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Uniform decay rates of a coupled suspension bridges with temperature

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Abstract In this paper, we investigate the decay properties of the thermoelastic suspension bridges model. We prove that the energy is decaying exponentially. To our knowledge, our result is new and our method of proof is based on the energy method to build the appropriate Lyapunov functional.

Mathematics Subject Classification 35B40 · 74D99 · 93D15 · 93D20.

1 Introduction

In recent years, increasing attention has been turned to the analysis of some classical engineering structures describing the dynamics of linear and nonlinear vibrations of suspension bridges. The stabilization of suspension bridges and their oscillations plays an important and crucial role nowadays.

In this paper, we consider the following coupled system of partial differential equations, given by:

$$\begin{cases} u_{tt} + u_{xxxx} + \left(\int_0^l (u^2 + v^2) dx \right) u + 2 \left(\int_0^l uv dx \right) v + \alpha \theta_{xx} = 0, & \text{in } (0, l) \times \mathbb{R}_+, \\ v_{tt} - v_{xx} + 2 \left(\int_0^l uv dx \right) u + \left(\int_0^l (u^2 + v^2) dx \right) v + \beta \varphi_x = 0, & \text{in } (0, l) \times \mathbb{R}_+, \\ \theta_t - \theta_{xx} - \alpha u_{txx} = 0, & \text{in } (0, l) \times \mathbb{R}_+, \\ \varphi_t - \varphi_{xx} + \beta v_{tx} = 0, & \text{in } (0, l) \times \mathbb{R}_+, \end{cases} \quad (1.1)$$

with the following initial data:

$$(u, u_t, v, v_t, \theta, \varphi)(x, 0) = (u_0, u_1, v_0, v_1, \theta_0, \varphi_0), \quad (1.2)$$

and the following boundary conditions:

$$\begin{aligned} u(0, t) = u(l, t) = u_{xx}(0, t) = u_{xx}(l, t) = v(0, t) = v(l, t) = 0, \quad \forall t \geq 0, \\ \theta(0, t) = \theta(l, t) = \varphi_x(0, t) = \varphi_x(l, t) = 0, \end{aligned} \quad (1.3)$$

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The given system models a one-dimensional suspension bridge with thermal effects, where u is the vertical displacement, v is the torsional angle, and the couple (θ, φ) is the two temperatures. We need to mention here that the modeling for the first two coupled equations in (1.1) was considered in [7] with intermediate piers, and the authors proved the well-posedness of the problem and investigated the stability of the solutions.

In recent years, the analysis and the stability of various nonlinear suspension bridge models have been attracting many researchers (see [1–10]) and the references therein.

In this paper, our interest is the stability properties of the coupling model (1.1). Therefore, from the mathematical point of view, it is important to study the stability properties of this system and determine whether the indirect dissipation coming via the two temperatures is enough to stabilize the whole system.

We prove that the coupling with the two temperatures is strong in a way that the solutions of System (1.1) decay exponentially. Our method of proof is based on multipliers' techniques and the construction of the appropriate Lyapunov functional.

The paper is organized as follows. In Sect. 2, we state and prove the exponential stability result for System (1.1). We give some concluding remarks in Sect. 3.

2 Exponential stability

In this section, we state and prove the exponential stability result for (1.1) with the boundary and initial conditions given by (1.3) and (1.2).

Throughout this paper, c is used to denote a generic positive constant. First, we introduce the following Hilbert spaces:

$$\begin{aligned} \widehat{H} &= \left\{ \varphi \in L^2(0, l) : \int_0^l \varphi(x) dx = 0 \right\}, \quad \widehat{H}_1 = H^1(0, l) \cap \widehat{H}, \\ H_0^2(0, l) &= H^2(0, l) \cap H_0^1(0, l), \text{ and } H_4 = \{u \in H^4(0, l) : u(0, t) = u(l, t) = u_{xx}(0, t) = u_{xx}(l, t) = 0\}, \\ \mathcal{H} &= H_0^2(0, l) \times L^2(0, l) \times H_0^1(0, l) \times L^2(0, l) \times L^2(0, l) \times \widehat{H}. \end{aligned}$$

The space $L^2(0, l)$ is the pivot space and the related standard inner product and norm will be denoted along the paper by $\langle u, v \rangle$ and $\|\cdot\|_2$.

As in [5], we introduce the biharmonic operator $A : D(A) \subset L^2(0, l) \longrightarrow L^2(0, l)$ defined as

$$Au = u_{xxxx} \text{ with } D(A) = H_4;$$

we consider the compactly nested family of Hilbert spaces generated by the powers of A (r will be always omitted whenever zero)

$$H_r = D\left(A^{\frac{r}{4}}\right), \quad \langle u, v \rangle_r = \left\langle A^{\frac{r}{4}}u, A^{\frac{r}{4}}v \right\rangle, \quad \|u\|_r = \left\| A^{\frac{r}{4}}u \right\|_2, \quad \text{for } r \in \mathbb{R}.$$

The well-posedness of (1.1, 1.2) is stated in the following theorem.

Theorem 2.1 Assume that $(u_0, u_1, v_0, v_1, \theta_0, \varphi_0) \in \mathcal{H}$. Then, Problem (1.1, 1.2) admits a unique weak solution $(u, u_t, v, v_t, \theta, \varphi) \in C([0, \infty), \mathcal{H})$.

Proof This theorem can be established using standard method such as Galerkin method following the same steps as in [3]. $\square$

Now, we introduce the first-order energy associated with (1.1, 1.2) given by

$$E(t) := \frac{1}{2} \left(\int_0^l (u_t^2 + v_t^2 + u_{xx}^2 + v_x^2) dx + \int_0^l (\theta^2 + \varphi^2) dx + \frac{1}{2} \left(\int_0^l (u^2 + v^2) dx \right)^2 + 2 \left(\int_0^l uv dx \right)^2 \right). \quad (2.1)$$

Our aim in this paper is to prove the uniform decay of the energy using the multipliers techniques.

Theorem 2.2 Let $(u_0, u_1, v_0, v_1, \theta_0, \varphi_0) \in \mathcal{H}$ and (u, v, θ, φ) be the solution of (1.1, 1.2). Then, there exist two positive constants C_0 and C_1 , such that

$$E(t) \leq C_1 e^{-C_0 t} \quad \forall \geq 0, \quad (2.2)$$

where C_0 and C_1 are two positive constants.

The proof of this theorem will be established through several lemmas.

Lemma 2.3 *Let $(u_0, u_1, v_0, v_1, \theta_0, \varphi_0) \in \mathcal{H}$ and (u, v, θ, φ) be the solution of (1.1, 1.2). Then, the energy functional E defined by (2.1) satisfies*

$$\frac{d}{dt} E(t) = -\frac{1}{2} \|\theta_x\|_2^2 - \frac{1}{2} \|\varphi_x\|_2^2. \quad (2.3)$$

Proof Multiplying the first equation of (1.1) by u_t , the second equation by v_t , the third equation by θ , the fourth equation by φ , integrating by parts over $(0, l)$, and using the boundary conditions (1.3), then summing up, we obtain the desired result. $\square$

Lemma 2.4 *Let $(u_0, u_1, v_0, v_1, \theta_0, \varphi_0) \in \mathcal{H}$ and (u, v, θ, φ) be the solution of (1.1, 1.2). Then, the functional*

$$I_1(t) := \int_0^l u u_t dx + \int_0^l v v_t dx,$$

satisfies for $\varepsilon_1 > 0$ the following estimate:

$$\begin{aligned} I_1'(t) &\leq \|u_t\|_2^2 + \|v_t\|_2^2 - (1 - \varepsilon_1) \|u_{xx}\|_2^2 - (1 - \varepsilon_1) \|v_x\|_2^2 \\ &\quad - \left(\int_0^l (u^2 + v^2) dx \right)^2 - 4 \left(\int_0^l uv dx \right)^2 + c \|\theta_x\|_2^2 + c \|\varphi_x\|_2^2. \end{aligned} \quad (2.4)$$

Proof Using the first and the second equations in (1.1), integrating by parts over $(0, l)$ and using the boundary conditions (1.3), we have

$$\begin{aligned} I_1'(t) &= \|u_t\|_2^2 + \|v_t\|_2^2 - \|u_{xx}\|_2^2 - \|v_x\|_2^2 \\ &\quad - \left(\int_0^l (u^2 + v^2) dx \right)^2 - 4 \left(\int_0^l uv dx \right)^2 - \int_0^l \theta u_{xx} dx + \int_0^l \varphi v_x dx. \end{aligned} \quad (2.5)$$

Applying Young and Poincaré's inequalities, we get, for any $\varepsilon_1 > 0$

$$\begin{aligned} \left| \int_0^l \theta u_{xx} dx \right| &\leq \varepsilon_1 \|u_{xx}\|_2^2 + c \|\theta\|_2^2 \leq \varepsilon_1 \|u_{xx}\|_2^2 + c \|\theta_x\|_2^2, \\ \left| \int_0^l \varphi v_x dx \right| &\leq \varepsilon_1 \|v_x\|_2^2 + c \|\varphi\|_2^2 \leq \varepsilon_1 \|v_x\|_2^2 + c \|\varphi_x\|_2^2; \end{aligned} \quad (2.6)$$

by (2.5) and (2.6), we deduce the result. $\square$

Lemma 2.5 *Let $(u_0, u_1, v_0, v_1, \theta_0, \varphi_0) \in \mathcal{H}$ and (u, v, θ, φ) be the solution of (1.1, 1.2). Then, the functional*

$$I_2(t) = \int_0^l v_t \left(\int_0^x \varphi(y, t) dy \right) dx,$$

satisfies for $\varepsilon_2, \varepsilon_3, \varepsilon_4 > 0$, the following estimate:

$$\begin{aligned} I_2'(t) &\leq - (1 - \varepsilon_2) \|v_t\|_2^2 + c \|\varphi_x\|_2^2 + \varepsilon_3 \|v_x\|_2^2 \\ &\quad + 4\varepsilon_4 E(0) \left(\int_0^l uv dx \right)^2 + \varepsilon_4 E(0) \left(\int_0^l (u^2 + v^2) dx \right)^2. \end{aligned} \quad (2.7)$$

Proof Using the first and the fourth equations in (1.1), integrating by parts over $(0, l)$ and using the boundary conditions (1.3), we have

$$\begin{aligned} I_2'(t) &= \int_0^l v_{tt} \left(\int_0^x \varphi(y, t) dy \right) dx + \int_0^l v_t \left(\int_0^x \varphi_t(y, t) dy \right) dx \\ &= - \int_0^l v_x \varphi dx - 2 \left(\int_0^l uv dx \right) \int_0^l u \left(\int_0^x \varphi(y, t) dy \right) dx \\ &\quad - \left(\int_0^l (u^2 + v^2) dx \right) \int_0^l v \left(\int_0^x \varphi(y, t) dy \right) dx + \|\varphi\|_2^2 - \|v_t\|_2^2 + \int_0^l v_t \varphi_x dx. \end{aligned} \quad (2.8)$$

Using Cauchy–Schwarz, Young and Poincaré’s inequalities, we get

$$\begin{aligned}
 \left| \int_0^l v_x \varphi \, dx \right| &\leq \varepsilon_3 \|v_x\|_2^2 + c \|\varphi_x\|_2^2, \\
 \left| \int_0^l v_t \varphi_x \, dx \right| &\leq \varepsilon_2 \|v_t\|_2^2 + c \|\varphi_x\|_2^2, \\
 \left| \left(\int_0^l uv \, dx \right) \int_0^l u \left(\int_0^x \varphi(y, t) \, dy \right) dx \right| &\leq \varepsilon_4 \left(\int_0^l uv \, dx \right)^2 \|u\|_2^2 + c \|\varphi_x\|_2^2 \\
 &\leq 2\varepsilon_4 E(0) \left(\int_0^l uv \, dx \right)^2 + c \|\varphi_x\|_2^2, \\
 \left| \left(\int_0^l (u^2 + v^2) \, dx \right) \int_0^l v \left(\int_0^x \varphi(y, t) \, dy \right) dx \right| &\leq \frac{\varepsilon_4}{2} \left(\int_0^l (u^2 + v^2) \, dx \right)^2 \|v\|_2^2 + c \|\varphi_x\|_2^2 \\
 &\leq \varepsilon_4 E(0) \left(\int_0^l (u^2 + v^2) \, dx \right)^2 + c \|\varphi_x\|_2^2;
 \end{aligned} \tag{2.9}$$

then from (2.8) and (2.9), we deduce that

$$\begin{aligned}
 I'_2(t) &\leq -(1 - \varepsilon_2) \|v_t\|_2^2 + c \|\varphi_x\|_2^2 + \varepsilon_3 \|v_x\|_2^2 \\
 &\quad + 4\varepsilon_4 E(0) \left(\int_0^l uv \, dx \right)^2 + \varepsilon_4 E(0) \left(\int_0^l (u^2 + v^2) \, dx \right)^2.
 \end{aligned}$$

□

Lemma 2.6 Let $(u_0, u_1, v_0, v_1, \theta_0, \varphi_0) \in \mathcal{H}$ and (u, v, θ, φ) be the solution of (1.1, 1.2). Then, the functional

$$I_3(t) = \langle u_t, \theta \rangle_{-1} := \int_0^l (\partial_{xxxx})^{-\frac{1}{4}} u_t (\partial_{xxxx})^{-\frac{1}{4}} \theta \, dx,$$

satisfies for $\varepsilon_5, \varepsilon_6, \varepsilon_7 > 0$, the following estimate:

$$\begin{aligned}
 I'_3(t) &\leq -(1 - \varepsilon_6) \|u_t\|_2^2 + \varepsilon_5 \|u_{xx}\|_2^2 + c \|\theta_x\|_2^2 \\
 &\quad + \varepsilon_7 E(0) \left(\int_0^l (u^2 + v^2) \, dx \right)^2 + 4\varepsilon_7 E(0) \left(\int_0^l uv \, dx \right)^2.
 \end{aligned} \tag{2.10}$$

Proof Using the first and the third equations in (1.1), integrating by parts over $(0, l)$ and taking into consideration the boundary conditions (1.3), we obtain

$$\begin{aligned}
 I'_3(t) &= \langle u_{tt}, \theta \rangle_{-1} + \langle u_t, \theta_t \rangle_{-1} \\
 &= -\langle u_{xxxx}, \theta \rangle_{-1} - \left(\int_0^l (u^2 + v^2) \, dx \right) \langle u, \theta \rangle_{-1} - 2 \left(\int_0^l uv \, dx \right) \langle v, \theta \rangle_{-1} \\
 &\quad - \langle \theta_{xx}, \theta \rangle_{-1} + \langle u_t, \theta_{xx} \rangle_{-1} + \langle u_t, u_{txx} \rangle_{-1} \\
 &= \int_0^l u_{xx} \theta \, dx - \left(\int_0^l (u^2 + v^2) \, dx \right) \langle u, \theta \rangle_{-1} - 2 \left(\int_0^l uv \, dx \right) \langle v, \theta \rangle_{-1} \\
 &\quad + \|\theta\|_2^2 - \int_0^l u_t \theta \, dx - \|u_t\|_2^2,
 \end{aligned} \tag{2.11}$$

by exploiting the following inequalities:

$$\begin{aligned}
 \left| \int_0^l u_{xx} \theta \, dx \right| &\leq \varepsilon_5 \|u_{xx}\|_2^2 + c \|\theta\|_2^2 \leq \varepsilon_5 \|u_{xx}\|_2^2 + c \|\theta_x\|_2^2, \\
 \left| \int_0^l u_t \theta \, dx \right| &\leq \varepsilon_6 \|u_t\|_2^2 + c \|\theta_x\|_2^2, \\
 \left| \left(\int_0^l (u^2 + v^2) \, dx \right) \langle u, \theta \rangle_{-1} \right| &\leq \left(\int_0^l (u^2 + v^2) \, dx \right) \|u\|_2^2 \|\theta\|_2^2 \\
 &\leq \varepsilon_7 E(0) \left(\int_0^l (u^2 + v^2) \, dx \right)^2 + c \|\theta_x\|_2^2, \\
 \left| \left(\int_0^l uv \, dx \right) \langle v, \theta \rangle_{-1} \right| &\leq \left| \left(\int_0^l uv \, dx \right) \right| \|v\|_2^2 \|\theta\|_2^2 \\
 &\leq 2\varepsilon_7 E(0) \left(\int_0^l uv \, dx \right)^2 + c \|\theta_x\|_2^2,
 \end{aligned} \tag{2.12}$$

we deduce from (2.11) and (2.12) that

$$\begin{aligned}
 I_3'(t) &\leq -(1 - \varepsilon_6) \|u_t\|_2^2 + \varepsilon_5 \|u_{xx}\|_2^2 + c \|\theta_x\|_2^2 \\
 &\quad + \varepsilon_7 E(0) \left(\int_0^l (u^2 + v^2) \, dx \right)^2 + 4\varepsilon_7 E(0) \left(\int_0^l uv \, dx \right)^2.
 \end{aligned}$$

□

To complete the proof of our main result, we introduce the following Lyapunov functional:

$$\mathcal{L}(t) := N E(t) + N_1 I_1(t) + N_2 I_2(t) + N_3 I_3(t), \tag{2.13}$$

where N , N_1 , N_2 , and N_3 are positive constants to be fixed later.

Lemma 2.7 *For N large enough, there exist two positive constants α_1 and α_2 , such that*

$$\alpha_1 E(t) \leq \mathcal{L}(t) \leq \alpha_2 E(t). \tag{2.14}$$

Proof Let us define the following functional:

$$\mathcal{L}_1(t) = N_1 I_1(t) + N_2 I_2(t) + N_3 I_3(t).$$

Using Cauchy–Schwarz, Young’s and Poincaré’s inequalities, we obtain

$$\begin{aligned}
 |\mathcal{L}_1(t)| &\leq \frac{N_1}{2} \|u\|_2^2 + \frac{1}{2} [N_1 + N_3] \|u_t\|_2^2 \\
 &\quad + \frac{N_1}{2} \|v\|_2^2 + \frac{1}{2} [N_1 + N_2] \|v_t\|_2^2 \\
 &\quad + cN_2 \|\varphi\|_2^2 + cN_3 \|\theta\|_2^2.
 \end{aligned}$$

Then, by (2.1), we get

$$|\mathcal{L}_1(t)| \leq c E(t).$$

Consequently

$$|\mathcal{L}(t) - N E(t)| \leq c E(t),$$

which implies that

$$(N - c) E(t) \leq \mathcal{L}(t) \leq (N + c) E(t).$$

Choosing N large enough, then we have (2.14). □

Proof of Theorem 2.2: Differentiating (2.13), and exploiting (2.4), (2.7) and (2.10), we obtain the following estimates:

$$\begin{aligned} \mathcal{L}'(t) \leq & -\left(\frac{N}{2} - (N_1 + N_3)c\right) \|\theta_x\|_2^2 - \left(\frac{N}{2} - (N_1 + N_2)c\right) \|\varphi_x\|_2^2 \\ & - (N_1 - (\varepsilon_4 N_2 + \varepsilon_7 N_3)E(0)) \left(\int_0^l (u^2 + v^2) dx\right)^2 \\ & - 4(N_1 - (\varepsilon_4 N_2 + \varepsilon_7 N_3)E(0)) \left(\int_0^l uv dx\right)^2 \\ & - ((1 - \varepsilon_1)N_1 - \varepsilon_5 N_3) \|u_{xx}\|_2^2 - ((1 - \varepsilon_1)N_1 - \varepsilon_3 N_2) \|v_x\|_2^2 \\ & - ((1 - \varepsilon_2)N_2 - N_1) \|v_t\|_2^2 - ((1 - \varepsilon_6)N_3 - N_1) \|u_t\|_2^2. \end{aligned} \quad (2.15)$$

Here, we choose carefully the constants as follows:

$$\begin{aligned} \varepsilon_4 = \frac{N_1}{4N_2 E(0)}, \quad \varepsilon_7 = \frac{N_1}{4N_3 E(0)}, \quad \varepsilon_1 = \frac{1}{4}, \quad \varepsilon_3 = \frac{N_1}{4N_2}, \quad \varepsilon_5 = \frac{N_1}{4N_3}, \\ N_1 = 1, \quad N_2 = 2N_1, \quad \varepsilon_2 = \frac{1}{4}, \quad N_3 = 2N_1, \quad \text{and} \quad \varepsilon_6 = \frac{1}{4}. \end{aligned}$$

Now, we choose N large enough and satisfying

$$\frac{N}{2} > 3N_1 c.$$

Consequently, using inequality (2.15) and (2.1), we get

$$\mathcal{L}'(t) \leq -k_0 E(t), \quad \forall t > 0, \quad (2.16)$$

where k_0 is a positive constant.

A combination of (2.14) and (2.16) gives

$$\mathcal{L}'(t) \leq -C_0 \mathcal{L}(t); \quad \forall t > 0.$$

A simple integration over $(0, t)$ yields

$$\mathcal{L}(t) \leq \mathcal{L}(0) e^{-C_0 t}; \quad \forall t > 0. \quad (2.17)$$

Thus, using (2.14) and (2.17), the conclusion of our main result follows.

3 Concluding Remark

In this work, we proved that the coupling of the suspension bridges model with two temperatures is strong enough to stabilize exponentially the solutions of the model. It will be interesting to study the case where the suspension bridges are coupled with only one heat equation.

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