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Rana S. Kahil · Mohammad N. Abdulrahim

On the representations of the braid group constructed by C. M. Egea and E. Galina

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Abstract In this paper, we determine a sufficient condition for the irreducibility of the family of representations of the braid group constructed by C. M. Egea and E. Galina without requiring that the representations are self-adjoint. Then, we construct a new subfamily of multi-parameter representations (ψ_m, V_m), $1 \leq m < n$, of dimension $V_m = \binom{n}{m}$. Finally, we study the irreducibility of (ψ_m, V_m).

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1 Introduction

Let B_n be the braid group on n strings. Ed. Formanek classified all irreducible representations of B_n of dimension at most n [3]. In [5], I. Sysoeva extended this classification to representations of dimensions n. It was shown that all irreducible complex specialization of the representations of B_n of dimension $n \geq 9$ are equivalent to the tensor product of a one-dimensional representation and a specialization of the standard representations. It was proved that for $n \geq 7$ every irreducible complex representation of B_n of corank two is equivalent to a specialization of the standard representation [5]. In [4], Larsen and Rowell proved that there are no irreducible unitary representations of B_n with dimension $n + 1$ for $n \geq 16$. In [6], I. Sysoeva proved that there are no irreducible representations of dimension $n + 1$ for $n \geq 10$.

In [2], Egea and Galina made a step forward in the classification of irreducible representations of the braid group B_n. They introduced a new family of finite-dimensional complex representations of B_n, which contains the standard representation. They gave a sufficient condition for members of this family to be irreducible. Moreover, they provided explicitly a subfamily of one parameter, self-adjoint representations (ϕ_m, V_m), $1 \leq m < n$. The question of irreducibility of this family was further studied.

In our work, we determine a sufficient condition for the irreducibility of the family of representations of the braid group constructed by Egea and Galina without requiring that the representations are self-adjoint. Then, we construct a multi-parameter family of representations whose irreducibility will be studied.

In Sect. 2, we present the results of the paper of Egea and Galina [2]. More precisely, we define the family of finite-dimensional representations of the braid group, which was constructed in [2]. Then, we present a theorem, given in [2], which gives a sufficient condition for members of this family to be irreducible. In the hypothesis of the theorem, the representation is assumed to be self-adjoint. A specific family of representations was computed in their work.

M. N. Abdulrahim (✉) · R. S. Kahil
Department of Mathematics and Computer Science, Beirut Arab University, P.O. Box 11-5020 Beirut, Lebanon
E-mail: mna@bau.edu.lb

R. S. Kahil
E-mail: rsk349@student.bau.edu.lb

In Sect. 3, we show that the self-adjoint condition required for members of this family is not needed to show their irreducibility. Moreover, we give explicitly another subfamily (ψ_m, V_m), $1 \leq m < n$, which contains non-self-adjoint representations. For all $1 \leq m < n$, (ψ_m, V_m) is a multi-parameter representation, where $\dim V_m = \binom{n}{m}$. Finally, we study the irreducibility of (ψ_m, V_m). More precisely, we show that for $n > 2$ and $n \neq 2m$, (ψ_m, V_m) is an irreducible representation of B_n for all $1 \leq m < n$, and if $n = 2m$, then (ψ_m, V_m) is the sum of two representations of B_n.

2 Definitions and known results

In this section, we list results of the paper of Egea and Galina. We present the family of finite-dimensional representations of the braid group containing the standard representation constructed in their work [2].

Definition 2.1 [1]. The braid group on n strings B_n is the abstract group with generators $\tau_1, \ldots, \tau_{n-1}$ satisfying the following conditions:

$$\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}, \qquad i = 1, 2, ..., n-2$$
$$\tau_i\tau_j = \tau_j\tau_i, \quad |i-j| \geq 2.$$

Definition 2.2 [7]. Let V be a finite-dimensional inner product space over $\mathbb{C}$ with inner product $\langle\, ,\rangle$. Given a linear operator $T \in \mathcal{L}(V)$, the adjoint of T is defined to be the operator $T^* \in \mathcal{L}(V)$ for which

$$\langle Tv, w\rangle = \langle v, T^*w\rangle, \text{ for all } v, w \in V.$$

A self-adjoint operator is an operator that is equal to its own adjoint. That is, T is self-adjoint if $T = T^*$. If, in addition, an orthonormal basis has been chosen, then the operator T is self-adjoint if and only if the matrix describing T with respect to this basis is Hermitian.

Definition 2.3 [7]. A self-adjoint representation π of a group G is a linear representation on a complex Hilbert space V, such that $\pi(g)$ is a self-adjoint operator for every $g \in G$.

Now, we introduce the family of representations of B_n constructed by Egea and Galina in [2]. Then, we present the main theorem that finds a sufficient condition for the irreducibility. A construction of such a family that satisfies the hypothesis of the theorem was made.

The authors in [2] considered n non-negative integers $z_1, z_2, \ldots, z_n$ not necessarily different; a set X which is the set of all the possible n-tuples obtained by permuting the coordinates of the fixed n-tuple $(z_1, z_2, \ldots, z_n)$. For example, if $n = 3$, then

$$X = \{(z_1, z_2, z_3), (z_1, z_3, z_2), (z_2, z_1, z_3), (z_2, z_3, z_1)(z_3, z_1, z_2), (z_3, z_2, z_1)\}.$$

They considered a complex vector space V with orthonormal basis $\beta = \{v_x : x \in X\}$. The dimension of V is the cardinality of X. Then, they defined a representation $\phi : B_n \to Aut(V)$, such that

$$\phi(\tau_k)(v_x) = q_{x_k,x_{k+1}} v_{\sigma_{k(x)}}.$$

Here, $q_{x_k,x_{k+1}}$ is a non-zero complex number that only depends on the places k and $k+1$ of $x = (x_1, \ldots, x_n)$, and

$$\sigma_k(x_1, ..., x_n) = (x_1, ..., x_{k-1}, x_{k+1}, x_k, x_{k+2}, ..., x_n).$$

With these notations, the authors in [2] obtained the following theorems.

Theorem 2.4 [2]. *(ϕ, V) is a representation of the braid group B_n.*

Theorem 2.5 [2]. *If $\phi(\tau_k)$ is a self-adjoint operator for all k, and for any pair $x, y \in X$, there exists j, $1 \leq j \leq n-1$, such that $|q_{x_j,x_{j+1}}|^2 \neq |q_{y_j,y_{j+1}}|^2$, then (ϕ,V) is an irreducible representation of the braid group B_n.*

Example 2.6 [2] Let $z_1, \ldots, z_n \in \{0, 1\}$, such that $z_1 = z_2 = \cdots = z_m = 1$ and $z_{m+1} = \cdots = z_n = 0$. If V_m is the vector space with basis $\beta_m = \{v_x, x \in X\}$, then $\dim V_m = \binom{n}{m}$. For each $x = (x_1, \ldots, x_n) \in X$, let

$$q_{x_k,x_{k+1}} = \begin{cases} 1 \quad if & x_k = x_{k+1} \\ t \quad if & x_k \neq x_{k+1} \end{cases}$$

where t is real number, $t \neq 0, 1, -1$. Let $\phi_m : B_n \to Aut(V_m)$ be a representation given by

$$\phi_m(\tau_k)(v_x) = q_{x_k,x_{k+1}} v_{\sigma_k(x)}.$$

The representation they obtained is self-adjoint. Thus, they used Theorem 2.2 to get the following result.

Theorem 2.7 [2]. *Let $n > 2$. Then, (ϕ_m, V_m) is an irreducible representation of B_n, for all $1 \leq m < n$, such that $2m \neq n$.*

If $n = 2m$ then (ϕ_m, V_m) is sum of two irreducible representations of B_n.

3 Construction and main theorems

In this section, we deal with the representation in Sect. 2 without requiring the condition "$\phi(\tau_k)$ is self- adjoint", as stated in the hypothesis of Theorem 2.2 and we still obtain that the representation is irreducible. We then construct a multi-parameter representation of high degree, and study its irreducibility.

Theorem 3.1 *If for any pair $x, y \in X$, there exists k, $1 \leq k \leq n-1$, such that $q_{x_k,x_{k+1}} q_{x_{k+1},x_k} \neq q_{y_k,y_{k+1}} q_{y_{k+1},y_k}$, then (ϕ, V) is an irreducible representation of the braid group B_n.*

Proof Let W be a non-zero invariant subspace of V. Consider $\beta = \{v_x;\, x \in X\}$ a basis for V. We now follow the steps adopted in [2] to show that if one of the basis vectors v_x belongs to W, then v_y belongs to W for any $v_y \in \beta$. Since elements in X are obtained by permutation of a fixed element, it follows that any two elements are permutations of each other. Thus, for any two elements $x, y \in X$, there exists a permutation σ formed of a product of transpositions, such that $\sigma(x) = y$. For $v_x \in W$. We let $\sigma = \sigma_{i_1} \ldots \sigma_{i_l}$. Then, $\tau = \tau_{i_1} \ldots \tau_{i_l}$ satisfies $\phi(\tau)(v_x) = \lambda v_y$ for some non-zero complex number λ. Hence, W contains v_y, and, therefore, W contains the basis β. Next, we show that v_x belongs to W for some $x \in X$.

We have

$$(\phi(\tau_k))^2(v_x) = \phi(\tau_k)(q_{x_k,x_{k+1}} v_{\sigma_k(x)}) = q_{x_k,x_{k+1}} q_{x_{k+1},x_k} v_{\sigma_k^2(x)} = q_{x_k,x_{k+1}} q_{x_{k+1},x_k} v_x.$$

Hence, the matrices $(\phi(\tau_k))^2$ are diagonal in the basis β for all k, $1 \leq k \leq n-1$. We now consider any of these diagonal matrices. Without loss of generality, we consider $(\phi(\tau_1))^2$. Since the matrix $(\phi(\tau_1))^2$ is diagonal, then any invariant subspace of V, in particular W, contains either an eigenvector v_x for some $x \in X$ or a linear combination of its eigenvectors corresponding to the same eigenvalue. If $v_x \in W$, then we are done. Otherwise, we assume that W contains a linear combination $a_1 v_{x^1} + a_2 v_{x^2} + \cdots + a_r v_{x^r}$, where at least two of the coefficients are non-zeros. Here, $x^1, \ldots, x^r \in X$ and $v_{x^1}, \ldots, v_{x^r} \in \beta$ are the eigenvectors of $(\phi(\tau_1))^2$ corresponding to the same eigenvalue. We now show that if any linear combination of such vectors belongs to W, then one of the basis elements belongs to W. In particular, we prove that if any such linear combination of r vectors, with r more than one, belongs to W, then we can obtain a non-zero linear combination of at most $r-1$ vectors that belongs to W. We consider

$$a_1 v_{x^1} + a_2 v_{x^2} + \cdots + a_r v_{x^r} \in W, \tag{1}$$

where $a_1, \ldots, a_r$ are different from zero. By hypothesis, for each pair of vectors in the basis β, say v_{x^1} and v_{x^2}, there exists k, $1 \leq k \leq n-1$, such that $q_{x_k^1,x_{k+1}^1} q_{x_{k+1}^1,x_k^1} \neq q_{x_k^2,x_{k+1}^2} q_{x_{k+1}^2,x_k^2}$. Since $a_1 v_{x^1} + a_2 v_{x^2} + \cdots + a_r v_{x^r} \in W$, it follows that:

$$\begin{aligned} &(\phi(\tau_k))^2(a_1 v_{x^1} + a_2 v_{x^2} + \cdots + a_r v_{x^r}) \\ &= a_1 q_{x_k^1,x_{k+1}^1} q_{x_{k+1}^1,x_k^1} v_{x^1} + a_2 q_{x_k^2,x_{k+1}^2} q_{x_{k+1}^2,x_k^2} v_{x^2} + \cdots + a_r q_{x_k^r,x_{k+1}^r} q_{x_{k+1}^r,x_k^r} v_{x^r} \in W. \end{aligned} \tag{2}$$

Multiplying (2) by $(q_{x_k^1,x_{k+1}^1} q_{x_{k+1}^1,x_k^1})^{-1}$ and subtracting it from (1), we get

$$(1-(q_{x_k^1,x_{k+1}^1} q_{x_{k+1}^1,x_k^1})^{-1}(q_{x_k^2,x_{k+1}^2} q_{x_{k+1}^2,x_k^2}))a_2 v_{x^2}+\cdots$$
$$+(1-(q_{x_k^1,x_{k+1}^1} q_{x_{k+1}^1,x_k^1})^{-1}(q_{x_k^r,x_{k+1}^r} q_{x_{k+1}^r,x_k^r}))a_r v_{x^r}\in W.$$

Having that

$$1-(q_{x_k^1,x_{k+1}^1} q_{x_{k+1}^1,x_k^1})^{-1}(q_{x_k^2,x_{k+1}^2} q_{x_{k+1}^2,x_k^2})\neq 0,$$

we get a non-zero linear combination of at most $r-1$ vectors that belongs to W. Repeating this process, we obtain one of the basis vectors in W. Therefore, $W=V$. □

Remark If $\phi(\tau_k)$ is self-adjoint, then the condition $q_{x_k,x_{k+1}} q_{x_{k+1},x_k}\neq q_{y_k,y_{k+1}} q_{y_{k+1},y_k}$ is equivalent to $|q_{x_j,x_{j+1}}|^2\neq|q_{y_j,y_{j+1}}|^2$, which is stated in the hypothesis of Theorem 2.2.

The previous theorem motivates the construction of new irreducible representations of the braid group. In [2], the authors constructed a subfamily of one parameter, self-adjoint representations (ϕ_m, V_m). We instead construct a subfamily of multi-parameter representations (ψ_m, V_m) which contains non-self-adjoint representations.

We consider n non-negative integers $z_1, z_2, \ldots, z_n$, not necessarily different; a set X which is the set of all the possible n-tuples obtained by permuting the coordinates of the fixed n-tuple $(z_1, z_2, \ldots, z_n)$. Let $z_1, \ldots, z_n \in \{0, 1\}$, such that $z_1 = z_2 = \cdots z_m = 1$ and $z_{m+1} = \cdots = z_n = 0$. Then, the cardinality of X is equal to

$$\binom{n}{m}=\frac{(n)!}{m!(n-m)!}\ .$$

If V_m is the vector space with basis $\beta_m=\{v_x, x\in X\}$, then $\dim V_m=\binom{n}{m}$. For each $x=(x_1,\ldots,x_n)\in X$, we let

$$q_{x_k,x_{k+1}}=\begin{cases} r_k & if \quad x_k = x_{k+1}\\ p_k & if \quad x_k > x_{k+1}\\ q_k & if \quad x_k < x_{k+1},\end{cases}$$

where $r_k, p_k, q_k\in\mathbb{R}-\{0\}$, $r_k^2\neq p_k q_k$, and $p_k q_k>0$ for any k.
Let $\psi_m: B_n\to Aut(V_m)$ be given by

$$\psi_m(\tau_k)(v_x)=q_{x_k,x_{k+1}} v_{\sigma_{k(x)}}.$$

We Now, consider the lexicographic order in X. Let $n=5$ and $m=3$. Then, we have $\dim V_m$ =10. The ordered basis in that case is

$$\beta := \{v_{(0,0,1,1,1)}, v_{(0,1,0,1,1)}, v_{(0,1,1,0,1)}, v_{(0,1,1,1,0)}, v_{(1,0,0,1,1)}, v_{(1,0,1,0,1)},$$
$$v_{(1,0,1,1,0)}, v_{(1,1,0,0,1)}, v_{(1,1,0,1,0)}, v_{(1,1,1,0,0)}\},$$

and the matrices in this basis are as follows:

$$\psi_{3(\tau_1)}=\begin{pmatrix} r_1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&p_1&0&0&0&0&0\\ 0&0&0&0&0&p_1&0&0&0&0\\ 0&0&0&0&0&0&p_1&0&0&0\\ 0&q_1&0&0&0&0&0&0&0&0\\ 0&0&q_1&0&0&0&0&0&0&0\\ 0&0&0&q_1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&r_1&0&0\\ 0&0&0&0&0&0&0&0&r_1&0\\ 0&0&0&0&0&0&0&0&0&r_1 \end{pmatrix},$$

$$\psi_{3(\tau_2)} = \begin{pmatrix} 0 & p_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ q_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & q_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_2 \end{pmatrix},$$

$$\psi_{3(\tau_3)} = \begin{pmatrix} r_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_3 & 0 \end{pmatrix},$$

$$\psi_{3(\tau_4)} = \begin{pmatrix} r_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & r_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & p_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_4 \end{pmatrix}.$$

As V_m is a complex vector space of finite dimension with an orthonormal basis β_m, then using Definition 2.2, we get that $\psi_m(\tau_k)$ is a self-adjoint operator iff $q_k = p_k$ for any k, $1 \leq k \leq n-1$.

We now study the irreducibility of (ψ_m, V_m). In what follows, p_k and q_k satisfy the conditions stated for (ψ_m, V_m). That is, $p_k, q_k \in \mathbb{R} - \{0\}$ and $p_k q_k > 0$ for any $1 \leq k \leq n-1$. To do so, we introduce some definitions and prove a lemma to prove Theorem 3.3.

Definition 3.2 Let $x = (x_1, \ldots, x_n)$ and $w = (w_1, \ldots, w_n)$ be any two elements in X. Let $\sigma = \sigma_{i_1} ... \sigma_{i_l}$ be a permutation sending x to w, where for $1 \leq j \leq l$ and $1 \leq i_j < n$, σ_{i_j} is the transposition acting on an element $x = (x_1, ..., x_n)$, by swapping the entries x_{i_j} and x_{i_j+1}, with $x_{i_j} \neq x_{i_j+1}$. We introduce $P_{x,w}$ a positive real number given by

$$P_{x,w} = P(\sigma_{i_1}) \ldots P(\sigma_{i_l}), \text{ where}$$

$$P(\sigma_{i_j}) = \begin{cases} q_{i_j} p_{i_j}^{-1} \quad if \;\; \sigma_{i_j} \text{ is acting on an element,} x = (x_1, ..., x_n) \text{ where} \\ x_{i_j} = 0 \text{ and } x_{i_j+1} = 1 \\ \\ p_{i_j} q_{i_j}^{-1} \quad if \;\; \sigma_{i_j} \text{is acting on an element} x = (x_1, ..., x_n) \text{ where} \\ x_{i_j} = 1 \text{ and } x_{i_j+1} = 0. \end{cases}$$

Given the vectors x and w. If we have two permutations that send x to w, then the additional transpositions in the longer permutation will be even in number, and their corresponding values $P(\sigma_{i_j})$'s will be cancelled out. Thus, it is easy to see that $P_{x,w}$ is independent of the choice of the permutation that sends x to w. For example, let $x = (0, 0, 1, 1)$ and $w = (0, 1, 1, 0)$. Consider the permutation $\sigma = \sigma_3\sigma_2$ that sends x to w. Thus, we have

$$x = (0, 0, 1, 1) \xrightarrow{\sigma_2} (0, 1, 0, 1) \xrightarrow{\sigma_3} (0, 1, 1, 0) = w.$$

We get that

$$P_{x,w} = P(\sigma_3)P(\sigma_2) = q_3 p_3^{-1} q_2 p_2^{-1}.$$

Considering another permutation, $\sigma = \sigma_1\sigma_3\sigma_1\sigma_2$ that also sends x to w. Then, we get

$$x = (0, 0, 1, 1) \xrightarrow{\sigma_2} (0, 1, 0, 1) \xrightarrow{\sigma_1} (1, 0, 0, 1) \xrightarrow{\sigma_3} (1, 0, 1, 0) \xrightarrow{\sigma_1} (0, 1, 1, 0) = w.$$

Now, $P_{x,w} = p_1 q_1^{-1} q_3 p_3^{-1} q_1 p_1^{-1} q_2 p_2^{-1} = q_3 p_3^{-1} q_2 q_2^{-1}$.

We consider the case $n = 2m$ and we give the following definitions and lemmas.

Definition 3.3 For $n = 2m$, we observe that for any $x \in X$, there is a $y_x \in X$, where y_x is obtained from x by replacing the zeros by ones and the ones by zeros. For example

$$\text{if} \quad x = (1, 0, 0, 1), \quad \text{then} \quad y_x = (0, 1, 1, 0).$$

For any $x \in X$, we define λ_x a non-zero positive real number defined as follows:

$$\lambda_x = \sqrt{P_{y_x,x}}\,.$$

Equivalently, we define λ_{y_x} a positive real number, as

$$\lambda_{y_x} = \sqrt{P_{x,y_x}}\,.$$

For example, we use the above definition to find $\lambda_{(1,\ldots,1,0,\ldots,0)}$, where $(1, \ldots, 1, 0, \ldots, 0)$ is the last element in the ordered set X.

Example 3.4 Let X be the set of all the possible n-tuples obtained by permuting the coordinates of the fixed n-tuple $(z_1, z_2, \ldots, z_n)$, and $z_1, \ldots, z_n \in \{0, 1\}$, such that $z_1 = z_2 = \cdots = z_m = 1$ and $z_{m+1} = \cdots = z_n = 0$. For $n = 2m$, let $x = (0, \ldots, 0, 1, \ldots, 1)$ and $y_x = (1, \ldots, 1, 0, \ldots, 0)$, be the first and last elements in the ordered set X, respectively. We consider a permutation that sends x to y_x in the following way: move each zero in the $(m-i)$th place to the $(n-i)$th place, starting from $i = 0$ till $i = m-1$. This is done by a sequence of transpositions starting from σ_{m-i} till σ_{n-i-1}, for each i. We then observe that this permutation will contain each of the transpositions σ_k, k times for $1 \le k \le m$, and $n-k$ times for $m+1 \le k \le n-1$. Having

$$P_{x,w} = P(\sigma_{i_1}) \ldots P(\sigma_{i_l}),$$

we obtain that

$$\lambda_{(1,\ldots,1,0,\ldots,0)} = \sqrt{(q_1 p_1^{-1})(q_2 p_2^{-1})^2 \ldots (q_m p_m^{-1})^m (q_{m+1} p_{m+1}^{-1})^{m-1} \ldots (q_{n-1} p_{n-1}^{-1})}\,.$$

Lemma 3.5 *For any two elements x and w in X, we have*

$$\lambda_x = \lambda_{y_x}^{-1} \tag{3}$$

and

$$\lambda_x P_{x,w} = \lambda_w. \tag{4}$$

Proof For any two elements x and w in X, let $\sigma = \sigma_{i_1} \ldots \sigma_{i_l}$ be the permutation that sends the element x to w. Since y_x is obtained from x by replacing the zeros with ones and the ones with zeros, and y_w is obtained from w in a similar way, it follows that σ also sends y_x to y_w. Moreover, according to the definition of $P_{x,w}$ and P_{y_x,y_w}, we observe that $P_{x,w} = P_{y_x,y_w}^{-1}$. Also, if σ is a permutation that sends the element x to w, then its inverse, $\sigma^{-1} = (\sigma_{i_1} \ldots \sigma_{i_l})^{-1} = \sigma_{i_l} \ldots \sigma_{i_1}$, is a permutation that sends w to x. The permutation σ^{-1} will contain the same transpositions as that of σ but by reversing the order of zeros and ones for each transposition. In other words, if, in σ, σ_{i_j} acts on an element where the i_j^{th} entry is 0 and i_{j+1}^{th} entry is 1, then in σ^{-1}, it will act on an element where the i_j^{th} entry is 1 and i_{j+1}^{th} entry is 0. Thus, we also have that $P_{x,w} = P_{w,x}^{-1}$. In particular,

$P_{y_x,x} = P^{-1}_{x,y_x}$. Therefore, $\lambda_x = \lambda^{-1}_{y_x}$. Given any three vectors x, y and w in X. Since $P_{x,w}$ is independent of the choice of the permutation sending x to w, it follows that $P_{y,w}P_{x,y} = P_{x,w}$. We get that

$$\lambda_x P_{x,w} = \sqrt{P_{y_x,x}}P_{x,w} = \sqrt{P_{y_x,x}}\sqrt{P^2_{x,w}} = \sqrt{P_{y_x,x}P_{x,w}P_{x,w}} =$$
$$= \sqrt{P_{y_x,w}P_{x,w}} = \sqrt{P_{y_x,w}P^{-1}_{y_x,y_w}} = \sqrt{P_{y_x,w}P_{y_w,y_x}} = \sqrt{P_{y_w,w}} = \lambda_w.$$

□

Theorem 3.6 *Let $n > 2$. Then, (ψ_m, V_m) is an irreducible representation of B_n for all $1 \le m < n$, such that $n \ne 2m$.*
If $n = 2m$, then (ψ_m, V_m) is the sum of two representations of B_n.

Proof Suppose that $n \ne 2m$. In this case, we follow the steps used in [2] to show the irreducibility of (ψ_m, V_m). Let $x \ne y \in X$, then there exists i, $1 \le i \le n$, such that $x_i \ne y_i$. If $i > 1$, we may suppose that $x_{i-1} = y_{i-1}$; thus, $q_{x_{i-1},x_i} \ne q_{y_{i-1},y_i}$ where one of them is equal to r_{i-1} and the other is either p_{i-1} or q_{i-1}. Thus, $q_{x_{i-1},x_i}q_{x_i,x_{i-1}} \ne q_{y_{i-1},y_i}q_{y_i,y_{i-1}}$, where one of them is equal to r^2_{i-1} and the other is equal to $p_{i-1}q_{i-1}$. If $i = 1$, $x_1 \ne y_1$, and $n \ne 2m$, then there exists $1 \le l < n$, such that $x_l \ne y_l$ and $x_{l+1} = y_{l+1}$. Then, $q_{x_l,x_{l+1}} \ne q_{y_l,y_{l+1}}$, where one of them is equal to r_l and the other is equal to p_{i-1} or q_{i-1}. Thus, $q_{x_l,x_{l+1}}q_{x_{l+1},x_l} \ne q_{y_l,y_{l+1}}q_{y_{l+1},y_l}$, where one of them is equal to r^2_l and the other is equal to p_lq_l. Therefore, by Theorem 3.1, ψ_m is an irreducible representation.

For $n = 2m$, $X = \{x, y_x : x \in Y, Y \subset X\}$, with y_x obtained from x by replacing the zeros by ones and the ones by zeros. The set Y can be considered as the set containing the first m elements of the ordered set X. For example, if

$$X := \{(0, 0, 1, 1), (0, 1, 0, 1), (0, 1, 1, 0), (1, 0, 0, 1), (1, 0, 1, 0), (1, 1, 0, 0)\},$$

then

$$Y := \{(0, 0, 1, 1), (0, 1, 0, 1), (0, 1, 1, 0)\}.$$

Given $x \in X$. It is easy to see that the vectors x and $y = y_x$ have the property that $q_{x_k,x_{k+1}}q_{x_{k+1},x_k} = q_{y_k,y_{k+1}}q_{y_{k+1},y_k}$ for any k, $1 \le k \le n-1$. Thus, the sufficient condition for irreducibility is not satisfied in the case $n = 2m$. Now, let $\beta_1 = \{v_x + \lambda_{y_x}v_{y_x}; x \in Y\}$ and $\beta_2 = \{v_x - \lambda_{y_x}v_{y_x}; x \in Y\}$. Let W_1 and W_2 be the vector spaces generated by β_1 and β_2, respectively. For $x \in Y$, $v_x + \lambda_{y_x}v_{y_x} \in W_1$, then $\lambda_{y_x}(\lambda^{-1}_{y_x}v_x + v_{y_x}) \in W_1$. Since $\lambda_x = \lambda^{-1}_{y_x}$, it follows that $v_{y_x} + \lambda_x v_x \in W_1$. We obtain that $v_x + \lambda_{y_x}v_{y_x} \in W_1$ for any $x \in X$. We claim that W_1 and W_2 are two invariant subspaces of V_m. We prove that W_1 is invariant and a similar proof follows for W_2.
Let $x = (x_1, x_2, \ldots, x_n) \in X$ and $v_x + \lambda_{y_x}v_{y_x}$ be an element in W_1. Given any $1 \le l < n$, and using Lemma 3.2, we consider the following cases:

Case 1: If $x_l = x_{l+1}$, then $y_{x_l} = y_{x_{l+1}}$. Thus

$$\phi(\tau_l)(v_x + \lambda_{y_x}v_{y_x}) = r_l(v_x + \lambda_{y_x}v_{y_x}) \in W_1.$$

Case 2: If $x_l = 0$ and $x_{l+1} = 1$, then $y_{x_l} = 1$ and $y_{x_{l+1}} = 0$. Thus

$$\phi(\tau_l)(v_x + \lambda_{y_x}v_{y_x}) = q_l v_{\sigma_l(x)} + p_l\lambda_{y_x}v_{\sigma_l(y_x)} = q_l(v_{\sigma_l(x)} + p_lq_l^{-1}\lambda_{y_x}v_{\sigma_l(y_x)}).$$

Since $\sigma_l(y_x) = y_{\sigma_l(x)}$ and $P_{y_x,\sigma_l(y_x)} = p_lq_l^{-1}$, it follows that:

$$p_lq_l^{-1}\lambda_{y_x} = \lambda_{y_x} \times P_{y_x,\sigma_l(y_x)} = \lambda_{\sigma_l(y_x)}.$$

Therefore, $q_l(v_{\sigma_l(x)} + p_lq_l^{-1}\lambda_{y_x}v_{\sigma_l(y_x)}) \in W_1$.

Case 3: If $x_l = 1$ and $x_{l+1} = 0$, then $y_{x_l} = 0$ and $y_{x_{l+1}} = 1$. Thus

$$\phi(\tau_l)(v_x + \lambda_{y_x}v_{y_x}) = p_l v_{\sigma_l(x)} + q_l\lambda_{y_x}v_{\sigma_l(y_x)} = p_l(v_{\sigma_l(x)} + q_lp_l^{-1}\lambda_{y_x}v_{\sigma_l(y_x)}).$$

Since $\sigma_l(y_x) = y_{\sigma_l(x)}$ and $P_{y_x,\sigma_l(y_x)} = q_l p_l^{-1}$, it follows that:

$$q_l p_l^{-1} \lambda_{y_x} = \lambda_{y_x} \times P_{y_x,\sigma_l(y_x)} = \lambda_{\sigma_l(y_x)}.$$

Therefore, $p_l(v_{\sigma_l(x)} + q_l p_l^{-1} \lambda_{y_x} v_{\sigma_l(y_x)}) \in W_1$.
Since β_1 is a basis for W_1, then the dimension of W_1 is equal to cardinality of β_1, which is half that of V_m. Thus, it is equal to $\dfrac{\binom{n}{m}}{2}$. □

Example 3.7 We consider the representation (ψ_m, V_m) as previously defined, but for $n = 4$ and $m = 2$. We compute explicitly the two invariant subspaces W_1 and W_2 in this case. Consider the following set:

$$X = \{(0, 0, 1, 1), (0, 1, 0, 1), (0, 1, 1, 0), (1, 0, 0, 1), (1, 0, 1, 0), (1, 1, 0, 0)\}$$
$$Y = \{(0, 0, 1, 1), (0, 1, 0, 1), (0, 1, 1, 0)\}$$

and the ordered basis

$$\beta := \{v_{(0,0,1,1)}, v_{(0,1,0,1)}, v_{(0,1,1,0)}, v_{(1,0,0,1)}, v_{(1,0,1,0)}, v_{(1,1,0,0)}\}.$$

The matrices in this basis are as follows:

$$\psi_{2(\tau_1)} = \begin{pmatrix} r_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_1 & 0 \\ 0 & q_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r_1 \end{pmatrix}, \psi_{2(\tau_2)} = \begin{pmatrix} 0 & p_2 & 0 & 0 & 0 & 0 \\ q_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p_2 \\ 0 & 0 & 0 & 0 & q_2 & 0 \end{pmatrix},$$

$$\psi_{2(\tau_3)} = \begin{pmatrix} r_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p_3 & 0 & 0 & 0 \\ 0 & q_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_3 & 0 \\ 0 & 0 & 0 & q_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r_3 \end{pmatrix},$$

where $r_k, p_k, q_k \in \mathbb{R} - \{0\}$, $r_k^2 \neq p_k q_k$, $p_k q_k > 0$ for any k. Next, we compute λ_x for each x in X.
We consider the permutation $P = \sigma_2\sigma_1\sigma_3\sigma_2$ that sends (0,0,1,1) to (1,1,0,0). Thus, for $\lambda_{(1,1,0,0)}$ in β_1, we have

$$\lambda_{(1,1,0,0)} = \sqrt{(q_1 p_1^{-1})(q_2 p_2^{-1})^2 (q_3 p_3^{-1})}.$$
$$\lambda_{(1,1,0,0)} = (q_1 p_1^{-1})^{\frac{1}{2}} (q_2 p_2^{-1})(q_3 p_3^{-1})^{\frac{1}{2}}.$$

Note that $\sigma_2(1, 1, 0, 0) = (1, 0, 1, 0)$, then

$$\lambda_{(1,0,1,0)} = \lambda_{(1,1,0,0)} \times (p_2 q_2^{-1}) = (q_1 p_1^{-1})^{\frac{1}{2}} (q_3 p_3^{-1})^{\frac{1}{2}}.$$

Since $\sigma_3(1, 0, 1, 0) = (1, 0, 0, 1)$, then

$$\lambda_{(1,0,0,1)} = \lambda_{(1,0,1,0)} \times (p_3 q_3^{-1}) = (q_1 p_1^{-1})^{\frac{1}{2}} (q_3 p_3^{-1})^{\frac{-1}{2}}.$$

Thus, we have the following two invariant subspaces:

$$W_1 = < v_{(0,0,1,1)} + (q_1 p_1^{-1})^{\frac{1}{2}} (q_2 p_2^{-1})(q_3 p_3^{-1})^{\frac{1}{2}} v_{(1,1,0,0)}, v_{(0,1,0,1)} + (q_1 p_1^{-1})^{\frac{1}{2}} (q_3 p_3^{-1})^{\frac{1}{2}} v_{(1,0,1,0)}, v_{(0,1,1,0)} + (q_1 p_1^{-1})^{\frac{1}{2}} (q_3 p_3^{-1})^{\frac{-1}{2}} v_{(1,0,0,1)} >$$
$$W_2 = < v_{(0,0,1,1)} - (q_1 p_1^{-1})^{\frac{1}{2}} (q_2 p_2^{-1})(q_3 p_3^{-1})^{\frac{1}{2}} v_{(1,1,0,0)}, v_{(0,1,0,1)} - (q_1 p_1^{-1})^{\frac{1}{2}} (q_3 p_3^{-1})^{\frac{1}{2}} v_{(1,0,1,0)}, v_{(0,1,1,0)} - (q_1 p_1^{-1})^{\frac{1}{2}} (q_3 p_3^{-1})^{\frac{-1}{2}} v_{(1,0,0,1)} > .$$

4 Conclusion

We weakened the conditions for the family of representations given by Egea and Galina to be irreducible. The sufficient condition obtained in our work is equivalent to that in [2] if we assume further that the representation is self-adjoint. We also constructed multi-parameter representations (ψ_m, V_m) of the braid group B_n, which satisfy the sufficient condition of irreducibility when $n \neq 2m$ $(1 \leq m < n)$, and is the sum of two representations of B_n when $n = 2m$. Such irreducible representations can be useful in the progress of the classification of irreducible representations of B_n.

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