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Common fixed points for $(\psi - \varphi)$ -weak contractions type in b -metric spaces

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Abstract The aim of this paper is to prove the existence and uniqueness of common fixed points for a pair of $(\psi - \varphi)$ -weak contractive self-maps in the setting of b -metric spaces satisfying the minimal requirement of weakly compatibility, and other weak commuting properties as compatibility, R -weakly commuting and R -weakly commuting of types (A_T) , (A_S) and (A_P) . Also, we will analyze the convergence and stability of the Jungck-Noor iterative scheme for this class of pairs of mappings on b -metric spaces endowed with a convex structure.

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1 Motivation

The beginning of the metric fixed point theory could be set in 1922 with the Banach's seminal paper [9], in which he introduced the so-called Banach contraction mappings and analyzed the existence and uniqueness of its fixed points in the setting of metric spaces. More precisely, Banach showed that a mapping $S : M \longrightarrow M$, where (M, d) is a complete metric space, satisfying that there exists $0 \leq \alpha < 1$ such that for all $x, y \in M$,

$$d(Sx, Sy) \leq \alpha d(x, y), \quad (1.1)$$

has a unique fixed point in M . This result now is known as the Banach contraction principle, (in short, BCP). Since then, several authors have obtained various extensions of the BCP by considering new contractive conditions, allowing, among other issues, the consideration of discontinuous maps, and posing the mappings on many different generalized metric spaces as well. Regarding this matter we cite the works [7, 26] its citations and references therein.

In 1984, Khan et al. [28] extended the BCP by redefining the class of functions introduced by Delbosco and Skof [18, 46], the so-called altering distance functions: Let $\mathbb{R}_+$ be the set of all non-negative real numbers, that is, $\mathbb{R}_+ := [0, +\infty)$.

Definition 1.1 A function $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is said to be an altering distance function if the following conditions are satisfied:

(ψ_1) ψ is a continuous mapping,

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- (ψ_2) ψ is a nondecreasing mapping,
 (ψ_3) $\psi(t) = 0$ iff $t = 0$, for $t \in \mathbb{R}_+$.

Henceforth, by Ψ we will denote the set of all altering distance functions.

Using this class of functions, Khan et al., in the same paper [28], proved that in a complete metric space (M, d) , a mapping $S : M \rightarrow M$ satisfying that there exist $\psi \in \Psi$ and $0 \leq \alpha < 1$ such that for all $x, y \in M$

$$\psi(d(Sx, Sy)) \leq \alpha \psi(d(x, y)), \quad (1.2)$$

has a unique fixed point. Notice that if we take $\psi(t) = t$, $t \in \mathbb{R}_+$ in (1.2) we obtain (1.1).

In 1997, Alber and Guerre-Delabriere [6] introduced a new generalization of the BCP, the so-called weakly contractive mappings for single-valued mappings in Hilbert spaces and proved some fixed point theorems in this setting. Afterwards, Rhoades [41] showed that most of the results of [6] hold in Banach spaces and in metric spaces as well.

Definition 1.2 Let (M, d) be a metric space. A mapping $S : M \rightarrow M$ is said to be a φ -weak contraction if there exists a continuous nondecreasing function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, with $\varphi(t) > 0$ for all $t > 0$, $\varphi(0) = 0$ and $\lim_{t \rightarrow +\infty} \varphi(t) = +\infty$, such that for all $x, y \in M$,

$$d(Sx, Sy) \leq d(x, y) - \varphi(d(x, y)).$$

Rhoades in his paper [41] extended the BCP to φ -weak contraction mappings without using the condition $\lim_{t \rightarrow +\infty} \varphi(t) = +\infty$, as follows:

Theorem 1.1 Let (M, d) be a complete metric space and let $S : M \rightarrow M$ be a φ -weak contraction, then S has a unique fixed point when φ is a nondecreasing continuous function with $\varphi(t) > 0$ for all $t > 0$ and $\varphi(0) = 0$.

In 2008, Dutta and Choudhury [19] using the altering distance functions generalized the concept of φ -weak contraction and proved the following result:

Theorem 1.2 Let (M, d) be a complete metric space and let S be a selfmap on M satisfying

$$\psi[d(Sx, Sy)] \leq \psi(d(x, y)) - \varphi(d(x, y)),$$

for all $x, y \in M$, where $\psi, \varphi \in \Psi$. Then S has a unique fixed point in M .

In 2009, Abbas and Khan [3] proved the following result on the existence of a common fixed point for two mappings, the so-called $(\psi - \varphi)$ -weak contraction mappings. To state it, for two mappings $S, T : M \rightarrow M$ we denote by $C(S, T)$ the set of coincidence points (CP) of S and T , that is,

$$C(S, T) = \{x \in M : Sx = Tx\};$$

and by $PC(S, T)$ we denote the set of point of coincidence (POC) of S and T , that is

$$PC(S, T) = \{z \in M : z = Sx = Tx, \text{ for some } x \in M\}.$$

Also, we recall that S and T are say to be (nontrivially) weakly compatible [25] if whenever $x \in C(S, T)$ implies that $STx = T Sx$.

Theorem 1.3 Let S and T be two selfmaps of a metric space (M, d) satisfying

$$\psi[d(Sx, Sy)] \leq \psi[d(Tx, Ty)] - \varphi[d(Tx, Ty)]$$

for all $x, y \in M$, where $\psi, \varphi \in \Psi$. If $SM \subset TM$ and $TM \subset M$ is a complete subspace, then S and T have a unique point of coincidence in M . Moreover, if S and T are weakly compatible, S and T have a unique common fixed point in M .

We would like to point out that many authors have extended and generalized the notion of $(\psi - \varphi)$ -weak contraction mappings and analyzed the existence of (common) fixed points for this class of mappings in different metric-like spaces. To mention some, we recommend the recent works [21, 32, 42].

On the other hand, effective and efficient algorithms to compute approximate common fixed points are crucial tools in the metric fixed point theory. The fixed point assured by the BCP and some of its extensions can be approximated computed by the basic, one-step, Picard iteration process (see Proposition 3.4 below for $(\psi - \varphi)$ -weak contraction type mappings). However, over the years, several more efficient iterative algorithms to compute approximate fixed points (and common fixed points) for contractive maps have been developed in the setting of linear spaces, such processes includes, among others, the so-called Krasnosel'skii (one-step), Mann (one-step) and Ishikawa (two-step) iteration, its extensions and generalizations. To a more comprehensible study about those algorithms, we recommend the seminal monograph [10], its citations and reference therein.

In this fashion, in 2002, Xu and Noor [49] defined a three-step iterative scheme in Banach spaces as follows: let x_0 fixed, define

$$\begin{aligned} z_n &= (1 - \alpha_n)x_n + \alpha_n T^n x_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n T^n z_n, \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T^n y_n, \quad n = 1, 2, \dots, \end{aligned}$$

where $(\alpha_n), (\beta_n), (\gamma_n) \subset [0, 1]$. Notice that Noor's iterative process includes Mann's and Ishikawa's iterations as special cases. Three-step schemes are natural generalization of the splitting methods to solve partial differential equations (inclusions). In 2006, Bnouhachem et al. [12] showed that three-step iterative schemes performs better than two-step and one-step methods for solving variational inequalities. Noor's iteration algorithm have been extended and generalized in different ways and its convergence analyzed for different classes of contractive-type mappings, see e.g., [20, 27] and references therein.

To define a Noor type iterative process in the framework of general metric spaces would require to provide the metric space with an extra structure. In 1970, Takahashi [47] introduce the notion of convex structure on metric spaces as follows:

Definition 1.3 Let (M, d) be a metric space. A mapping $\mathcal{W} : M \times M \times [0, 1] \longrightarrow M$ is said to be a convex structure on M if for each $(x, y, \lambda) \in M \times M \times [0, 1]$ and $z \in M$,

$$d(z, \mathcal{W}(x, y, \lambda)) \leq \lambda d(z, x) + (1 - \lambda)d(z, y).$$

A metric space (M, d) equipped with the convex structure $\mathcal{W}$ is called a convex metric space, and it is denoted by $(M, d, \mathcal{W})$.

This concept is a natural generalization of convexity in normed linear spaces (see [2]), nontrivial examples of convex metric spaces are the so-called CAT(0) spaces. Hence, convex metric spaces constitute a right setting to define and extend iterative schemes constructed by convex combinations, as mentioned above. Some investigations regarding to the application and convergence of Jungck–Mann, Jungck–Ishikawa, Jungck-type and Noor-type iterative procedures in convex metric spaces can be found, for instance, in [5, 31, 36].

Let $(M, d, \mathcal{W})$ be a convex metric space and let $S, T : N \longrightarrow M$ be two nonself mappings on a subset N of M such that $S(N) \subset T(N)$, where $T(N)$ is a complete subspace of M . For any $x_0 \in M$, the Jungck–Noor iterative procedure is then defined by the sequence (x_n) generated as:

$$\begin{aligned} Tx_{n+1} &= \mathcal{W}(Tx_n, Sz_n, \alpha_n), \\ Tz_n &= \mathcal{W}(Tx_n, Sy_n, \beta_n), \\ Ty_n &= \mathcal{W}(Tx_n, Sx_n, \gamma_n), \quad n = 0, 1, \dots, \end{aligned} \tag{1.3}$$

with sequences $(\alpha_n), (\beta_n), (\gamma_n) \subset [0, 1]$.

Motivated from these facts, in this paper we are going to prove the existence and uniqueness of common fixed points for a pair of $(\psi - \varphi)$ -weak contractive type self-maps in the setting of b -metric spaces satisfying the minimal requirement of *weakly compatibility*, and other weak commuting properties as *compatibility*, *R-weakly commuting* and *R-weakly commuting of types (A_T) , (A_S) and (A_P)* . The results proved in this paper extend and generalize the mentioned results, as well as, many others given in this line of research.

The strategy here is to prove the convergence of some sequences posed in the image of the mappings under consideration. Hence conditions like *completeness* of these sets will be naturally required, however, we are able to replace it by weak alternative notions as the so-called *b-property (EA)* and the *b-CLR_T-property*. Also we will analyze the convergence and stability of the Jungck–Noor iterative scheme for this class of pairs of mappings on b -metric spaces endowed with a convex structure.

2 Some fundamental results in b -metric spaces

In this section, we recall some definitions, results and properties of b -metric spaces that will be useful in this paper.

We recall that in 1989, Bakhtin [8] introduced in the theory of metric fixed point the concept of b -metric spaces (also known as quasimetric spaces), as a generalization of usual metric spaces and shows the BCP in this setting. After that, the b -metric spaces were widely utilized by Czerwik and others ([16] and [17]). Since then, some examples of b -metric spaces were given and several results about the existence of a fixed point for a single-valued and multivalued operator in b -metric spaces can be found, for instance, in Boriceanu [13], Berinde [11], Isutali [22] among many others authors.

Definition 2.1 Let M be a nonempty set and $s \geq 1$ be given a real number. A function

$$\rho : M \times M \longrightarrow \mathbb{R}_+$$

is said to be a b -metric if and only if for all $x, y, z \in M$ the following properties are satisfied:

- (bM1) $\rho(x, y) = 0$ iff $x = y$,
- (bM2) $\rho(x, y) = \rho(y, x)$,
- (bM3) $\rho(x, z) \leq s(\rho(x, y) + \rho(y, z))$.

In such a case, the pair (M, ρ) is called a b -metric space and the real number $s \geq 1$ is called the coefficient of (M, ρ) .

b -metric spaces (M, ρ) can be constructed from a metric space (M, d) , for instance, by considering the b -metric $\rho(x, y) := \sigma(d(x, y))$ with $\sigma(t) = t^p$, $1 \leq p < +\infty$. Notice that from the inequality $(a + b)^p \leq 2^{p-1}(a^p + b^p)$, we have that (M, ρ) is a b -metric space with coefficient $s = 2^{p-1}$. Also, there are b -metric spaces which are not metric spaces, as the Lebesgue spaces L^p ($0 < p < 1$). Other examples of b -metric spaces are the Orlicz spaces and spaces of homogeneous type, among other classical function spaces and scales of them used in functional analysis. See e.g. [15, 33].

The family of sets of the form $B(x, r) = \{y \in M : \rho(x, y) < r\}$ generates a topology on M (see [29, Proposition 1]), which is metrizable in virtue of the Alexandroff-Urysohn theorem. As a consequence of this metrization, much of the results given in metric spaces hold in this setting.

Now, we present the notions of a convergent sequence, Cauchy sequence and complete b -metric spaces.

Definition 2.2 Let (M, ρ) be a b -metric space with $s \geq 1$. Then, a sequence (x_n) in M is called:

1. b -convergent if there exists $x \in M$ such that $\lim_{n \rightarrow +\infty} \rho(x_n, x) = 0$. In this case we write $\lim_{n \rightarrow +\infty} x_n = x$ or $x_n \rightarrow x$.
2. b -Cauchy sequence if $\lim_{m, n \rightarrow +\infty} \rho(x_n, x_m) = 0$. If every b -Cauchy sequence in M is convergent, then (M, ρ) is said to be a complete b -metric space.

Proposition 2.1 Let (M, ρ) be a b -metric space with $s \geq 1$. The following assertions hold:

- (i) Any b -convergent sequence has a unique limit.
- (ii) The subsequences of a b -convergent sequence are also b -convergent to the limit of the original sequence.
- (iii) Every sequence which is b -convergent is also a b -Cauchy sequence.
- (iv) In general, a b -metric ρ is not necessarily continuous in all its variables.

Aghajani et al. [4] proved the following simple lemma about b -convergent sequences.

Lemma 2.2 Let (M, ρ) be a b -metric space with $s \geq 1$, and suppose that (x_n) and (y_n) are b -convergent sequences converging to x and y respectively. Then, we have,

$$\frac{1}{s^2} \rho(x, y) \leq \liminf_{n \rightarrow +\infty} \rho(x_n, y_n) \leq \limsup_{n \rightarrow +\infty} \rho(x_n, y_n) \leq s^2 \rho(x, y).$$

In particular, if $x = y$, then $\lim_{n \rightarrow +\infty} \rho(x_n, y_n) = 0$. Moreover, for each $z \in M$ we have

$$\frac{1}{s} \rho(x, z) \leq \liminf_{n \rightarrow +\infty} \rho(x_n, z) \leq \limsup_{n \rightarrow +\infty} \rho(x_n, z) \leq s \rho(x, z).$$

Now, we present some definitions and properties that a pair of mappings could satisfy and that will be needed in the statements of our main results.

Definition 2.3 Let (M, ρ) be a b -metric space with $s \geq 1$. The mappings $S, T : M \longrightarrow M$ are said to

1. be compatible [24] if $\lim_{n \rightarrow +\infty} \rho(STx_n, TSx_n) = 0$ whenever (x_n) is a sequence in M such that

$$\lim_{n \rightarrow +\infty} Sx_n = \lim_{n \rightarrow +\infty} Tx_n = t, \text{ for some } t \in M.$$

2. be non-compatible [24] if there exist at least one sequence (x_n) in M such that $\lim_{n \rightarrow +\infty} Sx_n = \lim_{n \rightarrow +\infty} Tx_n = t$ for some $t \in M$ but

$$\lim_{n \rightarrow +\infty} \rho(STx_n, TSx_n)$$

is either non-zero or non-existent.

3. Satisfy the b -property (EA) [1] if there exists a sequence (x_n) in M such that $\lim_{n \rightarrow +\infty} Sx_n = \lim_{n \rightarrow +\infty} Tx_n = t$ for some $t \in M$.
4. Satisfy the common b -limit range property with respect to T (in short, b -CLR $_T$ -property) [45] if there exists (x_n) in M such that

$$\lim_{n \rightarrow +\infty} Sx_n = \lim_{n \rightarrow +\infty} Tx_n = Tt$$

for some $t \in M$.

Remark 2.3 We list some comparison and properties of mappings satisfying the weak commuting forms given in Definition 2.3 (see, for instance, [14, 23, 40])

1. If S and T are compatible, then S and T commute at their coincidence points, hence S and T are weakly compatible.
2. If S and T are non-compatible, then S and T satisfy the b -property (EA).
3. Weak compatibility and b -property (EA) are independent to each other.
4. The b -CLR $_T$ -property avoid the requirement of the condition of closedness of the ranges of the involved mappings.

Example 2.4

1. Let $M = \mathbb{R}$ and let $\rho(x, y) = |x - y|^2$ for each $x, y \in M$. Let S and T be two selfmappings of M defined by $Sx = 2x - 1$ and $Tx = x^2$, $x \in M$. Then,
 - (i) $Sx = Tx$ iff $x = 1$. Hence $C(S, T) = \{1\}$, $S1 = T1$ and $ST1 = TS1$. Thus S and T commute at their coincidence point and, therefore, S and T are weakly compatible.
 - (ii) Consider the sequence $(x_n) = (1 + 1/n)$, $n \in \mathbb{N}$ in M , then

$$\begin{aligned} \lim_{n \rightarrow +\infty} Sx_n &= \lim_{n \rightarrow +\infty} S(1 + 1/n) = 1 \\ \lim_{n \rightarrow +\infty} Tx_n &= \lim_{n \rightarrow +\infty} T(1 + 1/n) = 1. \end{aligned}$$

Therefore, S and T have the b -property (EA). It is clear that

$$\lim_{n \rightarrow +\infty} \rho(STx_n, TSx_n) = \lim_{n \rightarrow +\infty} |STx_n - TSx_n|^2 = 0$$

hence, S and T are compatible selfmappings.

(iii) S and T satisfy the b -CLR $_T$ -property.

1. Let $M = [2, 20] \subset \mathbb{R}$ be with $\rho(x, y) = |x - y|^2$ for all $x, y \in M$. Let S and T be two selfmappings of M defined by

$$Sx = \begin{cases} 2, & \text{if } x = 2 \text{ or } x > 5 \\ 3, & \text{if } 2 < x \leq 5. \end{cases} \quad Tx = \begin{cases} 2, & \text{if } x = 2 \\ 12, & \text{if } 2 < x \leq 5 \\ \frac{x+1}{3}, & \text{if } x > 5. \end{cases}$$

Consider the sequence (x_n) defined by $(x_n) = \left(5 + \frac{1}{n}\right)$, $n \in \mathbb{N}$. Then,

- (i) $\lim_{n \rightarrow +\infty} Sx_n = \lim_{n \rightarrow +\infty} Tx_n = 2 = T2$. Hence, S and T have the b -property (EA) and $b\text{-CLR}_T$ -property. Moreover,

$$\lim_{n \rightarrow +\infty} \rho(STx_n, TSx_n) = \lim_{n \rightarrow +\infty} |STx_n - TSx_n|^2 = 1^2 \neq 0.$$

Therefore, S and T are non-compatible selfmaps.

- (ii) It is clear that $C(S, T) = 2$ and $ST2 = TS2 = 2$. Thus, S and T are weakly compatible selfmappings.

Definition 2.5 Let (M, ρ) be a b -metric space with $s \geq 1$. The mappings $S, T : M \rightarrow M$ are said to be,

1. R -weakly commuting [37] if there exists some $R > 0$ such that

$$\rho(STx, TSx) \leq R\rho(Sx, Tx), \text{ for all } x \in M.$$

2. R -weakly commuting mapping of type (A_T) [38] if there exists some $R > 0$ such that

$$\rho(STx, SSx) \leq R\rho(Sx, Tx), \text{ for all } x \in M.$$

3. R -weakly commuting mappings of type (A_S) [38] if there exists $R > 0$ such that

$$\rho(STx, TTx) \leq R\rho(Sx, Tx), \text{ for all } x \in M.$$

4. R -weakly commuting mapping of type (P) [30] if there exists $R > 0$ such that

$$\rho(SSx, TTx) \leq R\rho(Sx, Tx), \text{ for all } x \in M.$$

Remark 2.4 We list some comparison between the weaker forms of commuting properties given in Definition 2.5 (see [44]):

1. R -weakly commuting pairs of selfmappings are independent of R -weakly commuting of type (A_T) or type (A_S) .
2. Both compatible and non-compatible selfmappings can be R -weakly commuting of type (A_T) or type (A_S) , but the converse is not true.
3. If S and T are R -weakly commuting or R -weakly commuting of type (A_T) , or R -weakly commuting of type (A_S) , or R -weakly commuting of type (P) , then S and T are weakly compatible mapping.

Example 2.6 Let $M = \mathbb{R}$ and $\rho(x, y) = |x - y|^2$ for all $x, y \in M$. Let S and T be two selfmappings of M defined by $Sx = 2x - 1$ and $Tx = x^2$, $x \in M$. Then,

1. $\rho(STx, TSx) = |STx - TSx|^2 = (2|x - 1|^2)^2 \leq 5\rho(Sx, Tx)$. Thus, S and T are R -weakly commuting with $R = 5$.
2. $\rho(SSx, TSx) = (4|x - 1|^2)^2 \leq 16\rho(Sx, Tx) = 16(|x - 1|^2)^2$. Hence, S and T are R -weakly commuting of type (A_T) .
3. $\rho(STx, TTx) = (|x^2 - 1|^2)^2$ and $\rho(Sx, Tx) = (|x - 1|^2)^2$ if we take $x = 4$, we obtain $\rho(ST4, TT4) = 225^2$ and $\rho(S4, T4) = 81$. For $R = 16$, $\rho(ST4, TT4) > R\rho(S4, T4)$. Hence, S and T are not R -weakly commuting of type (A_S) .
4. $\rho(SSx, TTx) = |SSx - TTx|^2 = |x^4 - 4x + 3|^2$, $\rho(Sx, Tx) = (|x - 1|^2)^2$. If we take $x = 2$ for $R = 16$, we have $\rho(SS2, TT2) > R\rho(S2, T2)$. Thus, S and T are not R -weakly commuting of type (P) .

3 The class of $(\psi - \varphi)$ -weak contraction type mappings

In this section, we introduce the class of $(\psi - \varphi)$ -weak contraction type mappings in the framework of b -metric spaces, and we show the convergence of the Jungck iterative scheme. Also, we prove the existence and uniqueness of points of coincidence for mappings in this class.

We must recall that by Ψ we denoted the set of all altering distance functions. By Φ we will denote the set of all functions $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying the following conditions:

- (φ_1) $\varphi(t) > 0$, for all $t \in \mathbb{R}_+$,
- (φ_2) $\varphi(0) = 0$,
- (φ_3) φ is lower semicontinuous function.

Notice that $\Psi \subset \Phi$.

Definition 3.1 Let (M, ρ) be a b -metric spaces with $s \geq 1$ and let S and T be selfmappings of M . The mappings S and T are said to be of $(\psi - \varphi)$ -weak contractions type if there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that for all $x, y \in M$,

$$\psi[s\rho(Sx, Sy)] \leq \psi(\rho(Tx, Ty)) - \varphi(\rho(Tx, Ty)). \quad (3.1)$$

The following remark is very important and useful in the development of our work.

Remark 3.1

1. If we take $\varphi(t) = (1 - \alpha)\psi(t)$, $t \in \mathbb{R}_+$ and $0 \leq \alpha < 1$ in (3.1), then we obtain

$$\psi[s\rho(Sx, Sy)] \leq \alpha\psi(\rho(Tx, Ty)). \quad (3.2)$$

The functions S and T satisfying condition (3.2) are called Ψ -Jungck contraction mappings.

Notice that if we take $\psi = \text{Id}$ (the identity mapping) in (3.2), we obtain

$$s\rho(Sx, Sy) \leq \alpha\rho(Tx, Ty) \quad (3.3)$$

where $0 \leq \alpha < 1$. Since $s \geq 1$ then any pair of mappings (S, T) satisfying (3.3) satisfies the inequality

$$\rho(Sx, Sy) \leq \alpha\rho(Tx, Ty). \quad (3.4)$$

The maps S and T satisfying condition (3.4) are called Jungck contractions. If we take $T = \text{Id}$ in (3.4), we obtain the Banach contractions (1.1). Also notice that if any pair of mappings (S, T) satisfies (3.3), they are Jungck contractions with parameter $\alpha/s < 1$.

2. If we take $\psi = \text{Id}$ in (3.1), we obtain

$$s\rho(Sx, Sy) \leq \rho(Tx, Ty) - \varphi(\rho(Tx, Ty)). \quad (3.5)$$

The pair (S, T) satisfying condition (3.5), for $s = 1$, is called a φ -weak contractions pair.

If we take $w(t) = t - \varphi(t)$, $t \in \mathbb{R}_+$ it is clear that w is upper semicontinuous from the right function and we obtain

$$s\rho(Sx, Sy) \leq w(\rho(Tx, Ty)). \quad (3.6)$$

This type of mappings are called Boyd-Wong contractions type ($s = 1$).

If we take $T = \text{Id}$ in (3.6), we obtain

$$s\rho(Sx, Sy) \leq w(\rho(x, y))$$

which is called the Boyd-Wong contraction ($s = 1$).

3. If we take

$$k(t) = \begin{cases} 1 - \varphi(t)/t & \text{for } t > 0 \\ 0 & \text{for } t = 0. \end{cases}$$

in (3.5), we obtain

$$s\rho(Sx, Sy) \leq k(\rho(Tx, Ty))\rho(Tx, Ty) \quad (3.7)$$

and if we take $T = \text{Id}$ in (3.7) we obtain

$$s\rho(Sx, Sy) \leq k(\rho(x, y))\rho(x, y). \quad (3.8)$$

It is important to say that inequalities (3.7) and (3.8) are closely related to Reich contractions type.

In the next result we prove that the Jungck iterative process for $(\psi - \varphi)$ -weak contraction mappings type b -converges.

Proposition 3.2 Let (M, ρ) be a b -metric space with $s \geq 1$ and let S and T be selfmappings of M with $SM \subset TM$. If S and T are $(\psi - \varphi)$ -weak contraction type mappings, then for any $x_0 \in M$ the sequence defined by

$$y_n = Sx_n = Tx_{n+1}, \quad n = 0, 1, \dots \quad (3.9)$$

Satisfies:

- (i) $\lim_{n \rightarrow +\infty} \rho(y_n, y_{n+1}) = 0$.
- (ii) $(y_n) \subset M$ is a b -Cauchy sequence.

Proof To prove (i), let $x_0 \in M$ be an arbitrary point. Using the condition $SM \subset TM$, we construct a Jungck sequence defined by $y_n = Sx_n = Tx_{n+1}$, $n = 0, 1, \dots$. Suppose that $y_{n-1} = Tx_n \neq Tx_{n+1} = y_n$, for some $n \in \mathbb{N}$, since otherwise (y_n) would be a constant sequence for $n \geq n_0$. Now, using (3.1) we obtain:

$$\begin{aligned} \psi[\rho(y_n, y_{n+1})] &\leq \psi[s\rho(y_n, y_{n+1})] = \psi(s\rho(Sx_n, Sx_{n+1})) \\ &\leq \psi(\rho(Tx_n, Tx_{n+1})) - \varphi(\rho(Tx_n, Tx_{n+1})) \\ &\leq \psi(\rho(y_{n-1}, y_n)) - \varphi(\rho(y_{n-1}, y_n)) < \psi(\rho(y_{n-1}, y_n)). \end{aligned} \quad (3.10)$$

Since $\psi \in \Psi$, we obtain

$$\rho(y_n, y_{n+1}) < \rho(y_{n-1}, y_n).$$

Therefore, $(\rho(y_n, y_{n+1}))$ is a monotone decreasing sequence of positive real numbers, so there exists $L \geq 0$ such that $\lim_{n \rightarrow +\infty} \rho(y_n, y_{n+1}) = L$. We want to show that $L = 0$. Suppose that $L > 0$. From (3.10), we get that

$$\limsup_{n \rightarrow +\infty} \psi(\rho(y_n, y_{n+1})) \leq \limsup_{n \rightarrow +\infty} \psi(\rho(y_n, y_{n+1})) - \liminf_{n \rightarrow +\infty} \psi(\rho(y_n, y_{n+1})).$$

It follows that,

$$\psi(L) \leq \psi(L) - \varphi(L) < \psi(L)$$

which is not possible, therefore, $L = 0$. Thus we have,

$$\lim_{n \rightarrow +\infty} \rho(y_n, y_{n+1}) = 0.$$

To prove (ii), we want to show that (y_n) is a b -Cauchy sequence in M . Suppose the contrary, then there exists $\varepsilon > 0$ for which we can find sequences $(m(k))$ and $(n(k))$ with $n(k) > m(k) > k$ such that,

$$\rho(y_{m(k)}, y_{n(k)}) \geq \varepsilon \quad \text{and} \quad \rho(y_{m(k)}, y_{n(k)-1}) < \varepsilon$$

for all positive number k . Then, we have

$$\begin{aligned} \varepsilon &\leq \rho(y_{m(k)}, y_{n(k)}) \leq s\rho(y_{m(k)}, y_{m(k)-1}) + s\rho(y_{m(k)-1}, y_{n(k)}) \\ &\leq s\rho(y_{m(k)}, y_{m(k)-1}) + s^2\rho(y_{m(k)-1}, y_{n(k)-1}) + s^2\rho(y_{n(k)-1}, y_{n(k)}). \end{aligned}$$

From part (i) it follows that:

$$\begin{aligned} \frac{\varepsilon}{s^2} &\leq \lim_{k \rightarrow +\infty} \rho(y_{m(k)-1}, y_{n(k)-1}) \leq \limsup_{k \rightarrow +\infty} \rho(y_{m(k)-1}, y_{n(k)-1}) \\ &\leq s \limsup_{k \rightarrow +\infty} \rho(y_{m(k)-1}, y_{m(k)}) + s \limsup_{k \rightarrow +\infty} \rho(y_{m(k)}, y_{n(k)-1}) \leq s\varepsilon. \end{aligned}$$

Thus, we have

$$\frac{\varepsilon}{s^2} \leq \liminf_{k \rightarrow +\infty} \rho(y_{m(k)-1}, y_{n(k)-1}) \leq \limsup_{k \rightarrow +\infty} \rho(y_{m(k)-1}, y_{n(k)-1}) \leq s\varepsilon.$$

Putting $x = x_{m(k)}$ and $y = x_{n(k)}$ in (3.1), we obtain

$$\begin{aligned} 0 < \psi(s\varepsilon) &\leq \limsup_{k \rightarrow +\infty} \psi[s\rho(y_{m(k)}, y_{n(k)})] = \limsup_{k \rightarrow +\infty} \psi[s\rho(Sx_{m(k)}, Sy_{n(k)})] \\ &\leq \limsup_{k \rightarrow +\infty} \psi[\rho(y_{m(k)-1}, y_{n(k)-1})] - \liminf_{k \rightarrow +\infty} \varphi(\rho(y_{m(k)-1}, y_{n(k)-1})) \\ &\leq \psi(s\varepsilon) - \varphi(s\varepsilon) < \psi(s\varepsilon), \end{aligned}$$

which is a contradiction. This shows that (y_n) is a b -Cauchy sequence in M . $\square$

The uniqueness of POC's for $(\psi - \varphi)$ -weak contraction type mappings is proved in the next result.

Proposition 3.3 *Let (M, ρ) be a b -metric space with $s \geq 1$ and let S and T be selfmappings of M . Assume that S and T are $(\psi - \varphi)$ -weak contraction type mappings. If S and T have a POC in M , then it is unique.*

Proof Suppose that there are two different POC (z and w) of S and T . That is, there exist $u, v \in M$ such that $Su = Tu = z$ and $Sv = Tv = w$. Then, by inequality (3.1), we have

$$\begin{aligned} \psi(\rho(z, w)) &\leq \psi(s\rho(Su, Sv)) \leq \psi(\rho(Tu, Tv)) - \varphi(\rho(Tu, Tv)) \\ &\leq \psi(\rho(z, w)) - \varphi(\rho(z, w)) < \psi(\rho(z, w)), \end{aligned}$$

which is a contradiction. Therefore, $z = w$. $\square$

Now, we prove the existence of POC's for $(\psi - \varphi)$ -weak contraction type mappings.

Proposition 3.4 *Let (M, ρ) be a b -metric space with $s \geq 1$ and let S and T be selfmappings of M with $SM \subset TM$. Assume that S and T are $(\psi - \varphi)$ -weak contraction type mappings. If $TM \subset M$ is a complete subspace, then for any $x_0 \in M$ the Jungck sequence defined by (3.9) converges to $z \in M$, $C(S, T) \neq \emptyset$ and z is the unique POC of S and T .*

Proof By Proposition 3.2, the sequence (y_n) is a b -Cauchy in M , hence (Tx_{n+1}) is a b -Cauchy sequence in M . Since $TM \subset M$ is complete and $(Tx_{n+1}) \subset TM$, we have

$$\lim_{n \rightarrow +\infty} Tx_{n+1} = z = Tu, \text{ for some } u \in M.$$

We now show that $Su = Tu$. Suppose that $Su \neq Tu$. Using condition (3.1) and Lemma 2.2 we obtain,

$$\begin{aligned} \psi(\rho(Su, Tu)) &= \psi\left(\frac{s\rho(Su, Tu)}{s}\right) \leq \limsup_{n \rightarrow +\infty} \psi(s\rho(Su, Tx_{n+1})) \\ &\leq \limsup_{n \rightarrow +\infty} \psi(s\rho(Su, Sx_n)) \\ &\leq \limsup_{n \rightarrow +\infty} \psi(\rho(Tu, Tx_n)) - \liminf_{n \rightarrow +\infty} \varphi(\rho(Tu, Tx_n)) \\ &\leq \psi(s\rho(Tu, Tu)) - \varphi(s\rho(Tu, Tu)) = 0. \end{aligned}$$

It follows that $\psi(\rho(Su, Tu)) = 0$, hence $\rho(Su, Tu) = 0$, that is $Su = Tu$. Therefore, $C(S, T) \neq \emptyset$.

Now, since $Su = Tu = z$ we have that z is a POC of S and T . From Proposition 3.3 it is unique. $\square$

4 Common fixed points for $(\psi - \varphi)$ -weak contraction type mappings

In this section, we present the results concerning to the existence of common fixed points for $(\psi - \varphi)$ -weak contraction type mappings satisfying different non-commutative properties.

We recall that in 2006, Jungck and Rhoades [25] proved that any pair of weak compatible selfmaps having a point $z \in M$ as its unique POC, then this z is its unique common fixed point.

Lemma 4.1 *Let S and T be weak compatible selfmaps of a set $M \neq \emptyset$. If S and T have a unique POC, says, $z = Su = Tu$, then z is the unique common fixed point of S and T .*

On the other hand, weakly compatible is a minimal condition for the existence of common fixed points for contractive type of mappings (for a discussion on this, see, e.g., [34, 35]). In virtue of this, the existence of a unique common fixed point for weakly compatible maps is reduced to the existence of a unique point of coincidence.

Theorem 4.2 *Let (M, ρ) be a b -metric space with $s \geq 1$ and let S and T be weakly compatible $(\psi - \varphi)$ -weak contraction type selfmappings of M . If $SM \subset TM$ and $TM \subset M$ is a complete subspace of M , then S and T have a unique common fixed point.*

Proof We are in the hypotheses of Proposition 3.4, hence we know that (S, T) has a unique POC. Since (S, T) are weakly compatible, the conclusion then is obtained from Lemma 4.1. $\square$

Example 4.1 Let $M = [0, 1]$ and $\rho : M \times M \rightarrow \mathbb{R}_+$ be such that $\rho(x, y) = |x - y|^2$ for any $x, y \in M$. Let $S, T : M \rightarrow M$ be two functions defined by $Sx = \frac{1}{5}x$ and $Tx = \frac{3}{5}x$, $x \in M$. Then,

$$\begin{aligned}\rho(Sx, Sy) &= |Sx - Sy|^2 = \left| \left(\frac{1}{5}(x - y) \right) \right|^2 = \left(\frac{1}{5} \right)^2 |x - y|^2, \\ \rho(Tx, Ty) &= |Tx - Ty|^2 = \left| \frac{3}{5}(x - y) \right|^2 = (3/5)^2 |x - y|^2.\end{aligned}$$

Let $\psi, \varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be defined by $\psi(t) = 5\sqrt{t}$ and $\varphi(t) = \sqrt{t}$, $t \in \mathbb{R}_+$. Then,

$$\psi(\rho(Sx, Sy)) = |x - y|; \quad \psi(\rho(Tx, Ty)) = 3|x - y| \text{ and } \varphi(\rho(Tx, Ty)) = 3/5|x - y|.$$

It is not difficult to verify that

$$\psi(\rho(Sx, Sy)) \leq \psi(\rho(Tx, Ty)) - \varphi(\rho(Tx, Ty)), \quad x, y \in M.$$

Moreover, we have that

$$SM = [0, 1/5] \subset [0, 3/5] = TM$$

and $TM \subset M$ is complete. We have then,

$$C(S, T) = \{x \in M / Sx = Tx\} = \{0\}$$

and $ST0 = TS0 = 0$. Therefore, S and T are weakly compatible.

It is clear that S and T satisfy the conditions of Theorem 4.2, therefore, S and T have a unique common fixed point, $z = 0$.

In the following result using the b -property (EA) we drop the condition $SM \subset TM$.

Theorem 4.3 *Let (M, ρ) be a b -metric space with $s \geq 1$ and let S and T be $(\psi - \varphi)$ -weakly contraction type selfmappings of M satisfying the b -property (EA). If $SM \subset M$ is a closed subspace, then*

- (i) S and T have a unique POC and
- (ii) If S and T are weakly compatible, then S and T have a unique common fixed point in M .

Proof Since S and T satisfy the b -property (EA), there exists a sequence in M such that

$$\lim_{n \rightarrow +\infty} Sx_n = \lim_{n \rightarrow +\infty} Tx_n = z$$

for some $z \in M$. Since $SM \subset M$ is closed, we have $\lim_{n \rightarrow +\infty} Sx_n = z = Sa = \lim_{n \rightarrow +\infty} Tx_n$, for some $a \in M$. As in the proof of Proposition 3.4, we have that $z = Sa = Ta$ and z is the unique POC of S and T . Then, z is the unique common fixed point of S and T in virtue of Lemma 4.1. $\square$

Since two non-compatible selfmappings of a b -metric space (M, ρ) satisfy the b -property (EA), we get the following result.

Corollary 4.4 *Let (M, ρ) be a b -metric space with $s \geq 1$ and let S and T be non-compatible weakly compatible $(\psi - \varphi)$ -weakly contraction type selfmappings of M . If $SM \subset M$ is a closed subspace of M , then S and T have a unique common fixed point.*

Example 4.2 Let $M = \mathbb{R}$ be with $\rho(x, y) = |x - y|^2$, $x, y \in M$. Let S and T be selfmappings of M defined by $Sx = 2x - 1$ and $Tx = x^2$, $x \in M$. Then:

1. $\rho(Sx, Sy) = |Sx - Sy|^2 = |(2x - 1) - (2y - 1)|^2 = 4|x - y|^2$ and $\rho(Tx, Ty) = |Tx - Ty|^2 = |x^2 - y^2|^2$. Let $\psi, \varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be functions defined by $\psi(t) = 1/4\sqrt{t}$ and $\varphi(t) = \sqrt{t}$, $t \in \mathbb{R}_+$. Therefore, $\psi(\rho(Sx, Sy)) = |x - y|$, $\psi(\rho(Tx, Ty)) = \frac{1}{4}|x^2 - y^2|$ and $\varphi(\rho(Tx, Ty)) = |x^2 - y^2|$. Thus,

$$\psi(\rho(Sx, Sy)) \leq \psi(\rho(Tx, Ty)) - \varphi(\rho(Tx, Ty))$$

holds for all $x, y \in M$.

2. $C(S, T) = \{x \in M / Sx = Tx\} = \{1\}$ and $ST1 = 1 = TS1$. Hence S and T are weakly compatible.
3. From Example 2.4(1), we have that S and T satisfying the b -property (EA) and $TM = [0, +\infty) \subset M$ is closed.

Therefore, this example holds all the conditions of Theorem 4.3, hence S and T have a unique common fixed point $z = 1$.

In the next result we use the b -CLR $_T$ -property and we drop the conditions, $SM \subset TM$ and closedness of the range of any mapping.

Theorem 4.5 Let (M, ρ) be a b -metric space with $s \geq 1$ and let S and T be $(\psi - \varphi)$ -weak contraction type selfmappings of M satisfying the b -CLR $_T$ -property. Then,

- (i) S and T have a unique POC and
- (ii) If S and T are weakly compatible, then S and T have a unique common fixed point in M .

Proof Since the pair (S, T) satisfies the b -CLR $_T$ -property, there exists a sequence (x_n) in M such that

$$\lim_{n \rightarrow +\infty} Sx_n = \lim_{n \rightarrow +\infty} Tx_n = Ta \quad \text{for some } a \in M.$$

Hence, there exists $z \in M$ such that $z = Ta$. The rest of proof follows as the proof of Theorem 4.2. $\square$

Theorem 4.6 Theorems 4.2, 4.3 and 4.5 and Corollary 4.4 remains true if the weakly compatible property is replaced by any one (retaining the rest of hypotheses) of the following:

- (i) Compatible mappings.
- (ii) R -weakly commuting property.
- (iii) R -weakly commuting property of type (A_T) .
- (iv) R -weakly commuting property of type (A_S) .
- (v) R -weakly commuting property of type (P) .

Proof Let $x_0 \in M$ be an arbitrary point. Since $SM \subset TM$, we can construct the Jungck sequence defined by

$$y_n = Sx_n = Tx_{n+1}, \quad n = 0, 1, \dots$$

From Proposition 3.2, we conclude that $(y_n) \subset M$ is a b -Cauchy in TM . Since $TM \subset M$ is a complete subspace, there exists $z \in TM$ such that

$$\lim_{n \rightarrow +\infty} y_n = \lim_{n \rightarrow +\infty} Sx_n = \lim_{n \rightarrow +\infty} Tx_{n+1} = z.$$

Hence, there exists $u \in M$ such that $z = Tu$. As in Proposition 3.4, we prove that $z = Su = Tu$. Thus, u is a coincidence point of S and T . Therefore,

- (i) If S and T are compatible mappings, then $STu = TSu$, hence S and T are weakly compatible maps.
- (ii) If the pair (S, T) satisfies the R -communig property, then

$$\rho(STu, TSu) \leq R\rho(Su, Tu) = 0$$

which implies that $STu = TSu$. Thus, the pair (S, T) is weakly compatible.

(iii) If the pair (S, T) satisfies the R -commuting property of type (A_T) , then

$$\rho(TSu, SSu) \leq R\rho(Su, Tu) = 0$$

which implies that $TSu = SSu$. Now,

$$\rho(STu, TSu) \leq s[\rho(STu, SSu) + \rho(SSu, TSu)] = 0.$$

It follows that $STu = TSu$, then S and T are weakly compatible.

(iv) If the pair (S, T) satisfies the R -commuting property of type (A_S) , then:

$$\rho(STu, TTu) \leq R\rho(Su, Tu) = 0$$

so, $STu = TTu$ and

$$\rho(STu, TSu) \leq s[\rho(STu, TTu) + \rho(TTu, TSu)] = 0.$$

It follows that S and T are weakly compatible.

(v) If the pair (S, T) is a R -weakly mapping of type (P) , then

$$\rho(SSu, TTu) \leq R\rho(Su, Tu) = 0$$

which implies that $SSu = TTu$ and

$$\begin{aligned} \rho(STu, TSu) &\leq s[\rho(STu, SSu) + \rho(SSu, TSu)] \\ &\leq s(\rho(SSu, SSu) + \rho(TTu, TTu)) = 0. \end{aligned}$$

Thus, $STu = TSu$. Therefore, S and T are weakly compatible selfmappings. Now, in view of Theorem 4.2, in all cases, S and T have a unique common fixed point. This complete the proof. $\square$

5 Some consequences

The purpose of this section is to present some extensions and generalizations in the setting of b -metric spaces of well-known results given in the usual metric spaces.

Theorem 5.1 Let (M, ρ) be a b -metric space with $s \geq 1$ and let S and T be selfmappings of M satisfying the following inequality for all $x, y \in M$,

$$\psi(s\rho(Sx, Sy)) \leq \alpha\psi(\rho(Tx, Ty)) \quad (5.1)$$

where $\psi \in \Psi$. If $SM \subset TM$ and $TM \subset M$ is complete, then $C(S, T) \neq \emptyset$ and, if in addition, S, T are weakly compatible, then S and T have a unique common fixed point.

Proof Set $\varphi(t) = (1 - \alpha)\psi(t)$ where $\psi \in \Psi$ and $0 \leq \alpha < 1$. It is clear that $\varphi \in \Phi$. Inequality (5.1) implies that

$$\psi(s\rho(Sx, Sy)) \leq \psi(\rho(Tx, Ty)) - \varphi(\rho(Tx, Ty)).$$

The result follows from Theorem 4.2. $\square$

Theorem 5.2 Let (M, ρ) be a b -metric space with $s \geq 1$ and let S and T be selfmappings of M satisfying the following inequality for all $x, y \in M$,

$$s\rho(Sx, Sy) \leq \rho(Tx, Ty) - \varphi(\rho(Tx, Ty)), \quad (5.2)$$

where $\varphi \in \Phi$. If $SM \subset TM$ and $TM \subset M$ is complete, then $C(S, T) \neq \emptyset$ and also if S and T are weakly compatible, then S and T have a unique common fixed point.

Proof If we take $\psi = \text{Id}$, inequality (5.2) implies that

$$\psi(s\rho(Sx, Sy)) \leq \psi(\rho(Tx, Ty)) - \varphi(\rho(Tx, Ty)).$$

Therefore, the result follows from Theorem 4.2. $\square$

Corollary 5.3 *Let (M, ρ) be a b -metric space with $s \geq 1$ and let S and T be selfmappings of M with $SM \subset M$ and $TM \subset M$ complete. If the pair (S, T) satisfies any one of the following inequalities:*

(i)

$$s\rho(Sx, Sy) \leq w(\rho(Tx, Ty)), \quad (5.3)$$

where $w : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is upper semicontinuous from right, $w(0) = 0$, $w(t) > 0$, $t \in \mathbb{R}_+$.

(ii)

$$s\rho(Sx, Sy) \leq \alpha(\rho(Tx, Ty))\rho(Tx, Ty), \quad (5.4)$$

where $\alpha : \mathbb{R}_+ \rightarrow (0, 1)$ upper semicontinuous from right.

(iii)

$$s\rho(Sx, Sy) \leq \alpha\rho(Tx, Ty), \quad (5.5)$$

where $0 \leq \alpha < 1$, is a constant.

Then, $C(S, T) \neq \emptyset$ and if S and T are weakly compatible, then S and T have a unique common fixed point.

Proof (i) Set $\varphi(t) = t - w(t)$, then inequality (5.3) implies that

$$s\rho(Sx, Sy) \leq \rho(Tx, Ty) - \varphi(\rho(Tx, Ty)),$$

where $\varphi \in \Phi$.

(ii) Set $\varphi(t) = (1 - \alpha(t))t$, then inequality (5.4) implies that

$$s\rho(Sx, Sy) \leq \rho(Tx, Ty) - \varphi(\rho(Tx, Ty)),$$

where $\varphi \in \Phi$.

(iii) Set $\varphi(t) = (1 - \alpha)t$, then inequality (5.5) implies that

$$s\rho(Sx, Sy) \leq \rho(Tx, Ty) - \varphi(\rho(Tx, Ty)),$$

where $\varphi \in \Phi$.

In all cases the conditions of Theorem 5.2 hold, therefore, we obtain the result. $\square$

6 On the convergence and stability of the Jungck-Noor iteration scheme for $(\psi - \varphi)$ -weak contraction type mappings

From now on, as in [39], it is assumed that $(M, \rho, \mathcal{W})$ is a convex b -metric space with parameter $s \geq 1$. That is, there exists a mapping $\mathcal{W} : M \times M \times [0, 1] \rightarrow M$ such that for each $(x, y, \lambda) \in M \times M \times [0, 1]$ and $z \in M$,

$$\rho(z, \mathcal{W}(x, y, \lambda)) \leq \lambda\rho(z, x) + (1 - \lambda)\rho(z, y).$$

Also we are going to assume that $S, T : N \rightarrow M$ are two nonself mappings on a subset N of M such that $S(N) \subset T(N)$, where $T(N)$ is a complete subspace of M . The Jungck-Noor iteration scheme on $(M, \rho, \mathcal{W})$ is defined by similarity as in (1.3).

6.1 Convergence results

To prove the convergence of the Jungck-Noor iteration process we will impose some extra conditions to the function $\psi \in \Psi$. We recall that in 2007, Varošanec [48] introduce the following class of functions which generalize the classical convex functions, s -convex function and other convex-type functions.

Definition 6.1 [48] Let $h : J \supset (0, 1) \rightarrow \mathbb{R}$ be a non-negative function, $h \not\equiv 0$. The function $f : (a, b) \rightarrow \mathbb{R}$ is called an h -convex function, if f is non-negative and for all $x, y \in (a, b)$, $\alpha \in (0, 1)$ we have

$$f(\alpha x + (1 - \alpha)y) \leq h(\alpha)f(x) + h(1 - \alpha)f(y).$$

Theorem 6.1 Let $(M, \rho, \mathcal{W})$ be a convex b -metric space with parameter $s \geq 1$ and let $S, T : N \rightarrow M$ be $(\psi - \varphi)$ -weak contraction type nonself mappings on a subset N of M such that $S(N) \subset T(N)$, where $T(N)$ is a complete subspace of M . Let us assume that $\psi \in \Psi$ is a h -convex function satisfying $h(t) \leq Kt$, for all $t \in (0, 1)$, with $K > 0$. Let p be a POC point of S and T (that is, $p = Sw = Tw$, $w \in N$). For $x_0 \in N$, let (Tx_n) be the Jungck-Noor iteration process defined by (1.3). If any of the following conditions hold:

(i) $(\alpha_n), (\beta_n) \subset [0, 1/2]$ and

$$\sum_{j=0}^{+\infty} [K(1 + K)\beta_j + \alpha_j - (K + 1)^2 + 2] = +\infty,$$

or

(ii) $(\alpha_n), (\beta_n) \subset [1/2, 1]$ and

$$\sum_{j=0}^{+\infty} [-(1 + K)K\beta_j - \alpha_j - K + 2] = +\infty,$$

then, (Tx_n) converges to p .

Proof Let $x_0 \in N$ and define the sequence (Tx_n) by formula (1.3). Let $p = Sw = Tw$, then

$$\begin{aligned} \psi(s\rho(Tx_{n+1}, p)) &= \psi(s\rho(\mathcal{W}(Tx_n, Sz_n, \alpha_n), p)) \\ &\leq \psi(s[\alpha_n\rho(Tx_n, p) + (1 - \alpha_n)\rho(Sz_n, p)]) \\ &\leq h(\alpha_n)\psi(s\rho(Tx_n, p)) + h(1 - \alpha_n)\psi(s\rho(Sz_n, p)) \\ &\leq h(\alpha_n)\psi(s\rho(Tx_n, p)) + h(1 - \alpha_n)[\psi(\rho(Tz_n, p)) - \varphi(\rho(Tz_n, p))] \\ &\leq h(\alpha_n)\psi(s\rho(Tx_n, p)) + h(1 - \alpha_n)\psi(\rho(Tz_n, p)) \\ &\leq h(\alpha_n)\psi(s\rho(Tx_n, p)) + h(1 - \alpha_n)\psi(s\rho(Tz_n, p)). \end{aligned}$$

Now,

$$\begin{aligned} \psi(s\rho(Tz_n, p)) &= \psi(s\rho(\mathcal{W}(Tx_n, Sy_n, \beta_n), p)) \\ &\leq \psi(s[\beta_n\rho(Tx_n, p) + (1 - \beta_n)\rho(Sy_n, p)]) \\ &\leq h(\beta_n)\psi(s\rho(Tx_n, p)) + h(1 - \beta_n)\psi(s\rho(Sy_n, p)) \\ &\leq h(\beta_n)\psi(s\rho(Tx_n, p)) + h(1 - \beta_n)[\psi(\rho(Ty_n, p)) - \varphi(\rho(Ty_n, p))] \\ &\leq h(\beta_n)\psi(s\rho(Tx_n, p)) + h(1 - \beta_n)\psi(\rho(Ty_n, p)) \\ &\leq h(\beta_n)\psi(s\rho(Tx_n, p)) + h(1 - \beta_n)\psi(s\rho(Ty_n, p)). \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} \psi(s\rho(Ty_n, p)) &= \psi(s\rho(\mathcal{W}(Tx_n, Sx_n, \gamma_n), p)) \\ &\leq \psi(s[\gamma_n\rho(Tx_n, p) + (1 - \gamma_n)\rho(Sx_n, p)]) \\ &\leq h(\gamma_n)\psi(s\rho(Tx_n, p)) + h(1 - \gamma_n)\psi(s\rho(Sx_n, p)) \\ &\leq h(\gamma_n)\psi(s\rho(Tx_n, p)) + h(1 - \gamma_n)[\psi(\rho(Tx_n, p)) - \varphi(\rho(Tx_n, p))] \\ &\leq h(\gamma_n)\psi(s\rho(Tx_n, p)) + h(1 - \gamma_n)\psi(\rho(Tx_n, p)) \end{aligned}$$

$$\leq h(\gamma_n)\psi(s\rho(Tx_n, p)) + h(1 - \gamma_n)\psi(s\rho(Tx_n, p)).$$

Thus, assembling these bounds and the fact that $h(t) \leq Kt$, we have,

$$\begin{aligned} \psi(s\rho(Tx_{n+1}, p)) &\leq [h(\alpha_n) + h(1 - \alpha_n)\{h(\beta_n) + h(1 - \beta_n)(h(\gamma_n) + h(1 - \gamma_n))\}] \\ &\quad \times \psi(s\rho(Tx_n, p)) \\ &\leq [K\alpha_n + K(1 - \alpha_n)\{K\beta_n + K(1 - \beta_n)(K\gamma_n + K(1 - \gamma_n))\}] \\ &\quad \times \psi(s\rho(Tx_n, p)) \\ &= [K\alpha_n + K(1 - \alpha_n)\{K\beta_n + K^2(1 - \beta_n)\}] \psi(s\rho(Tx_n, p)). \end{aligned} \quad (6.1)$$

If $(\alpha_n), (\beta_n) \subset [0, 1/2]$, from estimate (6.1), we get

$$\begin{aligned} \psi(s\rho(Tx_{n+1}, p)) &\leq [K(1 - \alpha_n) + K(1 - \alpha_n)\{K(1 - \beta_n) + K^2(1 - \beta_n)\}] \\ &\quad \times \psi(s\rho(Tx_n, p)) \\ &= [1 + (1 + K)(1 - \beta_n)]K(1 - \alpha_n)\psi(s\rho(Tx_n, p)) \\ &\leq e^{K(1+K)(1-\beta_n)}e^{K-1}e^{-\alpha_n}\psi(s\rho(Tx_n, p)) \\ &= e^{-K(1+K)\beta_n-\alpha_n+(K+1)^2-2}\psi(s\rho(Tx_n, p)). \end{aligned} \quad (6.2)$$

Then, it follows recursively that:

$$\psi(s\rho(Tx_{n+1}, p)) \leq e^{-\sum_{j=0}^n [K(1+K)\beta_j + \alpha_j - (K+1)^2 + 2]} \psi(s\rho(Tx_0, p)). \quad (6.3)$$

For $(\alpha_n), (\beta_n) \subset [1/2, 1]$, we obtain, from (6.1), the estimate:

$$\begin{aligned} \psi(s\rho(Tx_{n+1}, p)) &\leq [K\alpha_n + K^2\alpha_n\beta_n + K^3\alpha_n\beta_n] \psi(s\rho(Tx_n, p)) \\ &= [1 + (1 + K)K\beta_n] K\alpha_n \psi(s\rho(Tx_n, p)) \\ &\leq e^{(1+K)K(\beta_n-1)}e^{(1+K)K}e^{K-1}e^{\alpha_n-1}\psi(s\rho(Tx_n, p)) \\ &= e^{-(1+K)K(1-\beta_n)-(1-\alpha_n)+(K+1)^2-2}\psi(s\rho(Tx_n, p)) \\ &= e^{(1+K)K\beta_n+\alpha_n+K-2}\psi(s\rho(Tx_n, p)) \\ &\quad \vdots \quad (\text{recursively}) \\ &\leq e^{-\sum_{j=0}^n [-(1+K)K\beta_j - \alpha_j - K + 2]} \psi(s\rho(Tx_0, p)). \end{aligned} \quad (6.5)$$

Taking limits as $n \rightarrow +\infty$ in (6.3) and (6.5), we conclude from hypotheses (i) and (ii), that in both cases $\lim_{n \rightarrow +\infty} \psi(s\rho(Tx_{n+1}, p)) = 0$, which implies that the Jungck-Noor sequence (Tx_n) converges to p . This finishes the proof. $\square$

In the next result we prove that the Jungck-Noor iteration process converges to the POC of S and T for any h -convex $\psi \in \Psi$ satisfying $h(t) \leq Kt$, $t \in (0, 1)$, with $0 < K < \sqrt{2} - 1$ and $(\alpha_n), (\beta_n) \subset [0, 1/2]$ or $(\alpha_n), (\beta_n) \subset [1/2, 1]$.

Proposition 6.2 *Let $(M, \rho, \mathcal{W})$ be a convex b -metric space with parameter $s \geq 1$ and let $S, T : N \rightarrow M$ be $(\psi - \varphi)$ -weak contraction type nonself mappings on a subset N of M such that $S(N) \subset T(N)$, where $T(N)$ is a complete subspace of M . Let us assume that $\psi \in \Psi$ is a h -convex function satisfying $h(t) \leq Kt$, for all $t \in (0, 1)$ with $0 < K < \sqrt{2} - 1$. Let p be a POC point of S and T and for $x_0 \in N$, let (Tx_n) be the Jungck-Noor iteration process (1.3). Then, (Tx_n) converges to p provided $(\alpha_n), (\beta_n) \subset [0, 1/2]$ or $(\alpha_n), (\beta_n) \subset [1/2, 1]$.*

Proof Notice, from estimate (6.2) in the proof of Theorem 6.1, that for $(\alpha_n), (\beta_n) \subset [0, 1/2]$ we have:

$$\begin{aligned} \psi(s\rho(Tx_{n+1}, p)) &\leq [1 + (1 + K)(1 - \beta_n)]K(1 - \alpha_n)\psi(s\rho(Tx_n, p)) \\ &\leq ((K + 1)^2 - 1)\psi(s\rho(Tx_n, p)) \\ &\quad \vdots \quad (\text{recursively}) \end{aligned}$$

$$\leq \kappa^n \psi(s\rho(Tx_0, p)), \quad \kappa := (K + 1)^2 - 1.$$

Notice that $0 < \kappa < 1$ for $0 < K < \sqrt{2} - 1$, then $\lim_{n \rightarrow +\infty} \psi(s\rho(Tx_{n+1}, p)) = 0$. Therefore, the Jungck-Noor iteration process converges to p in this case. The same conclusion is obtained for $(\alpha_n), (\beta_n) \subset [1/2, 1]$, by using estimate (6.4). $\square$

On the other hand, if in Definition 6.1, we consider the function $h(t) = t^s$, $s \in (0, 1]$, then the function f is called a s -convex function. Notice that in this situation, the function h does not satisfies the condition $h(t) \leq Kt$, since t^{s-1} is not bounded for t close to 0. In the next result we prove the convergence of the Jungck-Noor iteration for $\psi \in \Psi$ being s -convex.

Theorem 6.3 *Let $(M, \rho, \mathcal{W})$ be a convex b -metric space with parameter $s \geq 1$ and let $S, T : N \rightarrow M$ be $(\psi - \varphi)$ -weak contraction nonself mappings on a subset N of M such that $S(N) \subset T(N)$, where $T(N)$ is a complete subspace of M . Let us assume that $\psi \in \Psi$ is a s -convex function. Let p be a POC point of S and T (that is, $p = Sw = Tw$, $w \in N$). For $x_0 \in N$, let (Tx_n) be the Jungck-Noor iteration process defined by (1.3). If any of the following conditions hold:*

(i) $(\alpha_n), (\beta_n), (\gamma_n) \subset [0, 1/2]$ and

$$\sum_{j=0}^{+\infty} [s\alpha_j - (1 + 2(1 - \gamma_j)^s)(1 - \beta_j)^s] = +\infty,$$

or

(ii) $(\alpha_n), (\beta_n), (\gamma_n) \subset [1/2, 1]$ and

$$\sum_{j=0}^{+\infty} [1 - \alpha_j^s - \beta_j^s (1 + 2\gamma_j^s)] = +\infty,$$

then, (Tx_n) converges to p .

Proof From the proof of Theorem 6.1, we have the estimate

$$\begin{aligned} \psi(s\rho(Tx_{n+1}, p)) &\leq [h(\alpha_n) + h(1 - \alpha_n) \{h(\beta_n) + h(1 - \beta_n) (h(\gamma_n) + h(1 - \gamma_n))\}] \\ &\quad \times \psi(s\rho(Tx_n, p)). \end{aligned}$$

Replacing $h(t) = t^s$, we get

$$\begin{aligned} \psi(s\rho(Tx_{n+1}, p)) &\leq [(\alpha_n)^s + (1 - \alpha_n)^s \{(\beta_n)^s + (1 - \beta_n)^s ((\gamma_n)^s + (1 - \gamma_n)^s)\}] \\ &\quad \times \psi(s\rho(Tx_n, p)). \end{aligned}$$

Since the function $t \mapsto t^s$, $(s \in (0, 1])$ is increasing, we have: for $(\alpha_n), (\beta_n), (\gamma_n) \subset [0, 1/2]$

$$\begin{aligned} \psi(s\rho(Tx_{n+1}, p)) &\leq [(1 - \alpha_n)^s + (1 - \alpha_n)^s \{(1 - \beta_n)^s + (1 - \beta_n)^s (2(1 - \gamma_n)^s)\}] \\ &\quad \times \psi(s\rho(Tx_n, p)) \\ &= [1 + (1 + 2(1 - \gamma_n)^2)(1 - \beta_n)^2] (1 - \alpha_n)^s \psi(s\rho(Tx_n, p)) \\ &\leq e^{(1+2(1-\gamma_n)^s)(1-\beta_n)^s} e^{-s\alpha_n} \psi(s\rho(Tx_n, p)) \\ &\quad \vdots \quad (\text{recursively}) \\ &\leq e^{-\sum_{j=0}^n [s\alpha_j - (1+2(1-\gamma_j)^s)(1-\beta_j)^s]} \psi(s\rho(Tx_0, p)). \end{aligned} \tag{6.6}$$

Now, for $(\alpha_n), (\beta_n), (\gamma_n) \subset [1/2, 1]$ we get

$$\begin{aligned} \psi(s\rho(Tx_{n+1}, p)) &\leq [\alpha_n^s + \alpha_n^s \{\beta_n^s + \beta_n^s (2\gamma_n^s)\}] \psi(s\rho(Tx_n, p)) \\ &\leq \alpha_n^s e^{\beta_n^2(1+2\gamma_n^s)} \psi(s\rho(Tx_n, p)) \\ &\leq e^{\alpha_n^s-1} e^{\beta_n^2(1+2\gamma_n^s)} \psi(s\rho(Tx_n, p)) \end{aligned}$$

$$\begin{aligned} & \vdots \quad (\text{recursively}) \\ & \leq e^{-\sum_{j=0}^n [1 - \alpha_j^s - \beta_j^s(1 + 2\gamma_j^s)]} \psi(s\rho(Tx_0, p)). \end{aligned} \quad (6.7)$$

Taking limits as $n \rightarrow +\infty$ in (6.6) and (6.7), from the hypotheses, we conclude in both cases that $\lim_{n \rightarrow +\infty} \psi(s\rho(Tx_{n+1}, p)) = 0$. That is, the Jungck-Noor iteration converges to p . $\square$

6.2 Stability result

An iterative process is said numerically stable if and only if any sequence (y_n) approximately close to a convergent sequence (x_n) generated by the iteration converges to the same limit. The notion of stability of iterative procedures for a pair of self-maps of a metric space (M, d) was introduced in 2005 by Singh et al. [43].

Definition 6.2 Let $(M, \rho, \mathcal{W})$ be a convex b -metric space, let N be a subset of M , and let $S, T : N \rightarrow M$ be such that $S(N) \subset T(N)$. For any $x_0 \in N$, let the sequence (Tx_n) generated by the iterative procedure

$$Tx_{n+1} = f(Tx_n, Sx_n, \alpha_n), \quad 0 \leq \alpha_n \leq 1, \quad n = 0, 1, 2, \dots \quad (6.8)$$

converging to p . Also, let $(Ty_n) \subset M$ be an arbitrary sequence and let

$$\varepsilon_n = \rho(Ty_{n+1}, f(Ty_n, Sy_n, \alpha_n)).$$

The iterative process (6.8) will be called (S, T) -stable if $\lim_{n \rightarrow +\infty} \varepsilon_n = 0$ implies that $\lim_{n \rightarrow +\infty} Ty_n = p$.

The following fact concerning to recurrent inequalities will be useful to prove our (S, T) -stability result for the Jungck-Noor iterative scheme applied to $(\psi - \varphi)$ -weak contraction mappings. (See e.g. [10, Lemma 1.6].)

Lemma 6.4 Let $(a_n), (b_n)$ be sequences of nonnegative numbers and $0 \leq q < 1$, so that

$$a_{n+1} \leq qa_n + b_n, \quad \text{for all } n \geq 0.$$

If $\lim_{n \rightarrow +\infty} b_n = 0$, then $\lim_{n \rightarrow +\infty} a_n = 0$.

In the next result we will assume that the function h given in Definition 6.1 is bounded from above:

$$h(t) \leq M, \quad \text{for all } t \in (0, 1) \quad (6.9)$$

where $M > 0$ satisfies that

$$p_s(M) = 2M^3 + M^2 + \lfloor s \rfloor M - 1 < 0, \quad s \geq 1,$$

Here, $\lfloor s \rfloor$ denotes the smallest integer greater or equal to s . Notice that such a positive number M exists since the polynomial function $p_s(t) = 2t^3 + t^2 + \lfloor s \rfloor t - 1$ satisfies that $p_s(0) = -1$ and it is strictly increasing on $\mathbb{R}_+$ because $p'_s(t) > 0$ (due to the fact that $\lfloor s \rfloor \geq 1$). Also, p_s changes sign in $\mathbb{R}_+$ (take, for instance, $t = 1$). That means, there exists $t^* \in \mathbb{R}_+$ such that $p_s(t) < 0$, $t \in [0, t^*)$ and $p_s(t) \geq 0$, $t \in [t^*, +\infty)$.

Theorem 6.5 Let $(M, \rho, \mathcal{W})$ be a convex b -metric space with parameter $s \geq 1$, let S and T be $(\psi - \varphi)$ -weak contraction type selfmappings of M with $\psi \in \Psi$ a subadditive h -convex function such that h satisfies (6.9). Let p be a POC point of S and T (that is, $p = Sw = Tw$, $w \in M$). Let $x_0 \in M$ and suppose that the sequence (Tx_n) generated by Jungck-Noor iteration process (1.3) converges to p . Then the Jungck-Noor iteration process is (S, T) -stable.

Proof Let $p = Sw = Tw$ and let (Ty_n) be an arbitrary sequence. Define

$$\varepsilon_n = \rho(Ty_{n+1}, \mathcal{W}(Ty_n, Sz_n, \alpha_n)), \quad n = 0, 1, \dots,$$

where $Tz_n = \mathcal{W}(Ty_n, Sw_n, \beta_n)$ and $Tw_n = \mathcal{W}(Ty_n, Sy_n, \gamma_n)$. Now,

$$\begin{aligned} \rho(Ty_{n+1}, p) & \leq s\rho(Ty_{n+1}, \mathcal{W}(Ty_n, Sz_n, \alpha_n)) + s\rho(\mathcal{W}(Ty_n, Sz_n, \alpha_n), p) \\ & \leq s\varepsilon_n + \alpha_n s\rho(Ty_n, p) + (1 - \alpha_n)s\rho(Sz_n, p) \end{aligned}$$

$$= s\varepsilon_n + \alpha_n s\rho(Ty_n, p) + (1 - \alpha_n)s\rho(Sz_n, Sw).$$

Since $\psi \in \Psi$ is h -convex and subadditive, we have

$$\begin{aligned}\psi(\rho(Ty_{n+1}, p)) &\leq \psi(s\varepsilon_n) + h(\alpha_n)\psi(s\rho(Ty_n, p)) + h(1 - \alpha_n)\psi(s\rho(Sz_n, Sw)) \\ &\leq \psi(s\varepsilon_n) + h(\alpha_n)\lfloor s \rfloor \psi(\rho(Ty_n, p)) + h(1 - \alpha_n)\psi(s\rho(Sz_n, Sw)).\end{aligned}$$

In the last inequality we use the fact that a subadditive function ψ satisfies that $\psi(st) \leq \lfloor s \rfloor \psi(t)$, where, for $s \in \mathbb{R}$, $\lfloor s \rfloor$ denotes the smallest integer greater or equal to s .

Now, notice that

$$\psi(s\rho(Sz_n, Sw)) \leq \psi(\rho(Tz_n, p)) - \varphi(\rho(Tz_n, p)) < \psi(\rho(Tz_n, p)).$$

Therefore, we get

$$\psi(\rho(Ty_{n+1}, p)) < \psi(s\varepsilon_n) + h(\alpha_n)\lfloor s \rfloor \psi(\rho(Ty_n, p)) + h(1 - \alpha_n)\psi(\rho(Tz_n, p)).$$

On the other hand,

$$\begin{aligned}\psi(\rho(Tz_n, p)) &= \psi(\rho(\mathcal{W}(Ty_n, Sw_n, \beta_n), p)) \\ &\leq \psi(\beta_n \rho(Ty_n, p) + (1 - \beta_n)\rho(Sw_n, p)) \\ &\leq h(\beta_n)\psi(\rho(Ty_n, p)) + h(1 - \beta_n)\psi(\rho(Sw_n, Sw)) \\ &\leq h(\beta_n)\psi(\rho(Ty_n, p)) + h(1 - \beta_n)\psi(s\rho(Sw_n, Sw))\end{aligned}$$

where, as before,

$$\psi(s\rho(Sw_n, Sw)) \leq \psi(\rho(Tw_n, Tw)) - \varphi(\rho(Tw_n, Tw)) < \psi(\rho(Tw_n, p)).$$

Finally,

$$\begin{aligned}\psi(\rho(Tw_n, p)) &= \psi(\rho(\mathcal{W}(Ty_n, Sy_n, \gamma_n), p)) \\ &\leq \psi(\gamma_n \rho(Ty_n, p) + (1 - \gamma_n)\rho(Sy_n, p)) \\ &\leq h(\gamma_n)\psi(\rho(Ty_n, p)) + h(1 - \gamma_n)\psi(\rho(Sy_n, Sw)) \\ &\leq h(\gamma_n)\psi(\rho(Ty_n, p)) + h(1 - \gamma_n)\psi(s\rho(Sy_n, Sw))\end{aligned}$$

with,

$$\psi(s\rho(Sy_n, Sw)) \leq \psi(\rho(Ty_n, p)) - \varphi(\rho(Ty_n, p)) < \psi(\rho(Ty_n, p)).$$

These upper bounds allow us to conclude that

$$\begin{aligned}\psi(\rho(Ty_{n+1}, p)) &\leq \psi(s\varepsilon_n) + h(\alpha_n)\lfloor s \rfloor \psi(\rho(Ty_n, p)) + h(1 - \alpha_n) \\ &\quad \times [h(\beta_n)\psi(\rho(Ty_n, p)) + h(1 - \beta_n)\{h(\gamma_n)\psi(\rho(Ty_n, p)) \\ &\quad + h(1 - \gamma_n)\psi(\rho(Ty_n, p))\}] \\ &= \psi(s\varepsilon_n) + [h(\alpha_n)\lfloor s \rfloor + h(1 - \alpha_n)\{h(\beta_n) + h(1 - \beta_n)(h(\gamma_n) \\ &\quad + h(1 - \gamma_n))\}] \psi(\rho(Ty_n, p)) \\ &\leq \psi(s\varepsilon_n) + [M\lfloor s \rfloor + M\{M + M(M + M)\}] \psi(\rho(Ty_n, p)) \\ &= \psi(s\varepsilon_n) + [2M^3 + M^2 + \lfloor s \rfloor M] \psi(\rho(Ty_n, p)),\end{aligned}$$

where $q := 2M^3 + M^2 + \lfloor s \rfloor M < 1$. Now, from Lemma 6.4 we conclude that $\lim_{n \rightarrow +\infty} \psi(s\varepsilon_n) = 0$ implies $\lim_{n \rightarrow +\infty} \psi(\rho(Ty_n, p)) = 0$. Thus, the conclusion follows from the fact that $\psi \in \Psi$. $\square$

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