\\ &+\rho_2 l\int_0^L \psi_t w_t\,\mathrm{d}x + \frac{lk_3}{k_1}\left(k_2-g^0\right)\int_0^L (w_x-l\varphi)\psi_x\,\mathrm{d}x\\ &+\tfrac{\rho_1}{k_1}\int_0^L \varphi_t\int_0^{+\infty} g'(s)\eta_x\,\mathrm{d}s\,\mathrm{d}x + \frac{lk_3}{k_1}\int_0^L (w_x-l\varphi)\int_0^{+\infty} g(s)\eta_x\,\mathrm{d}s\,\mathrm{d}x.\end{aligned}$$

By applying (4.4), (4.5) and Young's inequality for the last four terms of the above equality, we deduce (4.7). □

Lemma 4.3 *Let*

$$K(t) = -\rho_1\int_0^L (\varphi_x+\psi+lw)w_t\,\mathrm{d}x - \frac{k_3\rho_1}{k_1}\int_0^L (w_x-l\varphi)\varphi_t\,\mathrm{d}x. \tag{4.9}$$

Then, for any $\epsilon_0>0$*, there exists* $c_{\epsilon_0}>0$ *such that*

$$\begin{aligned}K'(t) \le\; & lk_1\int_0^L (\varphi_x+\psi+lw)^2\,\mathrm{d}x - \frac{lk_3^2}{k_1}\int_0^L (w_x-l\varphi)^2\,\mathrm{d}x + c_{\epsilon_0}\int_0^L \psi_t^2\,\mathrm{d}x\\ &+\int_0^L \left(\frac{l\rho_1k_3}{k_1}\varphi_t^2 + (-l\rho_1+\epsilon_0)w_t^2\right)\mathrm{d}x + \rho_1\left(\frac{k_3}{k_1}-1\right)\int_0^L w_t\varphi_{xt}\,\mathrm{d}x.\end{aligned} \tag{4.10}$$

Proof Using the first and third equations in (1.1), integrating by parts, recalling (4.8) and using the boundary conditions, we find

$$\begin{aligned}K'(t) = &\,lk_1\int_0^L (\varphi_x+\psi+lw)^2\,\mathrm{d}x - \frac{lk_3^2}{k_1}\int_0^L (w_x-l\varphi)^2\,\mathrm{d}x + \rho_1\left(\frac{k_3}{k_1}-1\right)\int_0^L \varphi_{xt}w_t\,\mathrm{d}x\\ &-l\rho_1\int_0^L w_t^2\,\mathrm{d}x + \frac{lk_3\rho_1}{k_1}\int_0^L \varphi_t^2\,\mathrm{d}x - \rho_1\int_0^L \psi_t w_t\,\mathrm{d}x.\end{aligned}$$

By applying Young's inequality for the last four term of the above equality, we obtain (4.10). □

Lemma 4.4 *Let*

$$\begin{aligned}P(t) = &-\rho_1k_3\int_0^L (w_x-l\varphi)\int_0^x w_t(y,t)\,\mathrm{d}y\,\mathrm{d}x\\ &-\rho_1k_1\int_0^L \varphi_t\int_0^x (\varphi_x+\psi+lw)(y,t)\,\mathrm{d}y\,\mathrm{d}x.\end{aligned} \tag{4.11}$$

Then, for any $\epsilon_0,\ \delta_1>0$*, there exists* $c_{\epsilon_0}>0$ *such that*

$$\begin{aligned}P'(t) \le\; & k_1^2\int_0^L (\varphi_x+\psi+lw)^2\,\mathrm{d}x - k_3^2\int_0^L (w_x-l\varphi)^2\,\mathrm{d}x + c_{\epsilon_0}\int_0^L \psi_t^2\,\mathrm{d}x\\ &+\left(-\rho_1k_1+\epsilon_0+\tfrac{l\rho_1|k_3-k_1|\delta_1}{2}\right)\int_0^L \varphi_t^2\,\mathrm{d}x + \rho_1\left(k_3+\frac{\tilde{c}_0 l|k_3-k_1|}{2\delta_1}\right)\int_0^L w_t^2\,\mathrm{d}x.\end{aligned} \tag{4.12}$$

Proof By exploiting the first and third equations in (1.1), integrating by parts and using (2.20) and the boundary conditions, we get

$$\begin{aligned} P'(t) = & -\rho_1 k_3 \int_0^L w_t^2 \,\mathrm{d}x + \rho_1 k_1 \int_0^L \varphi_t^2 \,\mathrm{d}x - k_1^2 \int_0^L (\varphi_x + \psi + l w)^2 \,\mathrm{d}x \\ & + k_3^2 \int_0^L (w_x - l\varphi)^2 \,\mathrm{d}x + \rho_1 \int_0^L \varphi_t \int_0^x (k_1 \psi_t(y,t) + l(k_1 - k_3) w_t(y,t)) \,\mathrm{d}y \,\mathrm{d}x. \end{aligned} \tag{4.13}$$

Noting that the functions

$$x \mapsto \int_0^x \psi_t(y,t)\,\mathrm{d}y \quad \text{and} \quad x \mapsto \int_0^x w_t(y,t)\,\mathrm{d}y$$

vanish at 0 and L (because of (2.20)), then, applying (2.26), we have

$$\int_0^L \left(\int_0^x \psi_t(y,t)\,\mathrm{d}y \right)^2 \mathrm{d}x \le \tilde{c}_0 \int_0^L \psi_t^2 \,\mathrm{d}x \tag{4.14}$$

and

$$\int_0^L \left(\int_0^x w_t(y,t)\,\mathrm{d}y \right)^2 \mathrm{d}x \le \tilde{c}_0 \int_0^L w_t^2 \,\mathrm{d}x. \tag{4.15}$$

By applying Young's inequality for the last term in (4.13), and recalling (4.14) and (4.15), we conclude (4.12). □

Lemma 4.5 *Let*

$$R(t) = -\int_0^L (\rho_1 \varphi \varphi_t + \rho_2 \psi \psi_t + \rho_1 w w_t)\,\mathrm{d}x. \tag{4.16}$$

Then, for any $\delta_0 > 0$, *there exists* $c_{\delta_0} > 0$ *such that*

$$\begin{aligned} R'(t) \le & \int_0^L \left(\left(k_2 + \delta_0 - g^0\right) \psi_x^2 + k_1(\varphi_x + \psi + l w)^2 + k_3 (w_x - l\varphi)^2 \right) \mathrm{d}x \\ & - \int_0^L \left(\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + \rho_1 w_t^2 \right) \mathrm{d}x + c_{\delta_0} \int_0^L \int_0^{+\infty} g(s) \eta_x^2 \,\mathrm{d}s \,\mathrm{d}x. \end{aligned} \tag{4.17}$$

Proof By exploiting the first three equations in (1.1), integrating by parts and using the boundary conditions, we find

$$\begin{aligned} R'(t) = & \int_0^L \left(\left(k_2 - g^0\right) \psi_x^2 + k_1(\varphi_x + \psi + l w)^2 + k_3 (w_x - l\varphi)^2 \right) \mathrm{d}x \\ & - \int_0^L \left(\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + \rho_1 w_t^2 \right) \mathrm{d}x + \int_0^L \psi_x \int_0^{+\infty} g(s) \eta_x \,\mathrm{d}s \,\mathrm{d}x. \end{aligned}$$

By applying (4.4) for the last term in this equality, we arrive at (4.17). □

Lemma 4.6 *Let*

$$D(t) = -\rho_2 \int_0^L \psi_x \int_0^x \psi_t(y,t)\,\mathrm{d}y\,\mathrm{d}x. \tag{4.18}$$

Then, for any $\delta_0, \delta_2 > 0$, *there exists* $c_{\delta_0} > 0$ *such that*

$$\begin{aligned} D'(t) \le & \rho_2 \int_0^L \psi_t^2 \,\mathrm{d}x + \left(\frac{k_1}{2\delta_2} + g^0 + \delta_0 - k_2 \right) \int_0^L \psi_x^2 \,\mathrm{d}x \\ & + \frac{\tilde{c}_0 k_1 \delta_2}{2} \int_0^L (\varphi_x + \psi + l w)^2 \,\mathrm{d}x + c_{\delta_0} \int_0^L \int_0^{+\infty} g(s) \eta_x^2 \,\mathrm{d}s \,\mathrm{d}x. \end{aligned} \tag{4.19}$$

Proof By exploiting the second equation in (1.1), integrating by parts and using the boundary conditions, we find

$$\begin{aligned} D'(t) = \rho_2 \int_0^L \psi_t^2 \,\mathrm{d}x + \left(g^0 - k_2\right) \int_0^L \psi_x^2 \,\mathrm{d}x - \int_0^L \psi_x \int_0^{+\infty} g(s)\eta_x \,\mathrm{d}s \,\mathrm{d}x \\ +k_1 \int_0^L \psi_x \int_0^x (\varphi_x(y,t) + \psi(y,t) + lw(y,t))\,\mathrm{d}y\,\mathrm{d}x. \end{aligned} \tag{4.20}$$

Noting that the function

$$x \mapsto \int_0^x (\varphi_x(y,t) + \psi(y,t) + lw(y,t))\,\mathrm{d}y$$

vanishes at 0 and L (because of (2.20)), then, applying (2.26), we have

$$\int_0^L \left(\int_0^x (\varphi_x(y,t) + \psi(y,t) + lw(y,t))\,\mathrm{d}y \right)^2 \mathrm{d}x \le \tilde{c}_0 \int_0^L (\varphi_x + \psi + lw)^2 \,\mathrm{d}x. \tag{4.21}$$

Then, application of Young's inequality and (4.4) for the last two terms in (4.20), and use of (4.21) yield (4.19). □

Let $N,\ N_1,\ N_2,\ N_3,\ N_4,\ N_5 > 0$ and

$$F := NE + N_1 I + N_2 P + N_3 K + N_4 R + N_5 D + J. \tag{4.22}$$

Then, by combining (4.2), (4.7), (4.10), (4.12), (4.17) and (4.19), we obtain

$$\begin{aligned} F'(t) \le \int_0^L \left(l_1\varphi_t^2 + l_2\psi_t^2 + l_3 w_t^2 + l_4\psi_x^2 + l_5(w_x - l\varphi)^2 + l_6(\varphi_x + \psi + lw)^2\right) \mathrm{d}x \\ +NE'(t) + c_{N_1,N_4,N_5,\delta_0} \int_0^L \int_0^{+\infty} \left(g(s) - g'(s)\right) \eta_x^2 \,\mathrm{d}s\,\mathrm{d}x \\ +\delta_0 c_{N_1,N_4,N_5} \int_0^L \left(\psi_x^2 + (\varphi_x + \psi + lw)^2 + (w_x - l\varphi)^2 + \varphi_t^2 + \psi_t^2\right) \mathrm{d}x \\ + \left(\tfrac{k_2\rho_1}{k_1} - \rho_2\right) \int_0^L \psi_{xt}\varphi_t \,\mathrm{d}x + N_3\rho_1 \left(\frac{k_3}{k_1} - 1\right) \int_0^L w_t\varphi_{xt}\,\mathrm{d}x \\ +\epsilon_0 c_{N_2,N_3} \int_0^L \left(\varphi_t^2 + w_t^2\right) \mathrm{d}x + c_{N_2,N_3,\epsilon_0} \int_0^L \psi_t^2 \,\mathrm{d}x, \end{aligned} \tag{4.23}$$

where

$$\begin{aligned} l_1 &= -\rho_1 k_1 N_2 - \rho_1 N_4 + \frac{l\rho_1|k_3 - k_1|\delta_1 N_2}{2} + \frac{l\rho_1 k_3 N_3}{k_1}, \\ l_2 &= -\rho_2 g^0 N_1 - \rho_2 N_4 + \rho_2 N_5, \\ l_3 &= -l\rho_1 N_3 - \rho_1 N_4 + \rho_1 \left(k_3 + \frac{l\tilde{c}_0|k_3 - k_1|}{2\delta_1}\right) N_2, \\ l_4 &= -\left(k_2 - \frac{k_1}{2\delta_2}\right) N_5 + k_2 N_4 + \frac{lk_2k_3}{2k_1\epsilon_1} + g^0 \left(N_5 - N_4 + \frac{lk_3}{2k_1\epsilon_2}\right), \\ l_5 &= -k_3^2 N_2 - \frac{lk_3^2 N_3}{k_1} + k_3 N_4 + \frac{lk_2k_3\epsilon_1}{2k_1} + \frac{lk_3 g^0 \epsilon_2}{2k_1} \\ l_6 &= -k_1 + k_1^2 N_2 + lk_1 N_3 + k_1 N_4 + \frac{\tilde{c}_0 k_1 \delta_2 N_5}{2}. \end{aligned}$$

Using (2.23), (2.30), (3.10) and (3.11), we get from (4.23) that

$$F'(t) \le \int_0^L \left(l_1\varphi_t^2 + l_2\psi_t^2 + l_3 w_t^2 + l_4\psi_x^2 + l_5(w_x - l\varphi)^2 + l_6(\varphi_x + \psi + lw)^2\right) \mathrm{d}x \tag{4.24}$$

$$+\delta_0 c_{N_1,N_4,N_5}E(t)+(N-c_{N_1,N_4,N_5,\delta_0})E'(t)+c_{N_1,N_4,N_5,\delta_0}\int_0^L\int_0^{+\infty}g(s)\eta_x^2\,\mathrm{d}s\,\mathrm{d}x$$
$$+\left(\tfrac{k_2\rho_1}{k_1}-\rho_2\right)\int_0^L\psi_{xt}\varphi_t\,\mathrm{d}x+N_3\rho_1\left(\frac{k_3}{k_1}-1\right)\int_0^L w_t\varphi_{xt}\,\mathrm{d}x$$
$$+\epsilon_0 c_{N_2,N_3}\int_0^L\left(\varphi_t^2+w_t^2\right)\mathrm{d}x+c_{N_2,N_3,\epsilon_0}\int_0^L\psi_t^2\,\mathrm{d}x.$$

At this point, we choose carefully the constants $N,\ N_i,\ \delta_i$ and ϵ_i to get suitable values of l_i.

First, let us take

$$N_3=\delta_1=1,\quad \varepsilon_1=\frac{k_3}{k_2},\quad \varepsilon_2=\frac{k_3}{2g^0},\quad \delta_2=\frac{k_1}{k_2-g^0},\quad N_4=k_3N_2,\quad N_5=4k_3N_2;$$

thus, the l_i's take the forms

$$\begin{cases} l_1=-\rho_1(k_1+k_3)N_2+l\rho_1\left(\frac{|k_1-k_3|}{2}N_2+\frac{k_3}{k_1}\right),\\ l_2=-\rho_2(g^0N_1-3k_3N_2),\\ l_3=-l\rho_1\left(1-\frac{\tilde{c}_0|k_1-k_3|}{2}N_2\right),\\ l_4=-(k_2-g^0)k_3N_2+\frac{l}{k_1}\left(\frac{k_2^2}{2}+(g^0)^2\right),\\ l_5=-\frac{lk_3^2}{4k_1}<0,\\ l_6=-k_1\left(1-\left(k_1+k_3+\frac{2\tilde{c}_0k_1k_3}{k_2-g^0}\right)N_2\right)+lk_1. \end{cases}$$

Now, we choose $N_2>0$ so small that

$$1-\tilde{c}_0|k_1-k_3|N_2>0,\quad 1-\left(k_1+k_3+\frac{2\tilde{c}_0k_1k_3}{k_2-g^0}\right)N_2>0,$$

then, take $\varepsilon_0=\frac{1}{2c_{N_2,N_3}}l\rho_1$, so that we have

$$\begin{cases} \tilde{l}_1=l_1+\varepsilon_0c_{N_2,N_3}=-\rho_1(k_1+k_3)N_2+l\rho_1\left(\frac{1}{2}+\frac{|k_1-k_3|}{2}N_2+\frac{k_3}{k_1}\right),\\ \tilde{l}_2=l_2+c_{N_2,N_3,\varepsilon_0},\\ \tilde{l}_3=l_3+\varepsilon_0c_{N_2,N_3}=-\frac{l\rho_1}{2}\left(1-\tilde{c}_0|k_1-k_3|N_2\right)<0. \end{cases}$$

Next, we recall (3.2) to select $l>0$ small enough such that

$$\tilde{l}_1<0,\quad l_4<0,\quad l_6<0.$$

After that, we pick $N_1>0$ very large so that $\tilde{l}_2<0$. Then, we find that

$$\hat{l}:=2\max\left\{\frac{1}{\rho_1}\tilde{l}_1,\ \frac{1}{\rho_2}\tilde{l}_2,\ \frac{1}{\rho_1}\tilde{l}_3,\ \frac{1}{k_2}l_4,\ \frac{1}{k_3}l_5,\ \frac{1}{k_1}l_6\right\}<0$$

and, using (2.30) and (3.10),

$$\begin{aligned}
&\int_0^L\left(\tilde{l}_1\varphi_t^2+\tilde{l}_2\psi_t^2+\tilde{l}_3w_t^2+l_4\psi_x^2+l_5(w_x-l\varphi)^2+l_6(\varphi_x+\psi+lw)^2\right)\mathrm{d}x+\delta_0c_{N_1,N_4,N_5}E(t)\\
&\le\frac{\hat{l}}{2}\int_0^L\left(\rho_1\varphi_t^2+\rho_2\psi_t^2+\rho_1w_t^2+k_2\psi_x^2+k_3(w_x-l\varphi)^2+k_1(\varphi_x+\psi+lw)^2\right)\mathrm{d}x+\delta_0c_{N_1,N_4,N_5}E(t)\\
&\le(\hat{l}+\delta_0c_{N_1,N_4,N_5})E(t)+\frac{\hat{l}g^0}{2}\int_0^L\psi_x^2\,\mathrm{d}x-\frac{\hat{l}}{2}\int_0^L\int_0^{+\infty}g(s)\eta_x^2\,\mathrm{d}s\,\mathrm{d}x\\
&\le(\hat{l}+\delta_0c_{N_1,N_4,N_5})E(t)-\frac{\hat{l}}{2}\int_0^L\int_0^{+\infty}g(s)\eta_x^2\,\mathrm{d}s\,\mathrm{d}x.
\end{aligned}\tag{4.25}$$

Finally, we take $\delta_0 > 0$ small enough so that

$$\hat{l} + \delta_0 c_{N_1,N_2,N_5} < 0.$$

Consequently, we obtain from (4.24) and (4.25), for some positive constants $c, \tilde{c}_1$,

$$\begin{aligned} F'(t) \le -\tilde{c}_1 E(t) + (N-c)E'(t) + c\int_0^L \int_0^{+\infty} g(s)\eta_x^2 \,\mathrm{d}s\,\mathrm{d}x \\ + \left(\tfrac{k_2\rho_1}{k_1} - \rho_2\right)\int_0^L \psi_{xt}\varphi_t \,\mathrm{d}x + N_3\rho_1\left(\frac{k_3}{k_1} - 1\right)\int_0^L w_t\varphi_{xt}\,\mathrm{d}x. \end{aligned} \tag{4.26}$$

Now, we estimate the integral of $g\eta_x^2$ in (4.26).

Case $\xi \equiv$ **constant**. From (3.1), we have

$$\begin{aligned} \xi(t)\int_0^L \int_0^{+\infty} g(s)\eta_x^2 \,\mathrm{d}s\,\mathrm{d}x &= \int_0^L \int_0^{+\infty} \xi g(s)\eta_x^2 \,\mathrm{d}s\,\mathrm{d}x \\ &\le -\int_0^L \int_0^{+\infty} g'(s)\eta_x^2 \,\mathrm{d}s\,\mathrm{d}x, \end{aligned}$$

then, using (3.11), we find

$$\xi(t)\int_0^L \int_0^{+\infty} g(s)\eta_x^2 \,\mathrm{d}s\,\mathrm{d}x \le -2E'(t). \tag{4.27}$$

Case $\xi \ne$ **constant**. Following the arguments of [12] and [13], and using (3.1) and the fact that ξ is non-increasing, we get

$$\begin{aligned} \xi(t)\int_0^L \int_0^{t} g(s)\eta_x^2 \,\mathrm{d}s\,\mathrm{d}x &\le \int_0^L \int_0^{t} \xi(s) g(s)\eta_x^2 \,\mathrm{d}s\,\mathrm{d}x \\ &\le -\int_0^L \int_0^{t} g'(s)\eta_x^2 \,\mathrm{d}s\,\mathrm{d}x, \end{aligned}$$

then, recalling (3.11), we obtain

$$\xi(t)\int_0^L \int_0^{t} g(s)\eta_x^2 \,\mathrm{d}s\,\mathrm{d}x \le -2E'(t). \tag{4.28}$$

On the other hand, the definition of E, (2.23) and the fact that E is non-increasing imply that

$$\int_0^L \psi_x^2(x,t)\,\mathrm{d}x \le cE(0).$$

Therefore,

$$\begin{aligned} \int_0^L \eta_x^2 \,\mathrm{d}x &= \int_0^L \left(\eta_x^0(x,s-t) + \psi_x(x,t) - \psi_x(x,0)\right)^2 \mathrm{d}x \\ &\le c\left(E(0) + \sup_{s\in\mathbb{R}_+}\int_0^L \left(\eta_x^0(x,s)\right)^2 \mathrm{d}x\right). \end{aligned}$$

Then, using the boundedness condition on η^0 in (3.3), we deduce that

$$\xi(t)\int_0^L \int_t^{+\infty} g(s)\eta_x^2 \,\mathrm{d}s\,\mathrm{d}x \le c\xi(t)\int_t^{+\infty} g(s)ds. \tag{4.29}$$

Hence, by combining (4.28) and (4.29), we find

$$\xi(t)\int_0^L \int_0^{+\infty} g(s)\eta_x^2 \,\mathrm{d}s\,\mathrm{d}x \le -2E'(t) + c\xi(t)\int_t^{+\infty} g(s)ds. \tag{4.30}$$

Finally, multiplying (4.26) by $\xi(t)$ and combining with (4.27) and (4.30), we get for the two previous cases, for some $\tilde{c}_2 > 0$,

$$\begin{aligned} \xi(t)F'(t) &\leq -\tilde{c}_1\xi(t)E(t) + c\xi(t)\int_t^{+\infty} g(s)ds + (N-c)\xi(t)E'(t) - \tilde{c}_2E'(t) \\ &\quad + \left(\tfrac{k_2\rho_1}{k_1} - \rho_2\right)\xi(t)\int_0^L \psi_{xt}\varphi_t\,\mathrm{d}x + N_3\rho_1\left(\frac{k_3}{k_1} - 1\right)\xi(t)\int_0^L w_t\varphi_{xt}\,\mathrm{d}x. \end{aligned} \tag{4.31}$$

On the other hand, from (2.23), (2.30) and (3.10), we deduce that there exists a positive constant γ (independent of N) satisfying

$$|N_1 I + N_2 P + N_3 K + N_4 R + N_5 D + J| \leq \gamma E,$$

which, combined with (4.22), implies that

$$(N-\gamma)E \leq F \leq (N+\gamma)E. \tag{4.32}$$

Choosing N so that

$$N \geq c \quad \text{and} \quad N > \gamma,$$

noting that $E' \leq 0$ and using (4.31) and (4.32), we deduce that $F \sim E$ and

$$\begin{aligned} \tilde{F}'(t) &\leq -\tilde{c}_1\xi(t)E(t) + ch(t) + \xi'(t)F(t) \\ &\quad + \left(\tfrac{k_2\rho_1}{k_1} - \rho_2\right)\xi(t)\int_0^L \psi_{xt}\varphi_t\,\mathrm{d}x + N_3\rho_1\left(\frac{k_3}{k_1} - 1\right)\xi(t)\int_0^L w_t\varphi_{xt}\,\mathrm{d}x, \end{aligned} \tag{4.33}$$

where

$$\tilde{F} = \xi F + \tilde{c}_2 E \quad \text{and} \quad h(t) = \xi(t)\int_t^{+\infty} g(s)\mathrm{d}s.$$

From (4.32) and the relation $0 \leq \xi(t)F(t) \leq \xi(0)F(t)$, we see that

$$\tilde{c}_2 E \leq \tilde{F} \leq (\tilde{c}_2 + \xi(0)(N+\gamma))E. \tag{4.34}$$

Therefore, (4.33) implies that, for any $\alpha_0 \in]0, \beta_0[$, where $\beta_0 = \min\left\{1, \frac{\tilde{c}_1}{\tilde{c}_2+\xi(0)(N+\gamma)}\right\}$,

$$\begin{aligned} \tilde{F}'(t) &\leq -\alpha_0\xi(t)\tilde{F}(t) + ch(t) \\ &\quad + \left(\tfrac{k_2\rho_1}{k_1} - \rho_2\right)\xi(t)\int_0^L \psi_{xt}\varphi_t\,\mathrm{d}x + N_3\rho_1\left(\frac{k_3}{k_1} - 1\right)\xi(t)\int_0^L w_t\varphi_{xt}\,\mathrm{d}x, \end{aligned} \tag{4.35}$$

Since the last two terms in (4.35) vanish (thanks to (1.4)), then (4.35) implies that

$$\partial_t\left(\mathrm{e}^{\alpha_0\int_0^t \xi(s)\,\mathrm{d}s}\tilde{F}(t)\right) \leq c\mathrm{e}^{\alpha_0\int_0^t \xi(s)\,\mathrm{d}s} h(t).$$

Therefore, by integrating over $[0, T]$ with $T \geq 0$, we get

$$\tilde{F}(T) \leq \mathrm{e}^{-\alpha_0\int_0^T \xi(s)\,\mathrm{d}s}\left(\tilde{F}(0) + c\int_0^T \mathrm{e}^{\alpha_0\int_0^t \xi(s)\,\mathrm{d}s} h(t)\mathrm{d}t\right),$$

which implies, according to (4.34), that

$$E(T) \leq c\mathrm{e}^{-\alpha_0\int_0^T \xi(s)\,\mathrm{d}s}\left(1 + \int_0^T \mathrm{e}^{\alpha_0\int_0^t \xi(s)\,\mathrm{d}s} h(t)\,\mathrm{d}t\right). \tag{4.36}$$

Since

$$e^{\alpha_0 \int_0^t \xi(s)\,ds} h(t) = \frac{1}{\alpha_0}\partial_t\left(e^{\alpha_0 \int_0^t \xi(s)\,ds}\right)\int_t^{+\infty} g(s)\,ds,$$

then, by integration by parts, we obtain

$$\int_0^T e^{\alpha_0 \int_0^t \xi(s)\,ds} h(t)\,dt$$
$$= \frac{1}{\alpha_0}\left(e^{\alpha_0 \int_0^T \xi(s)\,ds}\int_T^{+\infty} g(s)\,ds - \int_0^{+\infty} g(s)\,ds + \int_0^T e^{\alpha_0 \int_0^t \xi(s)\,ds} g(t)\,dt\right).$$

Consequently, combining with (4.36), we arrive at

$$E(T) \le c\left(e^{-\alpha_0 \int_0^T \xi(s)\,ds} + \int_T^{+\infty} g(s)\,ds\right)$$
$$+ce^{-\alpha_0 \int_0^T \xi(s)\,ds}\int_0^T e^{\alpha_0 \int_0^t \xi(s)\,ds} g(t)\,dt. \tag{4.37}$$

On the other hand, (3.1) implies that

$$\partial_t\left(e^{\alpha_0 \int_0^t \xi(s)\,ds}(g(t))^{\alpha_0}\right) = \alpha_0 (g(t))^{\alpha_0-1}(\xi(t)g(t)+g'(t))e^{\alpha_0 \int_0^t \xi(s)\,ds} \le 0,$$

and, hence,

$$e^{\alpha_0 \int_0^t \xi(s)\,ds}(g(t))^{\alpha_0} \le (g(0))^{\alpha_0}.$$

Therefore,

$$\int_0^T e^{\alpha_0 \int_0^t \xi(s)\,ds} g(t)\,dt \le (g(0))^{\alpha_0}\int_0^T (g(t))^{1-\alpha_0}\,dt. \tag{4.38}$$

Finally, (3.10) and (4.38) give (3.4).

5 Proof of weak decay (3.7)

In this section, we treat the case when (1.4) does not hold but (3.6) holds. In this case, the last term in (4.35) vanishes. Therefore, we need to estimate

$$\left(\frac{k_2\rho_1}{k_1} - \rho_2\right)\xi(t)\int_0^L \psi_{xt}\varphi_t\,dx$$

using the following system resulting from differentiating (1.1) with respect to time t:

$$\begin{cases} \rho_1\varphi_{ttt} - k_1(\varphi_{xt} + \psi_t + lw_t)_x - lk_3(w_{xt} - l\varphi_t) = 0, \\ \rho_2\psi_{ttt} - k_2\psi_{xxt} + k_1(\varphi_{xt} + \psi_t + lw_t) + \displaystyle\int_0^{+\infty} g(s)\psi_{xxt}(x, t-s)\,\mathrm{d}s = 0, \\ \rho_1 w_{ttt} - k_3(w_{xt} - l\varphi_t)_x + lk_1(\varphi_{xt} + \psi_t + lw_t) = 0, \\ \varphi_t(0, t) = \psi_{xt}(0, t) = w_{xt}(0, t) = \varphi_t(L, t) = \psi_{xt}(L, t) = w_{xt}(L, t) = 0. \end{cases} \tag{5.1}$$

System (5.1) is well posed for initial data $U^0 \in D(\mathcal{A})$ thanks to Theorem 2.3, where $U_t \in C(\mathbb{R}_+; \mathcal{H})$. Let $U^0 \in D(\mathcal{A})$ and $\tilde{E}$ be the energy of (5.1) defined by

$$\tilde{E}(t) = \frac{1}{2}\|U_t(t)\|^2_{\mathcal{H}}. \tag{5.2}$$

Similarly to (3.11), we have

$$\tilde{E}'(t) = \frac{1}{2}\int_0^L \int_0^{+\infty} g'\eta_{xt}^2\,\mathrm{d}s\,\mathrm{d}x \le 0; \tag{5.3}$$

so $\tilde{E}$ is non-increasing. We use an idea introduced in [9] to get the following lemma.

Lemma 5.1 *For any $\epsilon > 0$, there exists $c_\epsilon > 0$ such that*

$$\left|\left(\frac{k_2\rho_1}{k_1} - \rho_2\right)\int_0^L \psi_{xt}\varphi_t\,\mathrm{d}x\right| \le c_\epsilon \int_0^L \int_0^{+\infty} g(s)\eta_{xt}^2\,\mathrm{d}s\,\mathrm{d}x + \epsilon E(t) - c_\epsilon E'(t). \tag{5.4}$$

Proof We have, by the definition of η,

$$\begin{aligned} \left(\tfrac{k_2\rho_1}{k_1} - \rho_2\right)\int_0^L \psi_{xt}\varphi_t\,\mathrm{d}x = \tfrac{1}{g^0}\left(\tfrac{k_2\rho_1}{k_1} - \rho_2\right)\int_0^L \varphi_t \int_0^{+\infty} g(s)\eta_{xt}\,\mathrm{d}s\,\mathrm{d}x \\ +\tfrac{1}{g^0}\left(\tfrac{k_2\rho_1}{k_1} - \rho_2\right)\int_0^L \varphi_t \int_0^{+\infty} g(s)\psi_{xt}(t-s)\,\mathrm{d}s\,\mathrm{d}x. \end{aligned} \tag{5.5}$$

Using (4.4) and (3.10), we get, for all $\epsilon > 0$,

$$\begin{aligned} \left|\tfrac{1}{g^0}\left(\tfrac{k_2\rho_1}{k_1} - \rho_2\right)\int_0^L \varphi_t \int_0^{+\infty} g(s)\eta_{xt}\,\mathrm{d}s\,\mathrm{d}x\right| \le \tfrac{\epsilon}{2}E(t) \\ +c_\epsilon \int_0^L \int_0^{+\infty} g(s)\eta_{xt}^2\,\mathrm{d}s\,\mathrm{d}x. \end{aligned} \tag{5.6}$$

On the other hand, by integrating with respect to s and using the definition of η, we obtain

$$\begin{aligned} \int_0^L \varphi_t \int_0^{+\infty} g(s)\psi_{xt}(t-s)\,\mathrm{d}s\,\mathrm{d}x &= -\int_0^L \varphi_t \int_0^{+\infty} g(s)\partial_s(\psi_x(t-s))\,\mathrm{d}s\,\mathrm{d}x \\ &= \int_0^L \varphi_t \left(g(0)\psi_x(t) + \int_0^{+\infty} g'(s)\psi_x(t-s)\,\mathrm{d}s\right)\mathrm{d}x \\ &= -\int_0^L \varphi_t \int_0^{+\infty} g'(s)\eta_x\,\mathrm{d}s\,\mathrm{d}x. \end{aligned}$$

Therefore, using (4.5) and (3.11),

$$\left|\frac{1}{g^0}\left(-\frac{k_2\rho_1}{k_1} - \rho_2\right)\int_0^L \varphi_t \int_0^{+\infty} g(s)\psi_{xt}(t-s)\,\mathrm{d}s\,\mathrm{d}x\right| \le \frac{\epsilon}{2}E(t) - c_\epsilon E'(t). \tag{5.7}$$

Inserting (5.6) and (5.7) into (5.5), we obtain (5.4). □

Now, using (3.6), combining (4.35) and (5.4), and choosing ϵ small enough, we find

$$\begin{aligned}\tilde{F}'(t) \leq -c\xi(t)E(t) + ch(t) - c\xi(t)E'(t)\\ +c\xi(t)\int_0^L \int_0^{+\infty} g(s)\eta_{xt}^2 \,\mathrm{d}s\,\mathrm{d}x.\end{aligned} \tag{5.8}$$

On the other hand, using the boundedness condition on η^0 in (3.5), we have (as for (4.27) and (4.30))

$$\xi(t)\int_0^L \int_0^{+\infty} g(s)\eta_{xt}^2 \,\mathrm{d}s\,\mathrm{d}x \leq -c\tilde{E}'(t) + ch(t). \tag{5.9}$$

Hence, combining (5.8) and (5.9), we have

$$\left(\tilde{F}(t) + c\tilde{E}(t) + c\xi(t)E(t)\right)' \leq -c\xi(t)E(t) + ch(t), \tag{5.10}$$

since ξ is non-increasing. Therefore, by integrating on $[0, T]$ and using the fact E is non-increasing, we get

$$cE(T)\int_0^T \xi(t)\,\mathrm{d}t \leq \tilde{F}(0) + c\tilde{E}(0) + c\xi(0)E(0) + c\int_0^T h(t)\,\mathrm{d}t,$$

which gives (3.7), since (3.10).

Comments.

1. This work generalizes the results of [6] and allows a wider class of relaxation functions.
2. Note that when $w = l = 0$, we obtain the Timoshenko system and our results reduce to those of [12].
3. It would be very interesting to obtain these general decay results without conditions (3.2) and (3.3).

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References

1. Alabau-Boussouira, F.; Muñoz Rivera, J.E.; Almeida Júnior, D.S.: Stability to weak dissipative Bresse system. J. Math. Anal. Appl. **374**, 481–498 (2011)
2. Alves, M.O.; Fatori, L.H.; Jorge Silva, M.A.; Monteiro, R.N.: Stability and optimality of decay rate for weakly dissipative Bresse system. Math. Methods Appl. Sci. **38**, 898–908 (2015)
3. Bresse, J.A.C.: Cours de Méchanique Appliquée. Mallet Bachelier, Paris (1859)
4. Charles, W.; Soriano, J.A.; Nascimentoc, F.A.; Rodrigues, J.H.: Decay rates for Bresse system with arbitrary nonlinear localized damping. J. Differ. Equ. **255**, 2267–2290 (2013)
5. Dafermos, C.M.: Asymptotic stability in viscoelasticity. Arch. Ration. Mech. Anal. **37**, 297–308 (1970)
6. De Lima Santos, M.; Soufyane, A.; Da Silva Almeida Júnior, D.: Asymptotic behavior to Bresse system with past history. Q. Appl. Math. **73**, 23–54 (2015)
7. Fatori, L.H.; Monteiro, R.N.: The optimal decay rate for a weak dissipative Bresse system. Appl. Math. Lett. **25**, 600–604 (2012)
8. Fatori, L.H.; Muñoz Rivera, J.E.: Rates of decay to weak thermoelastic Bresse system. IMA J. Appl. Math. **75**, 881–904 (2010)
9. Fernández Sare, H.D.; Muñoz Rivera, J.E.: Stability of Timoshenko systems with past history. J. Math. Anal. Appl. **339**, 482–502 (2008)
10. Guesmia, A.; Kafini, M.: Bresse system with infinite memories. Math. Methods Appl. Sci. **38**, 2389–2402 (2015)
11. Guesmia, A.; Kirane, M.: Uniform and weak stability of Bresse system with two infinite memories. ZAMP **67**, 1–39 (2016)

12. Guesmia, A.; Messaoudi, S.: A general stability result in a Timoshenko system with infinite memory: a new approach. Math. Methods Appl. Sci. **37**, 384–392 (2014)
13. Guesmia, A.; Messaoudi, S.: A new approach to the stability of an abstract system in the presence of infinite history. J. Math. Anal. Appl. **416**, 212–228 (2014)
14. Guesmia, A.: Non-exponential and polynomial stability results of a Bresse system with one infinite memory in the vertical displacement. Nonauton. Dyn. Syst. **4**, 78–97 (2017)
15. Guesmia, A.: Asymptotic stability of Bresse system with one infinite memory in the longitudinal displacements. Mediter. J. Math. **14**, 19 (2017)
16. Keddi, A.; Apalara, T.; Messaoudi, S.A.: Exponential and polynomial decay in a thermoelastic-Bresse system with second sound. Appl. Math. Optim. **77**, 315–341 (2018)
17. Komornik, V.: Exact Controllability and Stabilization. The Multiplier Method. Masson-John Wiley, Paris (1994)
18. Lagnese, J.E.; Leugering, G.; Schmidt, J.P.: Modelling of dynamic networks of thin thermoelastic beams. Math. Methods Appl. Sci. **16**, 327–358 (1993)
19. Lagnese, J.E., Leugering, G., Schmidt, J.P.: Modelling Analysis and Control of Dynamic Elastic Multi-Link Structures, Systems Control Found. Appl. Birkhauser Boston, Inc., Boston (1994) **(ISBN: 0-8176-3705-2)**
20. Liu, Z.; Rao, B.: Energy decay rate of the thermoelastic Bresse system. Z. Angew. Math. Phys. **60**, 54–69 (2009)
21. Najdi, N.; Wehbe, A.: Weakly locally thermal stabilization of Bresse systems. Electron. J. Differ. Equ. **2014**, 1–19 (2014)
22. Noun, N.; Wehbe, A.: Weakly locally internal stabilization of elastic Bresse system. C. R. Acad. Sci. Paris Sér. I **350**, 493–498 (2012)
23. Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer, New York (1983)
24. Qin, Y.; Yang, X.-G.; Ma, Z.: Global existence of solutions for the thermoelastic Bresse system. Commun. Pure Appl. Anal. **13**, 1395–1406 (2014)
25. Said-Houari, B.; Hamadouche, T.: The asymptotic behavior of the Bresse–Cattaneo system. Commun. Contemp. Math. **18**, 1–18 (2016)
26. Said-Houari, B.; Hamadouche, T.: The Cauchy problem of the Bresse system in thermoelasticity of type III. Appl. Anal. **95**, 2323–2338 (2016)
27. Said-Houari, B.; Soufyane, A.: The Bresse system in thermoelasticity. Math. Methods Appl. Sci. **38**, 3642–3652 (2015)
28. Soriano, J.A.; Charles, W.; Schulz, R.: Asymptotic stability for Bresse systems. J. Math. Anal. Appl. **412**, 369–380 (2014)
29. Soriano, J.A.; Muñoz Rivera, J.E.; Fatori, L.H.: Bresse system with indefinite damping. J. Math. Anal. Appl. **387**, 284–290 (2012)
30. Soufyane, A.; Said-Houari, B.: The effect of the wave speeds and the frictional damping terms on the decay rate of the Bresse system. Evolut. Equ. Control Theory **3**, 713–738 (2014)
31. Wehbe, A.; Youssef, W.: Exponential and polynomial stability of an elastic Bresse system with two locally distributed feedbacks. J. Math. Phys. **51**, 1–17 (2010)