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Stability result of a suspension bridge Problem with time-varying delay and time-varying weight

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Abstract In this paper, we study a plate equation as a model for a suspension bridge with time-varying delay and time-varying weights. Under some conditions on the delay and weight functions, we establish a stability result for the associated energy functional. The present work extends and generalizes some similar results in the case of wave or plate equations.

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1 Introduction

In this paper, we consider $\Omega = (0, \pi) \times (-\ell, \ell) \subset \mathbb{R}^2$ to be a thin rectangular plate with suspension hangers and suppose the plate is partially hinged on the vertical edges and free on the horizontal edges, see [7–9] for details on suspension bridge models through partially hinged plates. We are interested in studying the vertical displacement of the thin plate $u = u(x, y, t)$ in the presence of time-varying delay and time-varying weights, namely

$$u_{tt}(x, y, t) + \Delta^2 u(x, y, t) + h(u(x, y, t)) + \delta_1(t)u_t(x, y, t) + \delta_2(t)u_t(x, y, t - \tau(t)) = 0, \quad (1.1)$$

where $(x, t) \in \Omega \times (0, T)$ and subjected to the partially hinged boundary conditions

$$\begin{cases} u(0, y, t) = u_{xx}(0, y, t) = 0, & \text{for } (y, t) \in (-\ell, \ell) \times (0, T), \\ u(\pi, y, t) = u_{xx}(\pi, y, t) = 0, & \text{for } (y, t) \in (-\ell, \ell) \times (0, T), \\ u_{yy}(x, \pm\ell, t) + \sigma u_{xx}(x, \pm\ell, t) = 0, & \text{for } (x, t) \in (0, \pi) \times (0, T), \\ u_{yyy}(x, \pm\ell, t) + (2 - \sigma)u_{xxy}(x, \pm\ell, t) = 0, & \text{for } (x, t) \in (0, \pi) \times (0, T), \end{cases} \quad (1.2)$$

and initial data

$$\begin{cases} u(x, y, 0) = u_0(x, y), \quad u_t(x, y, 0) = u_1(x, y), & \text{in } \Omega, \\ u_t(x, y, t - \tau(0)) = g_0(x, y, t - \tau(0)), & \text{in } \Omega \times (0, \tau(0)). \end{cases} \quad (1.3)$$

Here, $0 < \sigma < \frac{1}{2}$ is the Poisson ratio, h is a non-linear function which represents the hangers restoring force, $\tau(t) > 0$ is the time-varying delay, $\delta_1(t)$ and $\delta_2(t)$ are nonlinear damping weights functions. Damping and internal feedback are important properties of great interest when modelling systems of natural phenomena in areas such as, physics, thermal science, biosciences and structural engineering. To have a better understanding of these phenomena, the present state and past history of occurrences are of great importance. A substantial number of articles have investigated wave or plate equations in the presence of constant time delay ($\tau(t) \equiv \tau$) and constant weights ($\delta_1(t) \equiv \delta_1$, $\delta_2(t) \equiv \delta_2$) and proved exponential or polynomial decay results for the

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associated energy functional, see [1, 13, 14, 22] and the references cited in them. When $\delta_2 \equiv 0$ and $\delta_1(t) \equiv \delta > 0$ in (1.1), Messaoudi and Mukiawa [17] showed that the associated energy functional decays exponentially. See similar results for wave equations in [4, 10, 11] and the references therein. If $\delta_2(t) = \delta > 0$ and $\delta_1(t) \equiv 0$ in (1.1), then the system becomes unstable, see for instance the results in [5, 18, 19] and the references cited. In the case of constant weights (δ_1 and δ_2 constants) and time-varying delay equations, not so much has been done. We mention among others, the works of Nicaise and Pignotti [20, 21], where they considered a wave equation with boundary or internal time-varying delay and established an exponential stability result. Recently, Enyi and Mukiawa [6] considered a viscoelastic plate equation when δ_1 and δ_2 are constants and time-varying delay and proved a general decay result provided $|\delta_2| < |\delta_1|\sqrt{1-d}$. For existence and stability results for the wave equation with time-varying weights and constant time feedback, see [2, 3, 12] and references therein.

Motivated by the result in [3], where an energy decay estimate is established for a wave equation with constant feedback ($\tau \equiv \text{constant}$) and $h \equiv 0$, we establish a stability result for Problem (1.1)–(1.3). The present paper improved and extended the results in [6, 17] for plate equations and the results in [3, 18] for wave equations. This paper is organized as follows: in Sect. 2, we recall some fundamental materials for the clarity of the reader. In Sect. 3, we state and prove our main stability result for Problem (1.1)–(1.3).

2 Functional setting and hypotheses

Here, we assume the hangers restoring force h satisfies the following, as given in [17]:

(A1) $h : \mathbb{R} \rightarrow \mathbb{R}$ is a locally Lipschitz non-decreasing function such that $h(0) = 0$ and the positive function

$$H \text{ defined by } H(s) = \int_0^s h(\tau) d\tau \text{ satisfying}$$

$$sh(s) - H(s) \geq 0, \quad \forall s \in \mathbb{R}.$$

Using the ideas of [3], we assume the time varying weights $\delta_1(t)$ and $\delta_2(t)$ satisfy

(A2) $\delta_1 : [0, +\infty) \rightarrow (0, +\infty)$ is a non-increasing C^1 -function and there exists a positive constant B_1 such that

$$\left| \frac{\delta_1'(t)}{\delta_1(t)} \right| \leq B_1, \quad \forall t \geq 0. \quad (2.1)$$

$\delta_2 : [0, +\infty) \rightarrow \mathbb{R}$ is a C^1 -function (which is not necessarily monotone or positive) and there exist some positive constants $0 < \alpha < \sqrt{1-d}$, B_2 such that

$$|\delta_2(t)| \leq \alpha \delta_1(t), \quad (2.2)$$

and

$$|\delta_2'(t)| \leq B_2 \delta_1(t). \quad (2.3)$$

As in [19], we assume the time varying internal feedback $\tau(t)$ satisfies

(A3) There exist $\tau_0, \tau_1 > 0$ such that

$$0 < \tau_0 \leq \tau(t) \leq \tau_1, \quad \forall t > 0, \quad (2.4)$$

$$\tau \in W^{2,\infty}(0, T), \quad \forall T > 0, \quad (2.5)$$

$$\tau'(t) \leq d < 1, \quad \forall t > 0. \quad (2.6)$$

As in [7], we introduce the Hilbert space

$$H_*^2(\Omega) = \{w \in H^2(\Omega) : w(0, y) = w(\pi, y) = 0, \quad \forall y \in (-\ell, \ell)\}$$

endowed with the inner product

$$(v, w)_{H_*^2(\Omega)} = \int_{\Omega} [(\Delta v \Delta w + (1 - \sigma)(2v_{xy}w_{xy} - v_{xx}w_{yy} - v_{yy}w_{xx}))] dx dy$$

and denote by $\mathcal{H}(\Omega)$ the dual of $H_*^2(\Omega)$. We now state some needed Lemmas:

Lemma 2.1 [7] Suppose $1 < p < +\infty$. Then, for any $w \in H_*^2(\Omega)$, there exists an embedding constant $C_e = C_e(\Omega, p) > 0$ such that

$$\|w\|_{L^p(\Omega)} \leq C_e \|w\|_{H_*^2(\Omega)}.$$

Lemma 2.2 [16] Let $u \in H^4(\Omega) \cap H_*^2(\Omega)$ and $v \in H_*^2(\Omega)$. Then,

$$(\Delta^2 u, v)_{L^2(\Omega)} = (u, v)_{H_*^2(\Omega)}. \quad (2.7)$$

Lemma 2.3 [15] Let $E : [0, +\infty) \rightarrow [0, +\infty)$ be a non-increasing function and $\psi : [0, +\infty) \rightarrow [0, +\infty)$ an increasing C^1 -function satisfying

$$\psi(0) = 0 \quad \text{and} \quad \lim_{t \rightarrow +\infty} \psi(t) = +\infty. \quad (2.8)$$

Suppose there exist $\beta > -1$ and $\gamma > 0$ so that

$$\int_s^{+\infty} E^{1+\beta}(t) \psi'(t) dt \leq \frac{1}{\gamma} E^\beta(s), \quad 0 \leq s < +\infty. \quad (2.9)$$

Then

$$\begin{cases} E(t) = 0, \quad \forall t \geq \frac{E^\beta(0)}{\gamma|\beta|}, \quad \text{if } -1 < \beta < 0 \\ E(t) \leq E(0) \left(\frac{1+\beta}{1+\gamma\beta\psi(t)} \right)^{\frac{1}{\beta}}, \quad \forall t \geq 0, \quad \text{if } \beta > 0 \\ E(t) \leq E(0)e^{1-\gamma\psi(t)}, \quad \forall t \geq 0, \quad \text{if } \beta = 0. \end{cases} \quad (2.10)$$

Similarly, as in Nicaise and Pignotti [18], we introduce the following change of variable:

$$z(x, y, \rho, t) = u_t(x, y, t - \tau(t)\rho), \quad \text{for } (x, y, \rho, t) \in \Omega \times (0, 1) \times (0, T). \quad (2.11)$$

Then simple calculations gives

$$\tau(t)z_t(x, y, \rho, t) + (1 - \tau'(t)\rho)z_\rho(x, y, \rho, t) = 0, \quad \text{for } (x, y, \rho, t) \in \Omega \times (0, 1) \times (0, T). \quad (2.12)$$

Therefore, Problem (1.1)–(1.3) becomes

$$\begin{cases} u_{tt} + \Delta^2 u + h(u) + \delta_1(t)u_t + \delta_2(t)z(x, y, 1, t) = 0, & \text{in } \Omega \times (0, T). \\ \tau(t)z_t(x, y, \rho, t) + (1 - \tau'(t)\rho)z_\rho(x, y, \sigma, t) = 0, & \text{in } \Omega \times (0, 1) \times (0, T) \end{cases} \quad (2.13)$$

with boundary conditions

$$\begin{cases} u(0, y, t) = u_{xx}(0, y, t) = 0, & \text{for } (y, t) \in (-\ell, \ell) \times (0, T), \\ u(\pi, y, t) = u_{xx}(\pi, y, t) = 0, & \text{for } (y, t) \in (-\ell, \ell) \times (0, T), \\ u_{yy}(x, \pm\ell, t) + \sigma u_{xx}(x, \pm\ell, t) = 0, & \text{for } (x, t) \in (0, \pi) \times (0, T), \\ u_{yyy}(x, \pm\ell, t) + (2 - \sigma)u_{xxy}(x, \pm\ell, t) = 0, & \text{for } (x, t) \in (0, \pi) \times (0, T), \\ z(x, y, 0, t) = u_t(x, y, t), & \text{for } (x, y, t) \in \Omega \times (0, T) \end{cases} \quad (2.14)$$

and initial data

$$\begin{cases} u(x, y, 0) = u_0(x, y), \quad u_t(x, y, 0) = u_1(x, y), & \text{in } \Omega, \\ z(x, y, \rho, 0) = u_t(x, y, -\tau(0)\rho) = g_0(x, y, -\tau(0)\rho), & \text{in } \Omega \times (0, 1). \end{cases} \quad (2.15)$$

Let $\bar{\xi}$ be a positive constant satisfying

$$\frac{\alpha}{\sqrt{1-d}} < \bar{\xi} < 2 - \frac{\alpha}{\sqrt{1-d}}. \quad (2.16)$$

The energy function of Problem (2.13)–(2.15) is defined by

$$E(t) = \frac{1}{2} \|u_t\|_{L^2(\Omega)}^2 + \frac{1}{2} \|u\|_{H_*^2(\Omega)}^2 + \int_{\Omega} H(u) dx dy + \frac{\xi(t)\tau(t)}{2} \|z\|_{L^2(\Omega \times (0, 1))}^2, \quad (2.17)$$

where $\xi(t) = \bar{\xi}\delta_1(t)$ is a non-increasing C^1 -function. The main existence and uniqueness result of (2.13)–(2.15) is the following.

Theorem 2.4 Assume (A1)–(A3) hold and let

$$(u_0, u_1, g_0) \in H^4(\Omega) \cap H_*^2(\Omega) \times H_*^2(\Omega) \times H^1(\Omega; H_0^1(0, 1))$$

be given and satisfies the compatibility condition $g_0(\cdot, 0) = u_1$. Then Problem (1.1)–(1.3) has a global unique solution u satisfying

$$\begin{aligned} u &\in L_{\text{loc}}^\infty((-\tau(0), T); H^4(\Omega) \cap H_*^2(\Omega)), \quad u_t \in L_{\text{loc}}^\infty((-\tau(0), T); H_*^2(\Omega)) \\ u_{tt} &\in L_{\text{loc}}^\infty((-\tau(0), T); L^2(\Omega)). \end{aligned} \quad (2.18)$$

Proof The result can be obtain by using the Galerkin approximation method and combining the ideas in ([17], Theorem 3.1) and ([3], Theorem 2.2). $\square$

Lemma 2.5 Assume (A1)–(A3) hold. Then the energy functional defined by (2.17) satisfies along the solution of Problem (2.13)

$$\begin{aligned} \frac{dE}{dt} &\leq -\delta_1(t) \left(1 - \frac{\bar{\xi}}{2} - \frac{\alpha}{2\sqrt{1-d}} \right) \|u_t\|_{L^2(\Omega)}^2 \\ &\quad - \delta_1(t) \left(\frac{\bar{\xi}}{2} - \frac{\bar{\xi}\tau'(t)}{2} - \frac{\alpha\sqrt{1-d}}{2} \right) \|z(\cdot, 1)\|_{L^2(\Omega)}^2 \\ &\leq 0. \end{aligned} \quad (2.19)$$

Proof First, we multiple (2.13)₁ by u_t , integrating over Ω and using condition (A1) and Lemma 2.2, we arrive at

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \|u_t\|_{L^2(\Omega)}^2 + \frac{1}{2} \|u\|_{H_*^2(\Omega)}^2 + \int_{\Omega} H(u) dx dy \right) \\ = -\delta_1(t) \|u_t\|_{L^2(\Omega)}^2 - \delta_2(t) \int_{\Omega} u_t z(x, y, 1, t) dx dy. \end{aligned} \quad (2.20)$$

Next, we multiple (2.13)₂ by $\xi(t)z$, integrate over $\Omega \times (0, 1)$, we get

$$\begin{aligned} \xi(t)\tau(t) \int_{\Omega} \int_0^1 z(x, y, \rho, t) z_t(x, y, \rho, t) d\rho dx dy \\ = -\xi(t) \int_{\Omega} \int_0^1 (1 - \tau'(t)\rho) z(x, y, \rho, t) z_{\rho}(x, y, \rho, t) d\rho dx dy, \end{aligned} \quad (2.21)$$

from which we obtain

$$\begin{aligned} \frac{d}{dt} \left(\frac{\xi(t)\tau(t)}{2} \|z\|_{L^2(\Omega \times (0,1))}^2 \right) \\ = -\frac{\xi(t)}{2} \int_{\Omega} \int_0^1 \frac{\partial}{\partial \rho} [(1 - \tau'(t)\rho) z^2(x, y, \rho, t)] d\rho dx dy \\ + \frac{\xi'(t)\tau(t)}{2} \int_{\Omega} \int_0^1 z^2(x, y, \rho, t) d\rho dx dy \\ = -\frac{\xi(t)(1 - \tau'(t))}{2} \|z(\cdot, 1)\|_{L^2(\Omega)}^2 + \frac{\xi(t)}{2} \|u_t\|_{L^2(\Omega)}^2 \\ + \frac{\xi'(t)\tau(t)}{2} \|z\|_{L^2(\Omega \times (0,1))}^2. \end{aligned} \quad (2.22)$$

Adding (2.20) and (2.22) leads to

$$\begin{aligned} \frac{dE(t)}{dt} = & - \left(\delta_1(t) - \frac{\xi(t)}{2} \right) \|u_t\|_{L^2(\Omega)}^2 - \delta_2(t) \int_{\Omega} u_t z(x, y, 1, t) dx dy \\ & - \frac{\xi(t)(1 - \tau'(t))}{2} \|z(\cdot, 1)\|_{L^2(\Omega)}^2 + \frac{\xi'(t)\tau(t)}{2} \|z\|_{L^2(\Omega \times (0, 1))}^2. \end{aligned}$$

Using Young's inequality, the fact that $\xi(t)$ is non-increasing and recalling (2.16), it follows that

$$\begin{aligned} \frac{dE(t)}{dt} \leq & - \left(\delta_1(t) - \frac{\xi(t)}{2} - \frac{|\delta_2(t)|}{2\sqrt{1-d}} \right) \|u_t\|_{L^2(\Omega)}^2 \\ & - \left(\frac{\xi(t)(1 - \tau'(t))}{2} - \frac{|\delta_2(t)|\sqrt{1-d}}{2} \right) \|z(\cdot, 1)\|_{L^2(\Omega)}^2 \\ \leq & -\delta_1(t) \left(1 - \frac{\bar{\xi}}{2} - \frac{\alpha}{2\sqrt{1-d}} \right) \|u_t\|_{L^2(\Omega)}^2 \\ & - \delta_1(t) \left(\frac{\bar{\xi}(1 - \tau'(t))}{2} - \frac{\alpha\sqrt{1-d}}{2} \right) \|z(\cdot, 1)\|_{L^2(\Omega)}^2 \\ \leq & 0. \end{aligned}$$

□

3 Stability result

The main stability result is the following.

Theorem 3.1 *Under the assumptions of Theorem 2.4, there exist positive constants m, λ such that the energy functional (2.17) satisfies*

$$E(t) \leq mE(0) \exp \left(-\lambda \int_0^t \delta_1(s) ds \right), \quad \forall t \geq 0. \quad (3.1)$$

Proof We start by defining the function ψ as follows:

$$\psi(t) = \int_0^t \delta_1(s) ds. \quad (3.2)$$

It follows from (A2) that ψ is non-decreasing and bounded C^1 -function such that

$$\psi(t) \longrightarrow +\infty \quad \text{as } t \longrightarrow +\infty.$$

Hence, ψ satisfies the conditions of Lemma 2.3. Now, multiplying (2.13)₁ by $E^q(t)\psi'(t)u$ and integrating over $\Omega \times (s, T)$, $T > s$, we obtain

$$\int_s^T \int_{\Omega} E^q(t) \psi'(t) u (u_{tt} + \Delta^2 u + h(u) + \delta_1(t) u_t + \delta_2(t) z(x, y, 1, t)) dx dy dt = 0.$$

Using Lemma 2.2 and some calculations, we obtain

$$\begin{aligned} & \left[E^q(t) \psi'(t) \int_{\Omega} u u_t dx dy \right]_s^T - \int_s^T [q E'(t) E^{q-1}(t) \psi'(t) + E^q(t) \psi''(t)] \int_{\Omega} u u_t dx dy dt \\ & + \int_s^T E^q(t) \psi'(t) \left(\frac{1}{2} \|u_t\|_{L^2(\Omega)}^2 + \frac{1}{2} \|u\|_{H_*^2(\Omega)}^2 + \int_{\Omega} H(u) dx dy \right) dt \\ & - \frac{3}{2} \int_s^T E^q(t) \psi'(t) \|u_t\|_{L^2(\Omega)}^2 dt + \frac{1}{2} \int_s^T E^q(t) \psi'(t) \|u\|_{H_*^2(\Omega)}^2 dt \end{aligned}$$

$$\begin{aligned}
& + \int_s^T E^q(t) \psi'(t) \int_{\Omega} (uh(u) - H(u)) \, dx \, dy \, dt + \int_s^T E^q(t) \psi'(t) \delta_1(t) \int_{\Omega} uu_t \, dx \, dy \, dt \\
& + \int_s^T E^q(t) \psi'(t) \delta_2(t) \int_{\Omega} uz(x, y, 1, t) \, dx \, dy \, dt = 0.
\end{aligned} \quad (3.3)$$

Next, we multiple (2.13)₂ by $E^q(t) \psi'(t) \xi(t) e^{-2\tau(t)\rho} z$ and integrate over $\Omega \times (0, 1) \times (s, T)$ to get

$$\int_s^T \int_{\Omega} \int_0^1 E^q(t) \psi'(t) \xi(t) e^{-2\tau(t)\rho} z [\tau(t) z_t + (1 - \tau'(t)\rho) z_{\rho}] \, d\rho \, dx \, dy \, dt = 0.$$

This gives us

$$\begin{aligned}
0 & = \left[\frac{1}{2} E^q(t) \psi'(t) \xi(t) \tau(t) \int_{\Omega} \int_0^1 e^{-2\tau(t)\rho} z^2 \, d\rho \, dx \, dy \right]_s^T \\
& - \frac{1}{2} \int_s^T \frac{d}{dt} [E^q(t) \psi'(t) \xi(t) \tau(t)] \int_{\Omega} \int_0^1 e^{-2\tau(t)\rho} z^2 \, d\rho \, dx \, dy \, dt \\
& + \int_s^T E^q(t) \psi'(t) \xi(t) \tau(t) \tau'(t) \int_{\Omega} \int_0^1 \rho e^{-2\tau(t)\rho} z^2 \, d\rho \, dx \, dy \, dt \\
& + \int_s^T E^q(t) \psi'(t) \xi(t) \int_{\Omega} \int_0^1 \left(\frac{1}{2} \frac{\partial}{\partial \rho} e^{-2\tau(t)\rho} (1 - \tau(t)\rho) z^2 \right) \, d\rho \, dx \, dy \, dt \\
& + \int_s^T E^q(t) \psi'(t) \xi(t) \tau(t) \int_{\Omega} \int_0^1 e^{-2\tau(t)\rho} (1 - \tau(t)\rho) z^2 \, d\rho \, dx \, dy \, dt \\
& + \int_s^T E^q(t) \psi'(t) \xi(t) \tau'(t) \int_{\Omega} \int_0^1 e^{-2\tau(t)\rho} z^2 \, d\rho \, dx \, dy \, dt.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
0 & = \left[\frac{1}{2} E^q(t) \psi'(t) \xi(t) \tau(t) \int_{\Omega} \int_0^1 e^{-2\tau(t)\rho} z^2 \, d\rho \, dx \, dy \right]_s^T \\
& - \frac{1}{2} \int_s^T \frac{d}{dt} [E^q(t) \psi'(t) \xi(t) \tau(t)] \int_{\Omega} \int_0^1 e^{-2\tau(t)\rho} z^2 \, d\rho \, dx \, dy \, dt \\
& + \int_s^T E^q(t) \psi'(t) \xi(t) \tau(t) \tau'(t) \int_{\Omega} \int_0^1 \rho e^{-2\tau(t)\rho} z^2 \, d\rho \, dx \, dy \, dt \\
& + \frac{1}{2} \int_s^T E^q(t) \psi'(t) \xi(t) e^{-2\tau(t)} (1 - \tau(t)) \int_{\Omega} z^2(x, y, 1, t) \, dx \, dy \, dt \\
& - \frac{1}{2} \int_s^T E^q(t) \psi'(t) \xi(t) \|u_t\|_{L^2(\Omega)}^2 \, dt \\
& + \int_s^T E^q(t) \psi'(t) \xi(t) \tau(t) \int_{\Omega} \int_0^1 e^{-2\tau(t)\rho} (1 - \tau(t)\rho) z^2 \, d\rho \, dx \, dy \, dt \\
& + \int_s^T E^q(t) \psi'(t) \xi(t) \tau'(t) \int_{\Omega} \int_0^1 e^{-2\tau(t)\rho} z^2 \, d\rho \, dx \, dy \, dt.
\end{aligned} \quad (3.4)$$

Recalling conditions (A1) and setting $k = \min \left\{ 1, e^{-2\tau_0}, \frac{d}{\tau_0} \right\}$, the summation of (3.3) and (3.4) leads to

$$\begin{aligned}
k \int_s^T E^{q+1}(t) \psi'(t) \, dt & \leq \int_s^T [q E'(t) E^{q-1}(t) \psi'(t) + E^q(t) \psi''(t)] \int_{\Omega} uu_t \, dx \, dy \, dt \\
& - \left[E^q(t) \psi'(t) \int_{\Omega} uu_t \, dx \, dy \right]_s^T + \frac{3}{2} \int_s^T E^q(t) \psi'(t) \|u_t\|_{L^2(\Omega)}^2 \, dt
\end{aligned}$$

$$\begin{aligned}
 & - \int_s^T E^q(t) \psi'(t) \delta_1(t) \int_{\Omega} u u_t dx dy dt \\
 & - \int_s^T E^q(t) \psi'(t) \delta_2(t) \int_{\Omega} u z(x, y, 1, t) dx dy dt \\
 & - \left[\frac{1}{2} E^q(t) \psi'(t) \xi(t) \tau(t) \int_{\Omega} \int_0^1 e^{-2\tau(t)\rho} z^2 d\rho dx dy \right]_s^T \\
 & + \frac{1}{2} \int_s^T \frac{d}{dt} [E^q(t) \psi'(t) \xi(t) \tau(t)] \int_{\Omega} \int_0^1 e^{-2\tau(t)\rho} z^2 d\rho dx dy dt \\
 & - \frac{1}{2} \int_s^T E^q(t) \psi'(t) \xi(t) e^{-2\tau(t)} (1 - \tau(t)) \int_{\Omega} z^2(x, y, 1, t) dx dy dt \\
 & + \frac{1}{2} \int_s^T E^q(t) \psi'(t) \xi(t) \|u_t\|_{L^2(\Omega)}^2 dt.
 \end{aligned} \tag{3.5}$$

Now, we estimate the terms on the right-hand side of (3.5). Using (A2)–(A3), Cauchy–Schwarz inequality, the embedding Lemma 2.1, (2.19) and assuming that ψ' is a non-negative and bounded function on $[0, +\infty)$, we obtain

$$\begin{aligned}
 & \left| \int_s^T q E'(t) E^{q-1}(t) \psi'(t) \int_{\Omega} u u_t dx dy dt \right| \leq -C \int_s^T E^q(t) E'(t) dt \leq C E^{q+1}(s), \\
 & \left| \int_s^T E^q(t) \psi''(t) \int_{\Omega} u u_t dx dy dt \right| \leq -C \int_s^T E^{q+1}(t) \psi''(t) dt \leq C E^{q+1}(s), \\
 & \left| E^q(t) \psi'(t) \int_{\Omega} u u_t dx dy \right| \leq C E^{q+1}(t), \\
 & \left| E^q(t) \psi'(t) \xi(t) \tau(t) \int_{\Omega} \int_0^1 e^{-2\tau(t)\rho} z^2 d\rho dx dy \right| \leq C E^{q+1}(t)
 \end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
 \frac{3}{2} \int_s^T E^q(t) \psi'(t) \|u_t\|_{L^2(\Omega)}^2 dt & \leq C \int_s^T E^q(t) \frac{\psi'(t)}{\delta_1(t)} \delta_1(t) \|u_t\|_{L^2(\Omega)}^2 dt \\
 & \leq -C \int_s^T E^q(t) \frac{\psi'(t)}{\delta_1(t)} E'(t) dt.
 \end{aligned} \tag{3.7}$$

Thus, using (3.7) and the definition of ψ in (3.2), we get

$$\frac{3}{2} \int_s^T E^q(t) \psi'(t) \|u_t\|_{L^2(\Omega)}^2 dt \leq -C \int_s^T E^q(t) E'(t) dt \leq C E^{q+1}(s). \tag{3.8}$$

Using (A2), Young's inequality, Lemma 2.1 and (2.20), we have for any $\epsilon > 0$

$$\begin{aligned}
 & \left| - \int_s^T E^q(t) \psi'(t) \delta_1(t) \int_{\Omega} u u_t dx dy dt \right| \\
 & \leq \epsilon \int_s^T E^q(t) \psi'(t) \|u\|_{L^2(\Omega)}^2 dt + C_{\epsilon} \int_s^T E^q(t) \psi'(t) \|u_t\|_{L^2(\Omega)}^2 dt \\
 & \leq \epsilon C_e \int_s^T E^q(t) \psi'(t) \|u\|_{H_*^2(\Omega)}^2 dt + C_{\epsilon} \int_s^T E^q(t) \psi'(t) \|u_t\|_{L^2(\Omega)}^2 dt \\
 & \leq \epsilon C_e \int_s^T E^{q+1}(t) \psi'(t) dt + C_{\epsilon} E^{q+1}(s)
 \end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
& \left| - \int_s^T E^q(t) \psi'(t) \delta_2(t) \int_{\Omega} u z(x, y, 1, t) dx dy dt \right| \\
& \leq \epsilon \int_s^T E^q(t) \psi'(t) \|u\|_{L^2(\Omega)}^2 dt + C_{\epsilon} \int_s^T E^q(t) \psi'(t) \|z(\cdot, 1)\|_{L^2(\Omega)}^2 dt \\
& \leq \epsilon C_e \int_s^T E^q(t) \psi'(t) \|u\|_{H_*^2(\Omega)}^2 dt + C_{\epsilon} \int_s^T E^q(t) \psi'(t) \|z(\cdot, 1)\|_{L^2(\Omega)}^2 dt \\
& \leq \epsilon C_e \int_s^T E^{q+1}(t) \psi'(t) dt + C_{\epsilon} E^{q+1}(s),
\end{aligned} \tag{3.10}$$

where C_e is the embedding constant in Lemma 2.1. By assumption (A3) and recalling that $\xi(t)$ is a non-increasing function, we obtain

$$\begin{aligned}
& \frac{1}{2} \int_s^T \frac{d}{dt} [E^q(t) \psi'(t) \xi(t) \tau(t)] \int_{\Omega} \int_0^1 e^{-2\tau(t)\rho} z^2 d\rho dx dy dt \\
& \leq \frac{1}{2} \int_s^T \frac{d}{dt} [E^q(t) \psi'(t)] \tau'(t) \int_{\Omega} \int_0^1 e^{-2\tau(t)\rho} z^2 d\rho dx dy dt \\
& \leq C \int_s^T E^q(t) (-E'(t)) dt \\
& \leq C E^{q+1}(s)
\end{aligned} \tag{3.11}$$

and using Cauchy–Schwarz inequality, we get

$$\begin{aligned}
& \left| -\frac{1}{2} \int_s^T E^q(t) \psi'(t) \xi(t) e^{-2\tau(t)} (1 - \tau(t)) \int_{\Omega} z^2(x, y, 1, t) dx dy dt \right| \\
& \leq C \int_s^T E^q(t) \xi(t) \|z(\cdot, 1)\|_{L^2(\Omega)}^2 dt \\
& \leq C E^{q+1}(s),
\end{aligned} \tag{3.12}$$

$$\frac{1}{2} \int_s^T E^q(t) \psi'(t) \xi(t) \|u_t\|_{L^2(\Omega)}^2 dt \leq C E^{q+1}(s). \tag{3.13}$$

A combination of (3.6)–(3.13) leads to

$$(k - \epsilon C_e) \int_s^T E^{q+1}(t) \psi'(t) dt \leq (C_{\epsilon} + C) E^{q+1}(s). \tag{3.14}$$

By selecting ϵ small enough so that $k - \epsilon C_e > 0$, we get

$$\int_s^T E^{q+1}(t) \psi'(t) dt \leq C E^{q+1}(s). \tag{3.15}$$

Letting $T \longrightarrow +\infty$ and applying Lemma 2.3, we obtain the result. $\square$

Examples

Let

$$\delta_1(t) = b(1+t)^{a-1}, \quad \delta_2(t) = \frac{b}{b+1} (1+t)^{a-2}, \quad 0 < a < 1, \quad b > 0,$$

where a and b are constants. Then

$$\left| \frac{\delta_1'(t)}{\delta_1(t)} \right| = \left| \frac{b(a-1)(1+t)^{a-2}}{b(1+t)^{a-1}} \right| = \frac{(1-a)}{1+t} \leq (1-a),$$

$$|\delta_2(t)| = \left| \frac{b}{b+1}(1+t)^{a-1} \frac{1}{(1+t)} \right| \leq \frac{1}{1+b} \delta_1(t)$$

and

$$|\delta_2'(t)| = \left| \frac{(a-2)b}{1+b}(1+t)^{a-3} \right| \leq \left| \frac{(a-2)b}{1+b} \right| \delta_1(t).$$

Therefore, δ_1 and δ_2 satisfy the assumptions in (A2). It follows from (3.1) and (3.2) that

$$E(t) \leq mE(0)e^{-\lambda_1(1+t)^a}, \quad \forall t \geq 0. \quad (3.16)$$

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