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Approximation of functions by Bernoulli wavelet and its applications in solution of Volterra integral equation of second kind

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Abstract In this paper, two new estimators $E_{2^{k-1},0}^{(1)}(f)$ and $E_{2^{k-1},M}^{(1)}(f)$ of characteristic function and an estimator $E_{2^{k-1},M}^{(2)}(f)$ of function of Hölder's class $H^\alpha[0, 1]$ of order $0 < \alpha \leq 1$ have been established using Bernoulli wavelets. A new technique has been applied for solving Volterra integral equation of second kind using Bernoulli wavelet operational matrix of integration as well as product operational matrix. These matrices have been utilized to reduce the Volterra integral equation into a system of algebraic equations, which are easily solvable. Some examples are illustrated to show the validity and efficiency of proposed technique of this research paper.

Mathematics Subject Classification 42C40 · 65T60 · 45D05

1 Introduction

Wavelet analysis is recent and advanced analysis in pure as well as applied mathematical research area. It has been applied in different branches of science and technology such as engineering, image processing, signal analysis, time frequency analysis, and fast numerical algorithms. There are several problems of science and technology in the form of integral equations. There are various types of methods to find the numerical solution of integral equations. Sometimes it is difficult to find the solution of the integral equations analytically. Therefore, there is need of approximate solution.

Several researchers such as Liang [6], Maleknejad [8], Keshvarz [3, 13] and Shah [14, 19] have been solving integral equations using wavelet methods. Lepik [7] and Yousefi [20] have presented numerical methods to solve the Fredholm integral equation by Haar and CAS wavelet method. Shah and Irfan [15, 16, 18] have used wavelet in bio-heat transfer model and population growth model.

Bernoulli wavelets have been applied to solve fractional order differential equations and calculus of variation problems by Keshavarz [3, 4]. We have presented Bernoulli wavelet method to solve linear Volterra integral equations of the form:

$$y(x) = f(x) + \int_0^x K(x, t)y(t)dt \quad (1.1)$$

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and nonlinear Volterra integral equation of the form:

$$y(x) = f(x) + \int_0^x K(x, t) F(y(t)) dt \quad (1.2)$$

where $f \in L^2[0, 1)$ and $K \in L^2[0, 1) \times L^2[0, 1)$ are known functions, F is a function of y and y is unknown function to be determined [1, 9–11].

The main objectives of this research paper are following:

1. To establish estimation of characteristic function and function of Hölder's class $H^\alpha[0, 1)$ of order $0 < \alpha \leq 1$.
2. To present a numerical method for solving linear Volterra integral equation (1.1) and nonlinear Volterra integral equation (1.2).
3. To compare the Bernoulli wavelet solution of integral equation (1.1) and (1.2) with its exact solution.

It is remarkable that the solution of (1.1) and (1.2) obtained by Bernoulli wavelet technique and its exact solution are almost same.

2 Definitions and preliminaries

2.1 Bernoulli wavelets

Bernoulli wavelets $B_{n,m}$ have four arguments: $\hat{n} = n - 1$, $n = 1, 2, \dots, 2^{k-1}$, where k is some positive integer, m is the order of Bernoulli polynomials and t is the normalized time. Then, $B_{n,m}$ are defined on the interval $[0, 1)$ as follows:

$$B_{n,m}(t) = \begin{cases} 2^{\frac{k-1}{2}} \tilde{\beta}_m(2^{k-1}t - \hat{n}), & \text{if } \frac{\hat{n}}{2^{k-1}} \leq t < \frac{\hat{n}+1}{2^{k-1}}, \\ 0, & \text{otherwise,} \end{cases} \quad (2.1)$$

where

$$\tilde{\beta}_m(t) = \begin{cases} 1, & \text{if } m = 0 \\ \frac{1}{\sqrt{\frac{(-1)^{m-1}(m!)^2}{(2m)!} \alpha_{2m}}} \beta_m(t), & \text{if } m > 0, \end{cases} \quad (2.2)$$

for $m = 0, 1, 2, \dots, M - 1$, and $n = 1, 2, \dots, 2^{k-1}$. The coefficient $\frac{1}{\sqrt{\frac{(-1)^{m-1}(m!)^2}{(2m)!} \alpha_{2m}}}$ is for normality, $\beta_m(t)$ is Bernoulli polynomial of order m , which is given by

$$\beta_m(t) = \sum_{i=0}^m \binom{m}{i} \alpha_{m-i} t^i$$

α_j are Bernoulli numbers and can be defined by the identity $\frac{t}{e^t - 1} = \sum_{i=0}^{\infty} \alpha_i \frac{t^i}{i!}$. The first few Bernoulli numbers are $\alpha_0 = 1$, $\alpha_1 = -\frac{1}{2}$, $\alpha_2 = \frac{1}{6}$, $\alpha_4 = -\frac{1}{30}$, $\alpha_6 = \frac{1}{42}$, $\alpha_8 = -\frac{1}{30}$, $\alpha_{10} = \frac{5}{66}$, ... and $\alpha_{2i+1} = 0$, $i = 1, 2, 3, \dots$

The first few Bernoulli polynomials are

$$\begin{aligned} \beta_0(t) &= 1, & \beta_1(t) &= t - \frac{1}{2}, & \beta_2(t) &= t^2 - t + \frac{1}{6}, \\ \beta_3(t) &= t^3 - \frac{3}{2}t^2 + \frac{1}{2}t, & \beta_4(t) &= t^4 - 2t^3 + t^2 - \frac{1}{30}, \\ \beta_5(t) &= t^5 - \frac{5}{2}t^4 + \frac{5}{3}t^3 - \frac{1}{6}t, & \beta_6(t) &= t^6 - 3t^5 + \frac{5}{2}t^4 - \frac{1}{2}t^2 + \frac{1}{42}, \end{aligned} \quad (\text{Shiralashetti}) [11, 12, 17].$$

2.2 Properties of Bernoulli polynomial [2, 5]

1. $\beta'_m(t) = m \beta_{m-1}(t)$
2. $\int_0^1 |\beta_m(t)| dt < 16 \frac{m!}{(2\pi)^{m+1}}$
3. $\beta_m(0) = (-1)^m \beta_m(1)$
4. $\int_0^1 \beta_m(x) dx = 0$

Fig. 1 Graph between m and normality factor.

2.3 Graph between m and the factor $\left| \frac{1}{\frac{(-1)^{m-1}(m!)^2}{(2m)!} \alpha_{2m}} \right|$

2.4 Function of Hölder's class $H^\alpha[0, 1]$

A function f is of Hölder's class $H^\alpha[0, 1]$ of order $0 < \alpha \leq 1$ if f satisfies the following condition:

$$|f(x) - f(y)| \leq A|x - y|^\alpha, \quad \forall x, y \in \mathbb{R} \quad (2.3)$$

for some positive constant A (Zheng and Wei [21]).

3 Approximation of function

A function $f \in L^2[0, 1]$ can be expanded as

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} B_{n,m}(t) \quad (3.1)$$

where the coefficients $c_{n,m}$ are given by

$$c_{n,m} = \langle f, B_{n,m} \rangle, \quad (3.2)$$

where $\langle \cdot, \cdot \rangle$ represents inner product in $L^2[0, 1]$.

$(2^{k-1}, M)$ th partial sum $S_{2^{k-1}, M}(f)(t)$ of infinite series (3.1) is given by

$$S_{2^{k-1}, M}(f)(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} B_{n,m}(t) = C^T B(t) \quad (3.3)$$

where C and $B(t)$ are given by

$$C = [c_{1,0}, c_{1,1}, \dots, c_{1,M-1}, c_{2,0}, \dots, c_{2,M-1}, \dots, c_{2^k,0}, \dots, c_{2^k,M-1}]^T = [c_1, c_2, \dots, c_{2^k-1,M}]^T$$

$$B(t) = [B_{1,0}(t), B_{1,1}(t), \dots, B_{1,M-1}(t), B_{2,0}(t), \dots, B_{2,M-1}(t), \dots, B_{2^k,0}(t), \dots, B_{2^{k-1},M-1}(t)]^T \\ = [B_1(t), B_2(t), \dots, B_{2^{k-1}M}(t)]^T.$$

Now $K(x, t) \in L^2[0, 1) \times L^2[0, 1)$ can be expanded as

$$K(x, t) = B^T(x) \mathbf{K} B(t) \quad (3.4)$$

where $\mathbf{K}$ is $2^{k-1}M \times 2^{k-1}M$ matrix with entries $\mathbf{K}_{i,j} = \langle B_i(x), \langle K(x, t), B_j(t) \rangle \rangle$. For $k = 3$ and $M = 3$, 12 Bernoulli wavelet basis functions are given by

$$\left. \begin{aligned} B_{1,0}(t) &= 2 \\ B_{1,1}(t) &= 2\sqrt{3} (8t - 1) \\ B_{1,2}(t) &= 2\sqrt{5} (96t^2 - 24t + 1) \end{aligned} \right\} 0 \leq t < \frac{1}{4},$$

$$\left. \begin{aligned} B_{2,0}(t) &= 2 \\ B_{2,1}(t) &= 2\sqrt{3} (8t - 3) \\ B_{2,2}(t) &= 2\sqrt{5} (96t^2 - 72t + 13) \end{aligned} \right\} \frac{1}{4} \leq t < \frac{1}{2},$$

$$\left. \begin{aligned} B_{3,0}(t) &= 2 \\ B_{3,1}(t) &= 2\sqrt{3} (8t - 5) \\ B_{3,2}(t) &= 2\sqrt{5} (96t^2 - 120t + 37) \end{aligned} \right\} \frac{1}{2} \leq t < \frac{3}{4},$$

$$\left. \begin{aligned} B_{4,0}(t) &= 2 \\ B_{4,1}(t) &= 2\sqrt{3} (8t - 7) \\ B_{4,2}(t) &= 2\sqrt{5} (96t^2 - 168t + 73) \end{aligned} \right\} \frac{3}{4} \leq t < 1.$$

3.1 Bernoulli wavelets operational matrix of integration

For $k = 3$ and $M = 3$, the $2^{k-1}M \times 2^{k-1}M$ Bernoulli wavelet operational matrix of integration given by Razzaghi [4] as follows:

$$\mathbf{P} = \frac{1}{4} \begin{bmatrix} \frac{1}{2} & \frac{1}{2\sqrt{3}} & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ -\frac{1}{2\sqrt{3}} & 0 & \frac{1}{2\sqrt{15}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2\sqrt{3}} & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2\sqrt{3}} & 0 & \frac{1}{2\sqrt{15}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2\sqrt{3}} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2\sqrt{3}} & 0 & \frac{1}{2\sqrt{15}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2\sqrt{3}} & 0 & \frac{1}{2\sqrt{15}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3.2 Bernoulli wavelets operational matrix of product

The product of two Bernoulli wavelet functions is given by $C^T B(t) B(t)^T \approx B(t) \tilde{C}$, where $\tilde{C}$ is a $2^{k-1}M \times 2^{k-1}M$ product operational matrix. The entries of $B(t) B(t)^T$ are given by $B_{ij} B_{rs} = 0$ for $i \neq r$, $B_{i0} B_{ij} = 2B_{ij}$,

$$B_{i1}B_{i1} = 2B_{i0} + \frac{4}{\sqrt{5}}B_{i2}, \quad B_{i1}B_{i2} = \frac{4}{\sqrt{5}}B_{i1}, \quad B_{i2}B_{i2} = 2B_{i0} + \frac{4\sqrt{5}}{7}B_{i2}.$$

Using these, we get

$$\tilde{C} = \begin{bmatrix} \tilde{C}_1 & 0 & 0 & 0 \\ 0 & \tilde{C}_2 & 0 & 0 \\ 0 & 0 & \tilde{C}_3 & 0 \\ 0 & 0 & 0 & \tilde{C}_4 \end{bmatrix} \quad \text{where} \quad \tilde{C}_i = \begin{bmatrix} 2c_{i0} & 2c_{i1} & 2c_{i2} \\ 2c_{i1} & 2c_{i0} + \frac{4}{\sqrt{5}}c_{i2} & \frac{4}{\sqrt{5}}c_{i1} \\ 2c_{i2} & \frac{4}{\sqrt{5}}c_{i1} & 2c_{i0} + \frac{4\sqrt{5}}{7}c_{i2} \end{bmatrix} \quad \text{for } i = 1, 2, 3, 4.$$

4 Results

Next, we prove the following Theorems:

Theorem 4.1 Let $f(x) = x \chi_{\left[\frac{n_1-1}{2^{k-1}}, \frac{n_1}{2^{k-1}}\right)}$ with $1 < n_1 < 2^{k-1}$ such that its Bernoulli wavelet expansion is given by series

$$f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} B_{n,m}(x), \quad (4.1)$$

and $(2^{k-1}, M)$ th partial sum $S_{2^{k-1},M}(f)$ given by

$$S_{2^{k-1},M}(f)(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} B_{n,m}(x). \quad (4.2)$$

Then approximation error $E_{2^{k-1},M}^{(1)}(f)$ of f by $S_{2^{k-1},M}(f)$ is given by

$$E_{2^{k-1},M}^{(1)}(f) = \begin{cases} o\left(\frac{1}{2^{\frac{1}{2}(k-1)}}\right), & \text{if } M = 0 \\ o\left(\frac{1}{2^{\frac{3}{2}(k-1)}\sqrt{M+1}}\right), & \text{if } M \geq 1. \end{cases} \quad (4.3)$$

Theorem 4.2 Let $f \in L^2[0, 1)$ be a function of Hölder's class $H^\alpha[0, 1)$ of order $0 < \alpha \leq 1$, such that its Bernoulli wavelet expansion is given by the series (4.1) with $(2^{k-1}, M)$ th partial sum $S_{2^{k-1},M}(f)$. Then approximation error $E_{2^{k-1},M}^{(2)}(f)$ of f by $S_{2^{k-1},M}(f)$ is given by

$$E_{2^{k-1},M}^{(2)}(f) = o\left(\frac{1}{\sqrt{M+1} 2^{(k-1)\alpha}}\right). \quad (4.4)$$

To prove the above theorems, first we prove the following Lemma:

Lemma 4.1 If $f \in L^2[0, 1)$ is a function such that its Bernoulli wavelet expansion is given by the series (4.1) with $(2^{k-1}, M)$ th partial sum $S_{2^{k-1},M}(f)(x)$ given by (4.2), then

$$\|f - S_{2^{k-1},M}(f)\|_2^2 = \sum_{n=1}^{2^{k-1}} \sum_{m=M}^{\infty} |c_{n,m}|^2 \quad (4.5)$$

Proof

$$\begin{aligned} f(x) - S_{2^{k-1},M}(f)(x) &= \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} B_{n,m}(x) - \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} B_{n,m}(x) \\ &= \sum_{n=1}^{2^{k-1}} \left(\sum_{m=0}^{M-1} + \sum_{m=M}^{\infty} \right) c_{n,m} B_{n,m}(x) - \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} B_{n,m}(x), \quad (\text{by def}^n \text{ of } B_{n,m}) \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} B_{n,m}(x) + \sum_{n=1}^{2^{k-1}} \sum_{m=M}^{\infty} c_{n,m} B_{n,m}(x) - \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} B_{n,m}(x) \\
&= \sum_{n=1}^{2^{k-1}} \sum_{m=M}^{\infty} c_{n,m} B_{n,m}(x) \\
(f(x) - S_{2^{k-1},M}(f)(x))^2 &= \sum_{n=1}^{2^{k-1}} \sum_{m=M}^{\infty} c_{n,m}^2 B_{n,m}^2(x) + 2 \sum_{1 \leq n \neq n' \leq 2^{k-1}} \sum_{M \leq m \neq m' < \infty} c_{n,m} c_{n',m'} B_{n,m}^T(x) B_{n',m'}(x) \\
\|f - S_{2^{k-1},M}(f)\|_2^2 &= \int_0^1 |f(x) - S_{2^{k-1},M}(f)(x)|^2 dx \\
&\leq \sum_{n=1}^{2^{k-1}} \sum_{m=M}^{\infty} |c_{n,m}|^2 \int_0^1 |B_{n,m}(x)|^2 dx + 2 \sum_{1 \leq n \neq n' \leq 2^{k-1}} \sum_{M \leq m \neq m' < \infty} |c_{n,m}| |c_{n',m'}| \\
&\quad \int_0^1 |B_{n,m}^T(x) B_{n',m'}(x)| dx = \sum_{n=1}^{2^{k-1}} \sum_{m=M}^{\infty} |c_{n,m}|^2, \text{ by orthonormality of } \{B_{n,m}\}.
\end{aligned}$$

□

Proof of Theorem 4.1 From (3.2), $c_{n,m} = \langle f(x), B_{n,m}(x) \rangle$

Case 1: $m = 0$

$$\begin{aligned}
c_{n_1,0} &= \int_{\frac{n_1-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} x \chi\left[\frac{n_1-1}{2^{k-1}}, \frac{n_1}{2^{k-1}}\right) B_{n_1,0}(x) dx \\
&= \int_{\frac{n_1-1}{2^{k-1}}}^{\frac{n_1}{2^{k-1}}} x 2^{\frac{k-1}{2}} \tilde{\beta}_0(2^{k-1}x - n_1 + 1) dx \\
&= 2^{\frac{-3}{2}(k-1)} \int_0^1 (u + n_1 - 1) \tilde{\beta}_0(u) du, \quad \left(\text{putting } 2^{k-1}x - n_1 + 1 = u\right) \\
&= 2^{\frac{-3}{2}(k-1)} \int_0^1 (u + n_1 - 1) du = 2^{\frac{-3}{2}(k-1)} \left(n_1 - \frac{1}{2}\right) \leq 2^{\frac{-1}{2}(k-1)}, \quad \left(\because 1 < n_1 < 2^{k-1}\right). \\
\|f - S_{2^{k-1},0}(f)\|_2^2 &= |c_{n_1,0}|^2 \leq 2^{-(k-1)} \quad (\text{by Lemma 4.1}) \\
E_{2^{k-1},0}^{(1)}(f) &= \|f - S_{2^{k-1},0}(f)\|_2 = o\left(\frac{1}{2^{\frac{1}{2}(k-1)}}\right).
\end{aligned}$$

Case 2: $m = M \geq 1$

$$\begin{aligned}
c_{n_1,m} &= \int_{\frac{n_1-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} x \chi\left[\frac{n_1-1}{2^{k-1}}, \frac{n_1}{2^{k-1}}\right) B_{n_1,m}(x) dx \\
&= \int_{\frac{n_1-1}{2^{k-1}}}^{\frac{n_1}{2^{k-1}}} x 2^{\frac{k-1}{2}} \tilde{\beta}_m(2^{k-1}x - n_1 + 1) dx \\
&= 2^{\frac{-3}{2}(k-1)} \frac{1}{\sqrt{\frac{(-1)^{m-1}(m!)^2}{(2m)!}} \alpha_{2m}} \int_0^1 (u + n_1 - 1) \beta_m(u) du, \quad \left(\text{putting } 2^{k-1}x - n_1 + 1 = u\right) \\
&= 2^{\frac{-3}{2}(k-1)} \frac{1}{\sqrt{\frac{(-1)^{m-1}(m!)^2}{(2m)!}} \alpha_{2m}} \int_0^1 (u + n_1 - 1) \frac{1}{m+1} \frac{d}{du} (\beta_{m+1}(u)) du, \quad (\text{by property (1)}) \\
&= 2^{\frac{-3}{2}(k-1)} \frac{1}{\sqrt{\frac{(-1)^{m-1}(m!)^2}{(2m)!}} \alpha_{2m}} \frac{1}{m+1} [n_1 \beta_{m+1}(1) - (n_1 - 1) \beta_{m+1}(0)], \quad (\text{integrating by part})
\end{aligned}$$

$$= \frac{2^{-\frac{3}{2}(k-1)}}{\sqrt{\frac{(-1)^{m-1}(m!)^2}{(2m)!}\alpha_{2m}}} \frac{1}{m+1} \beta_{m+1}(0) \leq 2^{-\frac{3}{2}(k-1)} \frac{1}{\sqrt{\frac{(-1)^{m-1}(m!)^2}{(2m)!}\alpha_{2m}}} \frac{1}{m+1}, \quad (\because \beta_m(1) = \beta_m(0))$$

$$\|f - S_{2^{k-1},M}(f)\|_2^2 = \sum_{m=M}^{\infty} |c_{n,m}|^2 = \sum_{m=M}^{\infty} 2^{-3(k-1)} \left| \frac{1}{\frac{(-1)^{m-1}(m!)^2}{(2m)!}\alpha_{2m}} \right| \frac{1}{(m+1)^2}. \quad (\text{by Lemma 4.1})$$

From the graph (2.3), we have $\left| \frac{1}{\frac{(-1)^{m-1}(m!)^2}{(2m)!}\alpha_{2m}} \right| \leq 3650$

$$\|f - S_{2^{k-1},M}(f)\|_2^2 \leq 3650 \left(2^{-3(k-1)} \right) \int_{x=M}^{\infty} \frac{1}{(x+1)^2} dx = o\left(\frac{1}{(2^{3(k-1)})(M+1)} \right)$$

$$E_{2^{k-1},M}^{(1)}(f) = \|f - S_{2^{k-1},M}(f)\|_2 = o\left(\frac{1}{2^{\frac{3}{2}(k-1)}\sqrt{M+1}} \right)$$

Hence, Theorem 4.1 has been established. $\square$

Proof of Theorem 4.2

$$\begin{aligned} c_{n,m} &= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f(t) B_{n,m}(t) dt \\ &= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left[(f(t) - f(\frac{n-1}{2^{k-1}})) B_{n,m}(t) + f\left(\frac{n-1}{2^{k-1}}\right) B_{n,m}(t) \right] dt \\ &= 2^{\frac{k-1}{2}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left[(f(t) - f(\frac{n-1}{2^{k-1}})) \beta_m(2^{k-1}t - n + 1) \right] dt \\ &= \frac{1}{2^{\frac{k-1}{2}}} \int_0^1 \left[(f(\frac{u+n-1}{2^{k-1}}) - f(\frac{n-1}{2^{k-1}})) \beta_m(u) \right] du, \quad (\text{putting } 2^{k-1}t - n + 1 = u) \\ &= \frac{1}{(m+1) 2^{\frac{k-1}{2}}} \int_0^1 \left(\frac{u}{2^{k-1}} \right)^{\alpha} \beta_{m+1}'(u) du \\ &= \frac{1}{(m+1) 2^{(\alpha+\frac{1}{2})(k-1)}} \beta_{m+1}(1) - \frac{1}{(\alpha+1) \sqrt{2\alpha+3}} \leq \frac{1}{(m+1) 2^{(\alpha+\frac{1}{2})(k-1)}} \\ \|f - S_{2^{k-1},M}(f)\|_2^2 &= \sum_{n=1}^{2^{k-1}} \sum_{m=M}^{\infty} |c_{n,m}|^2 \leq \sum_{n=1}^{2^{k-1}} \sum_{m=M}^{\infty} \frac{1}{(m+1)^2 2^{2(\alpha+1)(k-1)}}, \quad (\text{by Lemma 4.1}) \\ &= \frac{1}{2^{2\alpha(k-1)}} \sum_{m=M}^{\infty} \frac{1}{(m+1)^2} = \frac{1}{(M+1) 2^{2\alpha(k-1)}} \\ E_{2^{k-1},M}^{(2)}(f) &= \|f - S_{2^{k-1},M}(f)\|_2 = o\left(\frac{1}{\sqrt{M+1} 2^{(k-1)\alpha}} \right) \end{aligned}$$

Thus, Theorem 4.2 has been established. $\square$

5 Method of solution

5.1 Solution of linear Volterra integral equation of second kind

Consider the linear Volterra integral equation of second kind given by eqⁿ (1.1). Using eqⁿ (3.4), we have

$$\begin{aligned}
\int_0^x K(x, t)y(t) \, dt &= \int_0^x B^T(x) \mathbf{K} B(t) B^T(t) Y \, dt = B^T(x) \mathbf{K} \int_0^x B(t) B^T(t) Y \, dt \\
&= B^T(x) \mathbf{K} \int_0^x \tilde{Y} B(t) \, dt = B^T(x) \mathbf{K} \tilde{Y} P B(x) \\
\text{Then } y(x) &= f(x) + B^T(x) \mathbf{K} \tilde{Y} P B(x)
\end{aligned} \tag{5.1}$$

By evaluating the eqⁿ (5.1) at $2^{k-1}M$ points $\{x_i\}_{i=1}^{2^{k-1}M}$ in the intervals, we have the following system of linear equations:

$$B^T(x_i)Y = f(x_i) + B^T(x_i) \mathbf{K} \tilde{Y} P B(x_i) \tag{5.2}$$

Solving the system given by eqⁿ (5.2), we can find Y and hence the solution y(x).

5.2 Solution of nonlinear Volterra integral equation of second kind

Consider the Non-linear Volterra integral equation of second kind given by eqⁿ (1.2). Using the Bernoulli wavelet approximation, we have

$$B^T(x)Y = f(x) + \int_0^x k(x, t)F(B^T(t)Y)dt \tag{5.3}$$

Hence, we have following system of $2^{k-1}M$ equations:

$$B^T(x_i)Y = f(x_i) + \int_0^{x_i} k(x_i, t)F(B^T(t)Y)dt \tag{5.4}$$

Now, transform the integration in the interval $[0, x_i]$ as $t = \frac{x_i(u+1)}{2}$ and

$$H(x_i, t) = k(x_i, t)F(B^T(t)Y) \tag{5.5}$$

Using these, Eqⁿ (5.6) becomes

$$B^T(x_i)Y = f(x_i) + \frac{x_i}{2} \int_{-1}^1 H\left(x_i, \frac{x_i(u+1)}{2}\right) du \tag{5.6}$$

This is the nonlinear system of algebraic equations which can be solved by Newton's method. After obtaining the coefficients, we can find the solution of the integral equation (1.2).

6 Numerical examples

Some linear and nonlinear Volterra integral equations have been solved by Bernoulli wavelet technique discussed in this paper and are compared with their exact solutions. The graphs of these solutions are plotted. It is observed that exact solution and approximate solutions of Volterra integral equations are almost equal.

6.1 Example 1

Consider the linear Volterra integral equation

$$y(x) = 2x^3 - x^2 - e^{-x^2} + 1 + \int_0^x e^{-x^2+t^2} y(t) \, dt \tag{6.1}$$

$y(x) = 2x^3$ is the exact solution of the integral equation (6.1).

Table 1 shows a numerical comparison between exact solution and approximate solutions obtained by Bernoulli wavelet technique. In addition, Fig. 2 shows a graphical representation of exact and approximate solution of linear Volterra integral equation (6.1).

Table 1 Numerical comparison between Bernoulli wavelet solution and exact solution of example 6.1 for different values of x .

x	Exact Sol ⁿ	Bernoulli wavelet Sol ⁿ : $k = 2, m = 2$	Bernoulli wavelet Sol ⁿ : $k = 2, m = 3$	Bernoulli wavelet Sol ⁿ : $k = 3, m = 3$
0	0	0.0015	0.0009	0.0011
0.1	0.0020	0.0029	0.0035	0.0020
0.2	0.0160	0.0221	0.0109	0.0160
0.3	0.0540	0.0612	0.0405	0.0537
0.4	0.1280	0.1531	0.0945	0.1300
0.5	0.2500	0.3051	0.3072	0.2548
0.6	0.4320	0.4512	0.5960	0.4479
0.7	0.6860	0.0714	0.8809	0.7313
0.8	1.0240	0.8254	0.9800	0.9719
0.9	1.4580	1.3025	1.3881	1.5324
1	2	1.6025	1.6502	2.0157

Absolute error

x	For $k=2, m=2$	For $k=2, m=3$	For $k=3, m=3$
0	0.0015	0.0009	0.0011
0.1	0.0009	0.0015	0.0001
0.2	0.0061	0.0051	0.0004
0.3	0.0072	0.0135	0.0003
0.4	0.0251	0.0335	0.0020
0.5	0.0551	0.0572	0.0048
0.6	0.0192	0.1640	0.0159
0.7	0.6146	0.1949	0.0453
0.8	0.1986	0.0440	0.0521
0.9	0.1555	0.0699	0.0744
1	0.3975	0.3498	0.0157

Fig. 2 Comparison of Bernoulli wavelet solution and exact solution of example 6.1.

6.2 Example 2

Consider another linear Volterra integral equation given by

$$y(x) = \frac{x^2 e^x}{\pi} \cos(\pi x) + e^x \left(1 - \frac{x}{\pi^2}\right) \sin(\pi x) + \int_0^x x t e^{x-t} y(t) dt \quad (6.2)$$

$y(x) = e^x \sin(\pi x)$ is the exact solution of the integral equation (6.2).

Table 2 Numerical comparison between Bernoulli wavelet solution and exact solution of example 6.2 for different values of x .

x	Exact Sol ⁿ	Bernoulli wavelet Sol ⁿ : $k = 2, m = 2$	Bernoulli wavelet Sol ⁿ : $k = 2, m = 3$	Bernoulli wavelet Sol ⁿ : $k = 3, m = 3$
0	0	0.0101	0.0012	− 0.0058
0.1	0.3415	0.4255	0.3902	0.3424
0.2	0.7179	0.7545	0.7226	0.7173
0.3	1.0921	1.1902	1.1150	1.0912
0.4	1.4188	1.5166	1.4555	1.4112
0.5	1.6487	1.7823	1.7251	1.7068
0.6	1.7329	1.7703	1.8085	1.7723
0.7	1.6292	1.5892	1.5662	1.6114
0.8	1.3081	1.2456	1.1158	1.3105
0.9	0.7601	0.8278	0.5226	0.6247
1	0	0.3636	− 0.2051	− 0.3368

Absolute error

x	For $k=2, m=2$	For $k=2, m=3$	For $k=3, m=3$
0	0.0101	0.0012	0.0058
0.1	0.0840	0.0487	0.0009
0.2	0.0366	0.0047	0.0006
0.3	0.0981	0.0229	0.0009
0.4	0.0978	0.0367	0.0076
0.5	0.1336	0.0764	0.0581
0.6	0.0374	0.0756	0.0394
0.7	0.0400	0.0630	0.0178
0.8	0.0625	0.1923	0.0024
0.9	0.0677	0.2375	0.1354
1	0.3636	0.2051	0.3368

Fig. 3 Comparison of Bernoulli wavelet solution and exact solution of example 6.2.

Table 2 shows a numerical comparison between exact solution and approximate solution obtained by Bernoulli wavelet technique. In addition, Fig. 3 shows a graphical representation of exact and approximate solution of linear Volterra integral equation (6.2).

Table 3 Numerical comparison between Bernoulli wavelet solution and exact solution of example 6.3 for different values of x .

x	Exact Sol ⁿ	Bernoulli wavelet Sol ⁿ : $k = 2, m = 2$	Bernoulli wavelet Sol ⁿ : $k = 2, m = 3$	Bernoulli wavelet Sol ⁿ : $k = 3, m = 3$
0	0	− 0.01241	− 0.0042	0.0021
0.1	0.0100	0.0091	0.0052	0.0130
0.2	0.0400	0.0263	0.0290	0.0450
0.3	0.0900	0.0612	0.0699	0.0955
0.4	0.1600	0.1004	0.1400	0.1721
0.5	0.2500	0.1862	0.2096	0.2661
0.6	0.3600	0.2736	0.3006	0.3748
0.7	0.4900	0.3599	0.3911	0.5332
0.8	0.6400	0.4508	0.5010	0.6658
0.9	0.8100	0.5909	0.6436	0.8393
1	1	0.7542	0.7900	1.3390

Absolute error

x	For $k=2, m=2$	For $k=2, m=3$	For $k=3, m=3$
0	0.0124	0.0042	0.0021
0.1	0.0009	0.0048	0.0030
0.2	0.0137	0.0110	0.0050
0.3	0.0288	0.0201	0.0055
0.4	0.0596	0.0200	0.0121
0.5	0.0638	0.0404	0.0161
0.6	0.0864	0.0594	0.0148
0.7	0.1301	0.0989	0.0432
0.8	0.1892	0.1390	0.0258
0.9	0.2191	0.1664	0.0293
1	0.2458	0.2100	0.2390

Fig. 4 Comparison of Bernoulli wavelet solution and exact solution of example 6.3.

6.3 Example 3

Consider the following nonlinear Volterra integral equation:

$$y(x) = x^2 - \frac{x^5}{20} + \int_0^x (x-t) (y(t))^3 dt \quad (6.3)$$

$y(x) = x^2$ is its exact solution.

Table 3 shows a numerical comparison between exact solution and approximate solutions obtained by Bernoulli wavelet technique. In addition, Fig. 4 shows a graphical representation of exact and approximate solutions of nonlinear Volterra integral equation (6.3).

7 Remarks

1. From Theorem 4.1, we have

$$E_{2^{k-1},M}^{(1)}(f) = \begin{cases} o\left(\frac{1}{2^{\frac{1}{2}(k-1)}}\right), & \text{if } M = 0 \\ o\left(\frac{1}{2^{\frac{3}{2}(k-1)}\sqrt{M+1}}\right), & \text{if } M \geq 1. \end{cases}$$

hence $E_{2^{k-1},M}^{(1)}(f) \rightarrow 0$ as $M \rightarrow \infty$, $k \rightarrow \infty$.

2. From Theorem 4.2, we have

$$E_{2^{k-1},M}^{(2)}(f) = o\left(\frac{1}{\sqrt{M+1} 2^{\frac{1}{2}(k-1)\alpha}}\right) \rightarrow 0 \text{ as } M \rightarrow \infty, k \rightarrow \infty.$$

Therefore, $E_{2^{k-1},M}^{(1)}(f)$ and $E_{2^{k-1},M}^{(2)}(f)$ are the best possible estimators in wavelet analysis (Zygmund [22]).

3. Solution of linear and nonlinear Volterra integral equations, obtained by Bernoulli wavelet technique are compared with their exact solution with the help of tables and graphs. In addition, we observe that approximate solution and exact solution are almost same. This is a significant achievement of this research paper.
4. From the graphs of the solution, we observe that when we increase the values of k and m , the approximate solutions are more close to the exact solution.

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Declarations

Conflict of interest The first author declares that he has no conflict of interest. The second author declares that he has no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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