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Constraints in inflationary magnetogenesis

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Abstract We discuss popular models of inflationary and early post-inflationary magnetogenesis and present model-independent upper bounds on the strength of the resulting magnetic fields imposed by the considerations of weak coupling, back-reaction and Schwinger effect.

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1 Introduction

There is an ongoing debate as regards the origin of magnetic fields observed on different spatial scales in the universe (for reviews, see [15, 19, 23, 35, 57, 64]). In this paper, we focus on the widely discussed possibility of their cosmological origin [6]; specifically, that they were generated at the inflationary stage. Such a scenario naturally explains their large coherence length, which presumably can be comparable to the size of the large-scale structure [2, 11, 38, 59, 60].

A general class of models that were proposed for inflationary primordial magnetogenesis is described by a gauge-invariant action of the form pioneered in [41, 61]:

$$\mathcal{L}_{\text{em}} = -\frac{1}{4} \left(I^2 F_{\mu\nu} F^{\mu\nu} + f \tilde{F}_{\mu\nu} F^{\mu\nu} \right), \quad (1)$$

where $\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu}^{\alpha\beta} F_{\alpha\beta}$ is the Hodge dual of the electromagnetic strength tensor $F_{\mu\nu}$, and I and f are non-trivial functions of the cosmological time due to their possible dependence on the background fields such as the inflaton or the metric curvature. In [43], we have shown how such coupling between the curvature and inflaton can arise as a result of quantum corrections in the Starobinsky inflationary model [55]. But we will not specify such couplings here, making our consideration general and model independent.

The type of inflationary magnetogenesis under consideration is confronted by the issues of back-reaction and strong gauge coupling [10, 37, 48, 62], and by the Schwinger effect [12, 14, 22, 31, 33, 46, 47, 52–54, 56] that all reduce its efficiency. Many models of evolution of I and f were tried in the literature to overcome these difficulties and to obtain the highest possible value of primordial magnetic field. Given the diversity of these approaches, and based on our preliminary studies [49, 50], we suggested in [51] to seek for model-independent constraints imposed by all these limitations for the whole class of models described by (1). This is similar in spirit to the previous studies [10, 62], which, however, imposed a priori ansatzes for the evolution

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of the couplings and did not take into account the Schwinger effect. Here, we will review the results of our investigation in [49–51].

In our approach, we focus on a particular comoving spatial scale characterized by wavenumber k , and try to achieve the largest possible amplification of magnetic field on that particular scale disregarding of what is going on on other scales. In this way, we obtain a conservative upper bound for the magnetic field on that scale. We then show that this bound can, in fact, be reached by order of magnitude, i.e., that it is the least upper bound in a physical sense.

Amplification of electromagnetic field can start on the chosen comoving spatial scale well before its Hubble-radius crossing during inflation. One possible mechanism for this process was demonstrated in our publications [49–51] and will be reviewed below. It suffices to consider evolution of the chiral coupling $f(\eta)$ in (1) which is approximately linear in conformal time η in some time interval (it then evolves as $f \propto a^{-1}$ with the scale factor a during inflation). Then, only the modes of one helicity in a narrow band with $k \sim |f'(\eta)|/I^2$ will be exponentially amplified. The peak of electromagnetic field in this process is attained by the moment of Hubble-radius crossing during inflation, and the subsequent inflationary dilution of electromagnetic fields would then make them quite small by the end of inflation. Hence, it is necessary to amplify the fields also after the Hubble-radius crossing, during and after inflation. At all stages, one should take into account the constraints imposed by back-reaction and Schwinger effect. This eventually leads us to a simple model-independent upper bound for the final magnetic field [51]. This upper bound appears to be quite low for the prospects of inflationary magnetogenesis. A higher upper bound could only be obtained under the assumption that some unknown mechanism suppresses the Schwinger effect in the early universe.

2 Preliminaries

Magnetogenesis occurs in an expanding universe, which we describe by a spatially flat cosmological model with the metric

$$ds^2 = a^2(\eta) (d\eta^2 - \delta_{ij} dx^i dx^j), \quad (2)$$

where η is the conformal time, and a is the scale factor. As for the electromagnetic field with action (1), we adopt the longitudinal gauge $A_0 = 0$, $\partial^i A_i = 0$ for the vector potential. The transverse field variable A_i then satisfies the equation

$$(I^2 A'_i)' - I^2 \nabla^2 A_i + f' \epsilon_{ijk} \partial_j A_k = a^2 j_i, \quad (3)$$

where j_i is the electromagnetic current density, ϵ_{ijk} is the spatial Kronecker alternating tensor with $\epsilon_{123} = 1$, and the prime denotes the derivative with respect to the conformal time η . Assuming the Ohm's law in a medium with high electric conductivity σ , we have

$$j_i = \sigma E_i = -\frac{\sigma}{a} A'_i. \quad (4)$$

In the spatial Fourier representation, and in the constant (in space and time) normalized helicity basis $e^h(\mathbf{k})$, $h = \pm 1$, such that $i\mathbf{k} \times \mathbf{e}^h = h k \mathbf{e}^h$, we have $A_i = \sum_h \mathcal{A}_h e_i^h e^{i\mathbf{k}\cdot\mathbf{x}}$. Equations (3) and (4) then imply, for the helicity components $\mathcal{A}_h$,

$$(I^2 \mathcal{A}'_h)' + a\sigma \mathcal{A}'_h + (I^2 k^2 + h k f') \mathcal{A}_h = 0, \quad (5)$$

or, equivalently,

$$(I \mathcal{A}_h)'' + \frac{a\sigma}{I^2} (I \mathcal{A}_h)' + \left[k^2 + h k \frac{f'}{I^2} - \frac{I''}{I} - \frac{a\sigma I'}{I^3} \right] I \mathcal{A}_h = 0. \quad (6)$$

In the Lagrangian of charged matter, we assume the standard couplings of the form $|(\partial - iqA)\psi|^2$, where ψ is the matter field and q is its electric charge (we work in the system of units $\hbar = c = 1$). Hence, the spectral densities of quantum fluctuations of magnetic and electric field ‘felt’ by the charged matter are characterized, respectively, by the standard relations

$$\mathcal{P}_B \equiv \frac{d\langle B^2 \rangle}{d \ln k} = \frac{k^4}{8\pi^2 a^4} \sum_h |\mathcal{A}_h(\eta, k)|^2, \quad (7)$$

Fig. 1 Evolution of the form (10)

$$\mathcal{P}_E \equiv \frac{d\langle E^2 \rangle}{d \ln k} = \frac{k^4}{8\pi^2 a^4} \sum_h \left| \frac{\mathcal{A}'_h(\eta, k)}{k} \right|^2, \quad (8)$$

in which the amplitude of the vector potential is normalized so that $I\mathcal{A}_h \sim e^{-ik\eta}$ as $\eta \rightarrow -\infty$. The factor in front of the sums in (7) and (8) is the spectral density of the canonical vacuum fluctuations in each mode at the physical wavenumber k/a . In view of the presence of the factor I^2 in (1), the corresponding (unsubtracted) spectral energy densities per logarithmic interval of wavenumbers are given by

$$\frac{d\rho_B}{d \ln k} = I^2 \mathcal{P}_B, \quad \frac{d\rho_E}{d \ln k} = I^2 \mathcal{P}_E. \quad (9)$$

3 Amplification prior to Hubble-radius crossing

As discussed in the introduction, we focus on a particular comoving spatial scale with wave number k , and study the issue of general upper limits imposed by the back-reaction and Schwinger effect on the possible strength of magnetic field on this spatial scale. Amplification of the vacuum fluctuations of electromagnetic field can start prior to Hubble-radius crossing during inflation. An exactly solvable scenario for such a selective amplification was proposed in our works [49, 50] and used in [51]. Assuming a constant (or slowly evolving) kinetic coupling I , suppose that the chiral coupling evolves as (see Fig. 1)

$$f(\eta) = \frac{1}{2} \Delta f \tanh \left(\frac{2\eta - \eta_1 - \eta_2}{\Delta \eta} \right) + \text{const}, \quad (10)$$

in which $\Delta \eta = \eta_2 - \eta_1 > 0$ is its width in conformal time. One can write the general solution of (5) (with zero electric conductivity) in terms of the Ferrers functions [39, Chapter 14]. Then, the asymptotic behavior of the canonically normalized quantity $I\mathcal{A}_h \sim e^{-ik\eta}$ as $\eta \rightarrow -\infty$ determines the initial vacuum state, and the opposite asymptotic behavior $I\mathcal{A}_h \sim \alpha_k e^{-ik\eta} + \beta_k e^{ik\eta}$ in the future determines the Bogolyubov's coefficients α_k and β_k , and the mean number of quanta in a given mode [49–51]:

$$n_k = |\beta_k|^2 = \frac{\cos^2(\pi q/2)}{\sinh^2(\pi p/2)}, \quad (11)$$

where

$$p = k\Delta \eta, \quad q \equiv \sqrt{1 + h\Delta f p/I^2}. \quad (12)$$

For the helicity satisfying $h\Delta f < 0$, the quantity q , given by (12), is purely imaginary for $p > I^2/|\Delta f|$. In the approximation $p \gg \max \{I^2/|\Delta f|, 2/\pi\}$, we then obtain

$$n_k \approx e^{\pi(\sqrt{|\Delta f|p/I^2} - p)} = e^{\frac{\pi|\Delta f|}{I^2}(\sqrt{k/k_0} - k/k_0)}, \quad (13)$$

Fig. 2 Spectrum (11) on a logarithmic scale for the helicity satisfying $h\Delta f < 0$ and for $|\Delta f|/I^2 = 40$

where $k_0 = |\Delta f|/I^2 \Delta\eta$ is the wavenumber at which spectrum (13) reaches unity on the slope of its exponential decline. The exponent of this expression reaches maximum at $k_m = k_0/4$, with the maximum mean occupation number $n_m = e^{\pi|\Delta f|/4I^2}$, which is exponentially large for $|\Delta f|/I^2 \gg 1$. Spectrum (11) is plotted in Fig. 2 on a logarithmic scale for a typical value $|\Delta f|/I^2 = 40$ (see Sect. 5). The modes of the opposite helicity are not amplified, and their mean occupation numbers can be neglected.

The spectral densities (7) and (8) are of comparable magnitudes, and, since one of the helicities is dominating in n_k , we have, using (13),

$$\mathcal{P}_B \approx \mathcal{P}_E \approx \frac{k^4}{4\pi^2 a^4 I^2} \left[1 + e^{\frac{\pi|\Delta f|}{I^2}(\sqrt{k/k_0} - k/k_0)} \right]. \quad (14)$$

We observe that the spectral densities are peaked at the central value $k = k_m$ with width $\Delta k \simeq I k_m / |\Delta f|^{1/2} \ll k_m$ for $|\Delta f|/I^2 \gg 1$. Thus, electric and magnetic fields are generated in this scenario with similar spectra in the spectral region of amplification. The theory contains two free parameters, Δf and $\Delta\eta$, which can be adjusted to produce maximally helical electromagnetic fields of desirable strength with spectral density centered at a desirable wavenumber $k_m = |\Delta f|/4I^2 \Delta\eta$.

Solution for the electromagnetic field at the center of the spectral domain of amplification grows exponentially in the interval $\eta_1 < \eta < \eta_2$: $\mathcal{A}_h \propto e^{k_m \eta}$. Assuming $|k_m \eta_2| \lesssim 1$ and taking into account the factors $\propto a^{-4}$ in (7) and (8), we conclude that the peak of the electromagnetic spectral density is reached at the time where $|k_m \eta| = k_m / aH \approx 2$, i.e., a little before the Hubble-radius crossing.

4 Amplification after the Hubble-radius crossing

As we have seen in the previous section, amplification prior to the Hubble-radius crossing can produce a helical electromagnetic field with spectrum peaked at a specific comoving wavenumber k . Further amplification of electromagnetic field after the Hubble-radius crossing can take place due to evolution of the coupling I in (1). In this case, we assume the usual conditions

$$\left\{ k^2, k \frac{|f'|}{I^2} \right\} \ll \frac{|I''|}{I} \quad (15)$$

to be valid well after Hubble-radius crossing. These conditions mean that the evolution of the kinetic coupling I has to be sufficiently fast in order that a considerable amplification can be achieved. Under these conditions, the terms containing k in (6), hence, also in (5) can be neglected. Dropping also the terms with electric conductivity, we observe that equation (5) reduces to a simple equation $(I^2 \mathcal{A}_h')' = 0$, with general solution given by

$$\mathcal{A}_h \simeq \frac{1}{I_k} \left[Q_1 + Q_2 k \int \left(\frac{I_k}{I} \right)^2 d\eta \right] = \frac{1}{I_k} \left[Q_1 + Q_2 \int \left(\frac{I_k}{I} \right)^2 R_k dN \right]. \quad (16)$$

Here, Q_1 and Q_2 are integration constants that reflect evolution of the mode prior to Hubble-radius crossing, and, in the last expression, the integration variable was transformed as $kd\eta = kda/a^2H = kdN/aH = R_k dN$, where $R_k = k/aH < 1$ and $N = \ln a$ is the number of e -foldings. In passing, we note that, even if the coupling I stops evolving, mode (16) can still continue to grow as $\mathcal{A}_h \propto R_k \propto 1/aH$, with magnetic field evolving as $B \propto |\mathcal{A}_h|/a^2 \propto 1/a^3H$, which law was noted and used in [17,32,34].

5 Weak coupling, back-reaction and Schwinger effect

Focusing on the modes with a particular wavenumber k , we would like to obtain the largest possible amplitude for $\mathcal{A}_h$ in the end. The process of field amplification is constrained by the condition $I \gtrsim 1$ of weak kinetic coupling [10], and by the consideration of back-reaction and Schwinger effect.

The back-reaction constraint means that the electromagnetic energy density $\rho_{\text{em}} = I^2 (B^2 + E^2) / 2$ should not exceed the energy density of the rest of matter $\rho = M_{\text{P}}^2 H^2$, where $M_{\text{P}} = \sqrt{3/8\pi G} \approx 4.2 \times 10^{18} \text{ GeV}$ is the reduced Planck mass expressed through the Newton's gravitational constant G . This is required in order that the whole consideration is self-consistent. Hence, we should ensure

$$I(E + B) \lesssim M_{\text{P}} H. \quad (17)$$

Consider now the role of electric conductivity. The second term in Eq. (5) can be neglected under the condition

$$a\sigma \ll |(I^2)'| \sim aHI^2 \Rightarrow \frac{\sigma}{I^2} \ll H. \quad (18)$$

Here, we have assumed a typical evolution of the kinetic coupling on the Hubble timescale. In the opposite case, $\sigma/I^2 \gg H$, the term with conductivity in Eq. (5) will cause the derivative $\mathcal{A}'_h$ to exponentially decrease on timescale much exceeding the Hubble time, making the mode $\mathcal{A}_h$ approximately constant in time.

The main cause of electric conductivity during inflation is the Schwinger effect of production of charged particle-antiparticle pairs from the vacuum. For fermionic particles with mass m and electric charge q , the conductivity calculated in the approximation of homogeneous and slowly changing electromagnetic fields is given by [5,14,25,33]

$$\sigma \simeq \frac{q^3}{6\pi^2} \frac{B}{H} \coth\left(\frac{\pi B}{E}\right) e^{-\pi m^2/qE} \geq \frac{q^3}{6\pi^3} \frac{E}{H} e^{-\pi m^2/qE}, \quad (19)$$

where we have used the inequality $x \coth x \geq 1$.

During inflation, the fermion mass m is determined by its Yukawa coupling and by the fluctuations of the Higgs field $\phi \sim H$, so that $m = \gamma H$, where $\gamma \sim 10^{-6}$ for the electron. For such values of the mass, the exponent in (19) can be neglected, and inequality (18) together with (19) leads to the constraint

$$E \lesssim \frac{6\pi^3}{q^3} I^2 H^2 \simeq 10^3 I^2 H^2. \quad (20)$$

where we have taken into account that $q \approx \sqrt{4\pi/137} \approx 0.3$ for the unit charge.

We will estimate the Schwinger conductivity σ by (19) even at post-inflationary stage. This can be justified, e.g., by integrating [14, Eq. (4.11)], although the real behavior of the Schwinger current, especially after inflation, may be more complicated [22,54] (see also discussion in Sect. 8). Then, upper bound of the form (20) is applied for the electric field at all stages of cosmological evolution.

6 Upper bound on the magnetic field

In summary, from (17) and (20), we have the following constraints on solution (16):

$$IE \lesssim M_{\text{P}} H \Rightarrow |Q_2| \frac{k^2 I_k}{a^2 I} \lesssim M_{\text{P}} H, \quad (21)$$

Fig. 3 The back-reaction (24) and Schwinger-effect (25) constraints on the integrand in the last expression of (16). The back-reaction curve changes its slope after the end of inflation

$$IB \lesssim M_P H \Rightarrow \frac{k^2}{a^2} \frac{I}{I_k} \left[|Q_1| + |Q_2| \int \left(\frac{I_k}{I} \right)^2 R_k dN \right] \lesssim M_P H, \quad (22)$$

$$E \lesssim 10^3 I^2 H^2 \Rightarrow |Q_2| \frac{k^2}{a^2} \frac{I_k}{I} \lesssim 10^3 I^3 H^2. \quad (23)$$

These constraints take into account only contribution to the fields from the modes around the specified wave number k ; they are quite conservative in this sense. The role of the whole spectrum will be discussed in Sect. 7.

Conditions (21) and (23) are equivalent to the following respective constraints for the integrand in the last expression of (16):

$$\left(\frac{I_k}{I} \right)^2 R_k \lesssim \frac{M_P^2}{|Q_2|^2 H^2} R_k^{-3}, \quad (24)$$

$$\left(\frac{I_k}{I} \right)^2 R_k \lesssim \frac{(10I_k)^{3/2}}{|Q_2|^{1/2}}. \quad (25)$$

It is notable that the right-hand side of (24) monotonically increases with time, while the right-hand side of (25) is constant. If the constant threshold on the right-hand side of (25) is reached by the integrand before the conductivity from reheating becomes significant, then it is the Schwinger effect that stops the growth of the mode $\mathcal{A}_h$ by creating plasma with sufficiently high conductivity. In the opposite case, it is electric conductivity from reheating that stops the growth of the mode. This is schematically illustrated in Fig. 3.

Scenarios without significant amplification of magnetic field outside the Hubble radius, under the condition of weak gauge coupling, lead to a low upper bound on magnetic field [10, 51].¹ Hence, we assume that the main contribution to the final amplitude $\mathcal{A}_h$ comes from the integral part in (16), and constraints on the constant Q_1 is of no interest to us. The integrand in (16) reaches its maximal value on the time interval before electric conductivity from reheating becomes significant (see Fig. 3), which gives an estimate for the final amplitude up to a factor of order unity:

$$\mathcal{A}_h \simeq \frac{Q_2}{I_k} \int \left(\frac{I_k}{I} \right)^2 R_k dN \sim \frac{Q_2}{I_k} \left[\left(\frac{I_k}{I} \right)^2 R_k \right]_{\max} = Q_2 I_k \left(\frac{R_k}{I^2} \right)_{\max}, \quad (26)$$

where the label ‘max’ indicates the moment of time where the maximum of the integrand is reached on the specified time interval. The integrand is bounded from above by non-decreasing functions of time (24) and

¹ Our papers [49, 50] contain a mistake overestimating the generated magnetic field in this case.

(25), whence we have

$$|\mathcal{A}_h| \lesssim \min \left\{ \frac{1}{|Q_2| I_k} \left(\frac{M_P^2}{H^2} R_k^{-3} \right)_r, 10^{3/2} (|Q_2| I_k)^{1/2} \right\}, \quad (27)$$

where the label ‘r’ denotes quantities evaluated at the moment of reheating, where the conductivity becomes very large because of the plasma created during reheating, and electromagnetic amplification stops completely.

To obtain the largest possible upper bound for $|\mathcal{A}_h|$, we need to maximize the right-hand side of (27) with respect to the product $|Q_2| I_k$. This maximal upper bound for $|\mathcal{A}_h|$ is reached for

$$|Q_2| I_k \simeq 10^{-1} \left[\left(\frac{M_P}{H} \right)^{4/3} R_k^{-2} \right]_r, \quad (28)$$

and is given by

$$|\mathcal{A}_h| \lesssim 10 \left[\left(\frac{M_P}{H} \right)^{2/3} R_k^{-1} \right]_r. \quad (29)$$

The remaining constraint (22) can be shown to imply $R_k|_r = k/a_r H_r \lesssim 1$, i.e., our estimates are valid for the modes that are outside the Hubble radius by the moment of reheating. Also note that the value of I is bounded by (24) and (28), so that

$$I \gtrsim |Q_2| I_k \frac{H}{M_P} R_k^2 \gtrsim |Q_2| I_k \left(\frac{H}{M_P} R_k^2 \right)_r \simeq 10^{-1} \left(\frac{M_P}{H_r} \right)^{1/3} \gtrsim 1 \quad (30)$$

provided $H_r \lesssim 10^{-3} M_P$, or $T_r \lesssim 10^{-3} M_P$, which is practically the largest possible reheating temperature. Thus, we are in a safe weak-coupling regime for the field.

Assuming the adiabatic law $B \propto a^{-2}$, the current strength of magnetic field on the scale of interest is estimated from (7) and (29) as

$$B_0 \simeq \frac{k^2}{2\sqrt{2}\pi a_0^2} |\mathcal{A}_h| \lesssim \left(\frac{k}{a_0} \right) \left(\frac{a_r}{a_0} \right) M_P^{2/3} H_r^{1/3} \simeq \frac{k T_0}{a_0} g_r^{-1/6} \left(\frac{M_P}{T_r} \right)^{1/3}, \quad (31)$$

where g_r is the number of relativistic degrees of freedom in thermal equilibrium after reheating ($g_r \approx 100$ in the Standard Model), T_r is the reheating temperature, and $T_0 = 2.34 \times 10^{-4}$ eV is the current temperature of relic photons. Substituting here the physical numbers, we get

$$B_0 \lesssim 10^{-30} g_r^{-1/6} \left(\frac{M_P}{T_r} \right)^{1/3} \frac{\text{Mpc}}{\lambda} \text{ G}. \quad (32)$$

It is instructive to see whether the Schwinger effect played a significant role in our upper bound. The final amplitude for the mode for fixed Q_2 and I_k is estimated by (26). Now, condition (24) implies the inequality $|Q_2| I_k \lesssim M_P I / H R_k^2|_{\text{max}}$, using which, we get

$$|\mathcal{A}_h| \lesssim \left(\frac{M_P}{I H} R_k^{-1} \right)_{\text{max}} \lesssim \left(\frac{M_P}{H} R_k^{-1} \right)_r, \quad (33)$$

where we have taken into account the weak-coupling constraint $I \gtrsim 1$. Condition (22) gives a weaker constraint compared to (33).

The result (33) is larger than (29) by a factor $10^{-1} (M_P/H_r)^{1/3}$, which is significant for low temperatures of reheating. The estimate for the current magnetic field in this case is

$$B_0 \lesssim 10^{-31} g_r^{-1/3} \frac{M_P}{T_r} \frac{\text{Mpc}}{\lambda} \text{ G}. \quad (34)$$

This estimate would be applicable if the Schwinger particle production were suppressed, e.g., by enhancement of the masses of charged particles (perhaps, by some non-trivial coupling to I), as discussed in [34].

7 The role of spectrum

In the previous section, we derived the weakest possible upper bounds, considering electromagnetic field only around one particular scale k . If we take into account that magnetogenesis, in fact, takes place on all spatial scales, then our estimates may be strengthened. Let us show that this happens only mildly, i.e., that estimate (32) can be saturated by order of magnitude. We demonstrate this by constructing a scenario in which the spectrum for electric field remains to be peaked around one particular scale k_m .

The fastest possible growth of the integrand in (16) compatible with the back-reaction constraint (21) during inflation occurs under the law $I \propto a^{-2} \propto \eta^2$. The solution for the mode is then

$$\mathcal{A}_h \propto \int \frac{d\eta}{I^2} = C_1 + C_2 a^3. \quad (35)$$

The fields in the growing mode evolve as $E \propto a^2$ and $B \propto E/a \propto a$, growing with time.

Consider then scenario (35) with preliminary amplification as described in Sect. 3. This evolution of $\mathcal{A}_h$ takes place for all super-Hubble modes starting from the moment η_m of the beginning of evolution of I , which we take to be the time of Hubble-radius crossing of the mode with wavenumber k_m . As regards the modes with $k \gtrsim k_m$, the amplitude $|I\mathcal{A}_h|$ remains constant till the Hubble-radius crossing, where $a_k = k/H$, so that $\mathcal{A}_h \propto 1/I \propto a^2$, after which it evolves as (35). By matching the initial conditions at $a_m = a(\eta_m)$, we can obtain an interpolated solution for all k in the form [51]

$$\mathcal{A}_h(\eta, k) \simeq \mathcal{A}_h(\eta_m, k) \left[1 + \frac{k/k_m}{3 + (k/k_m)^2} \left(\frac{aH}{k_m} \right)^3 \right], \quad (36)$$

$$\frac{\mathcal{A}'_h(\eta, k)}{k} \simeq \mathcal{A}_h(\eta_m, k) \frac{1}{1 + k^2/3k_m^2} \left(\frac{aH}{k_m} \right)^4. \quad (37)$$

The spectral densities during inflation are then estimated as

$$\mathcal{P}_B = \frac{d\langle B^2 \rangle}{d \ln k} \simeq \frac{k^4}{8\pi^2 a^4} \left[1 + \frac{k/k_m}{3 + (k/k_m)^2} \left(\frac{aH}{k_m} \right)^3 \right]^2 \sum_h |\mathcal{A}_h(\eta_m, k)|^2, \quad (38)$$

$$\mathcal{P}_E = \frac{d\langle E^2 \rangle}{d \ln k} \simeq \frac{k^4}{8\pi^2 a^4} \frac{1}{(1 + k^2/3k_m^2)^2} \left(\frac{aH}{k_m} \right)^8 \sum_h |\mathcal{A}_h(\eta_m, k)|^2. \quad (39)$$

Due to the strong narrow peak of $Q(k) \equiv |I_k \mathcal{A}_h(\eta_k, k)| \approx (1 + n_k)^{1/2}$ [see Eqs. (13) and (14)] at the Hubble-radius crossing, the spectrum for electric field remains to be peaked at $k = k_m$ and is flat in the region $k \gg k_m$. Magnetic field is suppressed on all relevant scales $k \lesssim aH$ with respect to the electric one. This, then, does not strengthen much our estimates made in the previous section.

We also remark that, although our upper bounds (29) and (33) are independent of the value of $Q(k)$ at that particular wavenumber, the role of initial amplification inside the Hubble radius is significant, because it shapes the initial power spectrum, allowing it to be peaked at a given wavenumber and thus making our upper bounds virtually the least upper bounds.

8 Summary and discussion

Following our work [51], we have shown the existence of an upper bound [see (32)]

$$B_0 \lesssim 10^{-30} g_r^{-1/6} \left(\frac{M_P}{T_r} \right)^{1/3} \frac{\text{Mpc}}{\lambda} \text{G} \quad (40)$$

for the outcome of inflationary and post-inflationary magnetogenesis based on Lagrangian (1) under weak coupling $I \gtrsim 1$. This upper bound comes from the constraints imposed by the consideration of back-reaction and Schwinger effect. Our upper bound is quite conservative, taking into account only electromagnetic fields on a particular spatial scale of interest, and can only be lowered by considering the whole power spectrum. Still, we have shown in Sect. 7 that there exists a scenario in which the power spectrum is peaked around the

spatial scale of interest, which makes our bound, in some sense, virtually the least upper bound. Our model-independent result (40) is consistent with the estimates obtained in the literature for specific models (e.g., in [33, 34]).

The upper bound (40) allows one to appreciate the difficulty in producing sufficiently large magnetic fields on large spatial scales. Indeed, the lowest possible temperature of reheating lies in the MeV range [24], and even for such low temperature, one gets $B_0 \lesssim 10^{-23}$ G on the megaparsec scale. Estimate (40) is then valid for comoving spatial scales larger than the comoving Hubble radius at this temperature, i.e., for $\lambda \gtrsim 100$ pc.

In our treatment, we used the simple expression (19) for the electric conductivity induced by the Schwinger effect at the post-inflationary stage. Hoping that it gives correct estimates by order of magnitude, we should point out that recent elaborate calculations in the kinetic approach reveal an oscillatory and non-Markovian character of the electric current and field after inflation, failing to satisfy the simple Ohm's law and decaying slower than in the case of naïve application of (19) [22, 54]. This may call for revisions of our estimates in more refined treatments of the Schwinger effect.

If the Schwinger effect for some reason is suppressed in the early universe (e.g., because the masses of charged particles are sufficiently enlarged by some non-trivial mechanism), then only back-reaction is important, and our estimate for the final magnetic field becomes [see (34)]

$$B_0 \lesssim 10^{-31} g_r^{-1/3} \frac{M_P}{T_r} \frac{\text{Mpc}}{\lambda} \text{ G}. \quad (41)$$

This would allow one to obtain $B_0 \sim 10^{-15}$ G on the megaparsec scale by choosing $T_r \sim 10^2$ GeV. In some specific scenarios, generation of magnetic fields of this order of magnitude requires even lower reheating temperatures, of the order of GeV [17, 18].

Assuming the law $B \propto a^{-2}$ in the hot universe, we thereby neglected the possible chiral magnetohydrodynamics effects [4, 7–9, 20, 21, 26, 30, 40, 42, 44, 45, 57, 58] that may modify the evolution of the power spectrum. These effects are model-dependent and are not likely to modify our bounds significantly on large comoving spatial scales ($\lambda \sim \text{Mpc}$). For instance, the chiral magnetic effect [63] in the presence of chiral asymmetry in the charged-fermion distribution leads to the scaling behavior [58] $\lambda_c^5 B_c^9 \approx \text{const}$, $\lambda_c \propto a^{1/2}$, for the comoving correlation length $\lambda_c = \lambda/a$ of the comoving magnetic field $B_c = a^2 B$, so the product $B\lambda$ has an extra growth factor $a^{2/9}$, and estimate (40) is replaced by

$$B_0 \lesssim \left(\frac{T_i}{T_f} \right)^{2/9} 10^{-30} g_r^{-1/6} \left(\frac{M_P}{T_r} \right)^{1/3} \frac{\text{Mpc}}{\lambda} \text{ G}, \quad (42)$$

where T_i is the temperature of the beginning of this process, and $T_f \sim 100$ MeV is the temperature of its end. For characteristic temperatures $T_i \sim 10$ TeV, this gives an extra factor of 10 in the bound.

We did not take into account the contribution of electromagnetic field to the primordial power spectrum of energy-density fluctuations, but the corresponding constraints on the magnetic field [36] appear to be much weaker than those discussed in this paper.

Finally, we note that primordial helical hypermagnetic fields may also be responsible for generating baryon asymmetry of the universe [1, 3, 13, 16, 27–29]. This imposes a typical constraint [13, 29]

$$B_0 \lesssim 10^{-21} \left(\frac{\text{Mpc}}{\lambda} \right)^{1/2} \text{ G} \quad (43)$$

on the admissible values of B_0 on the spatial scale λ for the fields of *maximal* helicity that existed prior to the electroweak transition. This constraint is weaker than our upper bound (40) for not very small spatial scales.

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