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Certain results on invariant submanifolds of an almost Kenmotsu (κ, μ, ν) -space

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Abstract In the present paper, we study invariant submanifolds of almost Kenmotsu structures whose Riemannian curvature tensor has (κ, μ, ν) -nullity distribution. Since the geometry of an invariant submanifold inherits almost all properties of the ambient manifold, we research how the functions κ, μ and ν behave on the submanifold. In this connection, necessary and sufficient conditions are investigated for an invariant submanifold of an almost Kenmotsu (κ, μ, ν) -space to be totally geodesic under some conditions.

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1 Introduction

It is well known that a $(2n + 1)$ -dimensional contact metric manifold $\tilde{M}$ admits an almost contact metric structure (ϕ, ξ, η, g) , i.e., it admits a global vector field ξ , called the characteristic vector field or the Reeb vector field, its dual is η , a tensor ϕ of type $(1, 1)$ and the Riemannian metric tensor g such that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad (1)$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (2)$$

for all $X, Y \in \Gamma(T\tilde{M})$, where $\Gamma(T\tilde{M})$ denotes the set of differentiable vector fields on $\tilde{M}$ [2].

The manifold $\tilde{M}$ together with the structure tensor (ϕ, ξ, η, g) is called a contact metric manifold and we will denote it by $\tilde{M}^{(2n+1)}(\phi, \xi, \eta, g)$ in the rest of this paper.

By $\tilde{\nabla}$, we denote the Levi-Civita connection of g , then the Riemannian curvature tensor of $\tilde{R}$ of $\tilde{M}^{2n+1}(\phi, \xi, \eta, g)$ is given by

$$\tilde{R}(X, Y) = \tilde{\nabla}_X \tilde{\nabla}_Y - \tilde{\nabla}_Y \tilde{\nabla}_X - \tilde{\nabla}_{[X, Y]},$$

for all $X, Y \in \Gamma(T\tilde{M})$.

On the other hand, we define the tensor field $(1, 1)$ -type by h

$$2hX = (\ell_\xi \phi)X,$$

for all $X \in \Gamma(T\tilde{M})$, where ℓ_ξ is the Lie-derivative in the direction of ξ . Then the tensor field h is self-adjoint and satisfies

$$\phi h + h\phi = 0, \quad \text{tr} h = \text{tr} \phi h = 0, \quad h\xi = 0. \quad (3)$$

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We have also these formulas for a contact metric manifold

$$\tilde{\nabla}_X \xi = \phi X - \phi h X, \quad \tilde{\nabla}_\xi \phi = 0. \quad (4)$$

A contact metric manifold for which ξ is Killing vector field is called a K-contact manifold. It is well known that a contact metric manifold is K-contact iff $h = 0$.

The (κ, μ) -nullity distribution of a contact metric manifold $\tilde{M}^{2n+1}(\phi, \xi, \eta, g)$ for the pair $(\kappa, \mu) \in \mathbb{R}^2$ is distribution

$$\begin{aligned} \tilde{M}(\kappa, \mu) : p &\longrightarrow \tilde{M}_p(\kappa, \mu) = \{Z_p \in T_{\tilde{M}}(p) : \\ \tilde{R}(X, Y)Z &= \kappa\{g(Y, Z)X - g(X, Z)Y\} + \mu\{g(Y, Z)hX - g(X, Z)hY\}, \end{aligned}$$

for all $X, Y \in \Gamma(T\tilde{M})$. So if the characteristic vector field ξ belongs to the (κ, μ) -nullity distribution, then

$$\tilde{R}(X, Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\},$$

and the manifold $\tilde{M}^{2n+1}(\phi, \xi, \eta, g)$ is called (κ, μ) -contact metric manifold. If κ and μ are non-constant smooth functions on $\tilde{M}$, then the manifold $\tilde{M}^{2n+1}(\phi, \xi, \eta, g)$ is called generalized (κ, μ) -contact metric manifold [3].

Going beyond generalized (κ, μ) -space, T. Koufogiorgos, M. Markellos and V. J. Papantoniou introduced in [2] the notion of (κ, μ, ν) -contact metric manifold, its Riemannian curvature tensor $\tilde{R}$ is given by

$$\begin{aligned} \tilde{R}(X, Y)\xi &= \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\} \\ &\quad + \nu\{\eta(Y)\phi hX - \eta(X)\phi hY\}, \end{aligned} \quad (5)$$

for all $X, Y \in \Gamma(T\tilde{M})$, where κ, μ, ν are smooth functions on $\tilde{M}^{2n+1}$.

It is well known that an almost contact metric manifold is an almost Kenmotsu if $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$, where $\Phi(X, Y) = g(X, \phi Y)$ is the fundamental 2-form of $\tilde{M}^{2n+1}$. If an almost Kenmotsu manifold $\tilde{M}^{2n+1}(\phi, \eta, \xi, g)$ has a (κ, μ, ν) -nullity distribution, it is called an almost Kenmotsu (κ, μ, ν) -space [5].

Proposition 1.1 *Given $\tilde{M}^{2n+1}(\phi, \eta, \xi, g)$ an almost Kenmotsu (κ, μ, ν) -space, then*

$$h^2 = (\kappa + 1)\phi^2, \quad \kappa \leq -1, \quad (6)$$

$$\xi(\kappa) = 2(\kappa + 1)(\nu - 2) \quad (7)$$

$$(\tilde{\nabla}_X \phi)Y = g(\phi X + hX, Y)\xi - \eta(Y)(\phi X + hX) \quad (8)$$

$$\tilde{\nabla}_X \xi = -\phi^2 X - \phi h X \quad (9)$$

$$S(X, \xi) = 2n\kappa\eta(X) \quad (10)$$

$$\begin{aligned} \tilde{R}(\xi, X)Y &= \kappa\{g(X, Y)\xi - \eta(Y)X\} + \mu\{g(hX, Y)\xi - \eta(Y)hX\} \\ &\quad + \nu\{g(\phi hX, Y)\xi - \eta(Y)\phi hX\}. \end{aligned} \quad (11)$$

They proved that this type of manifold is intrinsically related to the harmonicity of the Reeb vector on contact metric 3-manifolds. Some authors have studied manifolds satisfying condition (5) but a non-contact metric structure. In this connection, P. Dacko and Z. Olszak defined an almost cosymplectic (κ, μ, ν) -space as an almost cosymplectic manifold that satisfies (5), but with κ, μ and ν functions varying exclusively in the direction of ξ in [6]. Later examples have been given for this type of manifold [7].

In modern analysis, the geometry of submanifolds has become a subject of growing interest for its significant applications in applied mathematics and theoretical physics. For instance, the notion of invariant submanifold is used to discuss properties of non-linear autonomous systems. Also, the notion of geodesic plays an important role in the theory of relativity. For totally geodesic submanifolds, the geodesics of the ambient manifolds remain geodesics in the submanifolds. Hence, totally geodesic submanifolds have also importance in mathematics as well as in physical sciences. There have been several papers on contact metric manifolds which admit covector field ξ tangent to the submanifold. In this connection, we refer to [11, 12, 14, 16].

Now, in this part of the study, we are especially interested in an invariant submanifold of an almost Kenmotsu (κ, μ, ν) -space to be pseudoparallel, 2-pseudoparallel, Ricci-generalized pseudoparallel and 2-generalized pseudoparallel.

Pseudoparallel submanifolds have been studied in different structures and working on [8–10]. On the other hand, the study of the geometry of invariant submanifolds was introduced by Bejancu and Papaghuic [10]. In general, the geometry of an invariant submanifold inherits almost all properties of the ambient manifold.

In the present paper, we generalize the ambient space and investigate the conditions under which invariant pseudoparallel submanifolds of an almost Kenmotsu (κ, μ, ν) -space are totally geodesic.

Now, let M be an immersed submanifold of an almost Kenmotsu (κ, μ, ν) -space $\tilde{M}^{2n+1}$. By $\Gamma(TM)$ and $\Gamma(T^\perp M)$, we denote the tangent and normal subspaces of M in $\tilde{M}$. Then the Gauss and Weingarten formulae are, respectively, given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \quad (12)$$

and

$$\tilde{\nabla}_X V = -A_V X + \nabla_X^\perp V, \quad (13)$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, where ∇ and $\nabla^\perp$ are the induced connections on M and $\Gamma(T^\perp M)$ and σ and A are called the second fundamental form and shape operator of M , respectively, $\Gamma(TM)$ denotes the set differentiable vector fields on M . They are related by

$$g(A_V X, Y) = g(\sigma(X, Y), V). \quad (14)$$

The first covariant derivative of the second fundamental form σ is defined by

$$(\tilde{\nabla}_X \sigma)(Y, Z) = \nabla_X^\perp \sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z), \quad (15)$$

for all $X, Y, Z \in \Gamma(TM)$. If $\tilde{\nabla} \sigma = 0$, then the submanifold is said to be its second fundamental form is parallel.

By R , we denote the Riemannian curvature tensor of the submanifold M , we have the following Gauss equation

$$\begin{aligned} \tilde{R}(X, Y)Z &= R(X, Y)Z + A_{\sigma(X, Z)}Y - A_{\sigma(Y, Z)}X + (\tilde{\nabla}_X \sigma)(Y, Z) \\ &\quad - (\tilde{\nabla}_Y \sigma)(X, Z), \end{aligned} \quad (16)$$

for all $X, Y, Z \in \Gamma(TM)$.

$\tilde{R} \cdot \sigma$ is given by

$$\begin{aligned} (\tilde{R}(X, Y) \cdot \sigma)(U, V) &= R^\perp(X, Y)\sigma(U, V) - \sigma(R(X, Y)U, V) \\ &\quad - \sigma(U, R(X, Y)V), \end{aligned} \quad (17)$$

where the Riemannian curvature tensor of normal bundle $\Gamma(T^\perp M)$ is given

$$R^\perp(X, Y) = [\nabla_X^\perp, \nabla_Y^\perp] - \nabla_{[X, Y]}^\perp$$

On the other hand, the concircular curvature tensor for Riemannian manifold (M^{2n+1}, g) is given by

$$\begin{aligned} \mathcal{C}(X, Y)Z &= \tilde{R}(X, Y)Z \\ &\quad - \frac{\tau}{2n(2n+1)}\{g(Y, Z)X - g(X, Z)Y\}, \end{aligned} \quad (18)$$

where τ denotes the scalar curvature of M .

Similarly, the tensor $\mathcal{C} \cdot \sigma$ is defined by

$$\begin{aligned} (\mathcal{C}(X, Y) \cdot \sigma)(U, V) &= R^\perp(X, Y)\sigma(U, V) - \sigma(\mathcal{C}(X, Y)U, V) \\ &\quad - \sigma(U, \mathcal{C}(X, Y)V), \end{aligned} \quad (19)$$

for all $X, Y, U, V \in \Gamma(TM)$.

For a $(0, k)$ -type tensor field T , $k \geq 1$ and a $(0, 2)$ -type tensor field A on a Riemannian manifold (M, g) , $Q(A, T)$ -tensor field is defined by

$$\begin{aligned} Q(A, T)(X_1, X_2, \dots, X_k; X, Y) &= -T((X \wedge_A Y)X_1, X_2, \dots, X_k) \\ &\quad \dots - T(X_1, X_2, \dots, X_{k-1}, (X \wedge_A Y)X_k), \end{aligned} \quad (20)$$

for all $X_1, X_2, \dots, X_k, X, Y \in \Gamma(TM)$ [8], where

$$(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y. \quad (21)$$

Definition 1.2 Let M be a submanifold of a Riemannian manifold $(\tilde{M}, g)$. If there exist functions L_1, L_2, L_3 and L_4 on $\tilde{M}$ such that

$$\tilde{R} \cdot \sigma = L_1 Q(g, \sigma), \quad (22)$$

$$\tilde{R} \cdot \tilde{\nabla} \sigma = L_2 Q(g, \tilde{\nabla} \sigma), \quad (23)$$

$$\tilde{R} \cdot \sigma = L_3 Q(S, \sigma) \quad (24)$$

$$\tilde{R} \cdot \tilde{\nabla} \sigma = L_4 Q(S, \tilde{\nabla} \sigma), \quad (25)$$

then M is, respectively, pseudoparallel, 2-pseudoparallel, Ricci-generalized pseudoparallel and 2-Ricci-generalized pseudoparallel submanifold. In particular, if $L_1 = 0$ (resp., $L_2 = 0$), then M is said to be semiparallel (resp. 2-semiparallel) [9].

Kowalczyk studied the semi-Riemannian manifolds satisfying $Q(S, R) = 0$ and $Q(S, g) = 0$ [17]. Also, De and Majhi investigated the invariant submanifolds of Kenmotsu manifolds and showed that geometric conditions of invariant submanifolds of Kenmotsu manifolds are totally geodesic [13]. Recently, Hu and Wang obtained the geometric conditions of invariant submanifolds of a trans-Sasakian manifold to be totally geodesic [15]. Furthermore, the geometry of invariant submanifolds of different manifolds was studied by many geometers [8, 9, 11–16].

Motivated by the above studies, we make an attempt to study the invariant submanifolds of an almost Kenmotsu (κ, μ, ν) -space satisfying some geometric conditions such as $\tilde{R} \cdot \sigma = L_1 Q(g, \sigma)$, $\tilde{R} \cdot \tilde{\nabla} \sigma = L_2 Q(g, \tilde{\nabla} \sigma)$, $\tilde{R} \cdot \sigma = L_3 Q(S, \sigma)$ and $\tilde{R} \cdot \tilde{\nabla} \sigma = L_4 Q(S, \tilde{\nabla} \sigma)$.

Finally, we show that the submanifold is either totally geodesic or the functions L_i, κ, μ and ν functions are restricted.

2 Invariant Submanifolds of an almost Kenmotsu (κ, μ, ν) -Space

Now, let $\tilde{M}^{2n+1}(\phi, \xi, \eta, g)$ be an almost Kenmotsu (κ, μ, ν) -space and M be an immersed submanifold of $\tilde{M}^{2n+1}$. If $\phi(T_x M) \subseteq T_x M$, for each point at $x \in M$, then M is said to be an invariant submanifold of $\tilde{M}^{2n+1}(\phi, \xi, \eta, g)$ with respect to ϕ . From (3), one can easily see that an invariant submanifold with respect to ϕ is also invariant with respect to h .

Proposition 2.1 Let M be an invariant submanifold of an almost Kenmotsu (κ, μ, ν) -space $\tilde{M}^{2n+1}(\phi, \xi, \eta, g)$ such that ξ is tangent to M . Then the following equalities hold on M ;

$$R(X, Y)\xi = \kappa[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY] + \nu[\eta(Y)\phi hX - \eta(X)\phi hY] \quad (26)$$

$$(\nabla_X \phi)Y = g(\phi X + hX, Y)\xi - \eta(Y)(\phi X + hX) \quad (27)$$

$$\nabla_X \xi = -\phi^2 X - \phi hX \quad (28)$$

$$\phi \sigma(X, Y) = \sigma(\phi X, Y) = \sigma(X, \phi Y), \quad \sigma(X, \xi) = 0, \quad (29)$$

where ∇, σ and R denote the induced Levi-Civita connection on M , the shape operator and Riemannian curvature tensor of M , respectively.

Proof We omit the proof as it is a result of direct calculations. $\square$

In the rest of this paper, we will assume that M is an invariant submanifold of an almost Kenmotsu (κ, μ, ν) -space $\tilde{M}^{2n+1}(\phi, \xi, \eta, g)$. In this case, from (5), we have

$$\phi hX = -h\phi X, \quad (30)$$

for all $X \in \Gamma(TM)$, that is, M is also invariant with respect to the tensor field h .

We need the following lemma to guarantee that the second fundamental form σ is not always identically zero.

Lemma 2.2 Let M be an invariant submanifold of an almost Kenmotsu (κ, μ, ν) -space $\tilde{M}^{2n+1}(\phi, \xi, \eta, g)$. Then the second fundamental form σ of M is parallel if and only if M is totally geodesic provided $\kappa \neq 0$.

Proof Let us suppose that σ is parallel. From (16), we have

$$(\tilde{\nabla}_X \sigma)(Y, Z) = \nabla_X^\perp \sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z) = 0, \quad (31)$$

for all vector fields X, Y and Z on M^{2n+1} . Setting $Z = \xi$ in (31) and taking into account (28) and (29), we have

$$\sigma(\nabla_X \xi, Y) = -\sigma(\phi^2 X + \phi hX, Y) = 0,$$

that is,

$$\sigma(X, Y) - \phi\sigma(hX, Y) = 0. \quad (32)$$

Writing hX of X in (30) and using (7) and (27), we obtain

$$\begin{aligned} \sigma(hX, Y) - \phi\sigma(h^2 X, Y) &= 0, \\ \sigma(hX, Y) + (1 + \kappa)\phi\sigma(X, Y) &= 0. \end{aligned} \quad (33)$$

From (32) and (33), we conclude that $\kappa\sigma(X, Y) = 0$, which proves our assertion. The converse is obvious. $\square$

Theorem 2.3 *Let M be an invariant submanifold of an almost Kenmotsu (κ, μ, ν) -space $\tilde{M}^{2n+1}(\phi, \eta, \xi, g)$. If M is a pseudoparallel submanifold of an almost Kenmotsu (κ, μ, ν) -space $\tilde{M}^{2n+1}(\phi, \eta, \xi, g)$, then M is either totally geodesic or the function L_1 satisfies*

$$L_1 = \kappa \mp \sqrt{(\kappa + 1)(\nu^2 - \mu^2)}, \quad \mu \cdot \nu(\kappa + 1) = 0. \quad (34)$$

Proof Let M be an invariant pseudoparallel submanifold of an almost Kenmotsu (κ, μ, ν) -space $\tilde{M}^{2n+1}(\phi, \eta, \xi, g)$. That means

$$L_1 Q(g, \sigma)(U, V; X, Y) = (\tilde{R}(X, Y) \cdot \sigma)(U, V),$$

for all $X, Y, U, V \in \Gamma(TM)$, which implies that

$$\begin{aligned} -L_1 \{\sigma((X \wedge_g Y)U, V) + \sigma(U, (X \wedge_g Y)V)\} &= R^\perp(X, Y)\sigma(U, V) \\ -\sigma(R(X, Y)U, V) - \sigma(U, R(X, Y)V). \end{aligned}$$

Substituting $X = U = \xi$ in the last equality, and taking into account Proposition 2.1, we have

$$L_1 \sigma(Y, V) = -\sigma(R(\xi, Y)\xi, V) = \kappa\sigma(Y, V) + \mu\sigma(hY, V) + \nu\sigma(\phi hY, V),$$

that is,

$$(L_1 - \kappa)\sigma(Y, V) = \mu\sigma(hY, V) + \nu\phi\sigma(hY, V). \quad (35)$$

Substituting hY for Y in (35), by view of (6) and (29), we have

$$\begin{aligned} (L_1 - \kappa)\sigma(hY, V) &= \mu\sigma(h^2 Y, V) + \nu\phi\sigma(h^2 Y, V) \\ &= -(\kappa + 1)[\mu\sigma(Y, V) + \nu\phi\sigma(Y, V)]. \end{aligned} \quad (36)$$

From (35) and (36), one can easily see that

$$[(\kappa + 1)(\mu^2 - \nu^2) + (L_1 - \kappa)^2]\sigma(Y, V) + 2(\kappa + 1)\mu\nu\phi\sigma(Y, V) = 0.$$

This tell us that M is either totally geodesic submanifold or

$$(\kappa + 1)(\mu^2 - \nu^2) + (L_1 - \kappa)^2 = \mu\nu(\kappa + 1) = 0, \quad (37)$$

which is equivalent to (34). $\square$

From Theorem 2.3, we have the following corollary.

Corollary 2.4 *Let M be an invariant submanifold of an almost Kenmotsu (κ, μ, ν) -space $\tilde{M}^{2n+1}(\phi, \eta, \xi, g)$. M is a semiparallel submanifold if and only if M is totally geodesic provided*

$$(\kappa + 1)(\mu^2 - \nu^2) + \kappa^2 \neq 0 \text{ or } \mu\nu(\kappa + 1) \neq 0.$$

Theorem 2.5 *Let M be an invariant submanifold of an almost Kenmotsu (κ, μ, ν) -space $\tilde{M}^{2n+1}(\phi, \eta, \xi, g)$. If M is a Ricci-generalized pseudoparallel submanifold of an almost Kenmotsu (κ, μ, ν) -space $\tilde{M}^{2n+1}(\phi, \eta, \xi, g)$, then M is either totally geodesic or the function L_3 satisfies*

$$L_3 = \frac{1}{2n} \left[1 \mp \frac{1}{\kappa} \sqrt{(\kappa + 1)(\nu^2 - \mu^2)} \right], \quad \mu \cdot \nu = 0. \quad (38)$$

Proof Since M is an invariant Ricci-generalized pseudoparallel, there exists a function L_3 on M such that

$$(\tilde{R}(X, Y) \cdot \sigma)(U, V) = L_3 Q(S, \sigma)(U, V; X, Y),$$

for all $X, Y, U, V \in \Gamma(TM)$. This yields to

$$\begin{aligned} -L_3 \{ \sigma((X \wedge_S Y)U, V) + \sigma(U, (X \wedge_S Y)V) \} &= R^\perp(X, Y)\sigma(U, V) \\ -\sigma(R(X, Y)U, V) - \sigma(U, R(X, Y)V). \end{aligned} \quad (39)$$

Expanding by (39) and inserting $X = V = \xi$, making use of (29) and Proposition 2.1, we obtain

$$\begin{aligned} 2n\kappa L_3 \sigma(U, Y) &= -\sigma(R(\xi, Y)\xi, U) = \kappa \sigma(Y, U) + \mu \sigma(hY, U) \\ &\quad + \nu \phi \sigma(hY, U), \end{aligned}$$

that is,

$$\kappa(2nL_3 - 1)\sigma(U, Y) = \mu \sigma(hY, U) + \nu \phi \sigma(hY, U). \quad (40)$$

If hY is taken instead of Y in (40) and by virtue of (6) and Proposition 2.1, we reach at

$$\kappa(2nL_3 - 1)\sigma(U, hY) = -(\kappa + 1)[\mu \sigma(hY, U) + \nu \phi \sigma(hY, U)]. \quad (41)$$

From (40) and (41), we conclude that

$$[\kappa^2(2nL_3 - 1)^2 + (\kappa + 1)(\mu^2 - \nu^2)]\sigma(U, V) + 2\mu\nu\phi\sigma(U, V) = 0.$$

This completes the proof. $\square$

Theorem 2.6 *Let M be an invariant submanifold of an almost Kenmotsu (κ, μ, ν) -space $\tilde{M}^{2n+1}(\phi, \eta, \xi, g)$. If M is a 2-pseudoparallel submanifold of $\tilde{M}^{2n+1}(\phi, \eta, \xi, g)$, then M is either totally geodesic or the function L_2 satisfies the condition*

$$L_2 = \kappa \mp \sqrt{(\kappa + 1)(\nu^2 - \mu^2)}, \quad \mu\nu(\kappa + 1) = 0. \quad (42)$$

Proof If M is an invariant 2-pseudoparallel of an almost Kenmotsu (κ, μ, ν) -space $\tilde{M}^{2n+1}(\phi, \eta, \xi, g)$, then there exists a function L_2 such that

$$L_2 Q(g, \tilde{\nabla}\sigma)(U, V, Z; X, Y) = (\tilde{R}(X, Y) \cdot \tilde{\nabla}\sigma)(U, V, Z),$$

for all $X, Y, U, V, Z \in \Gamma(TM)$, which implies that

$$\begin{aligned} -L_2 \{ (\tilde{\nabla}_{(X \wedge_g Y)U}\sigma)(V, Z) + (\tilde{\nabla}_U\sigma)((X \wedge_g Y)V, Z) \\ + (\tilde{\nabla}_U\sigma)(V, (X \wedge_g Y)Z) \} &= R^\perp(X, Y)(\tilde{\nabla}_U\sigma)(V, Z) \\ -(\tilde{\nabla}_{R(X, Y)U}\sigma)(V, Z) - (\tilde{\nabla}_U\sigma)(R(X, Y)V, Z) \\ -(\tilde{\nabla}_U\sigma)(V, R(X, Y)Z). \end{aligned} \quad (43)$$

Here, substituting ξ for $X = V$ in (43), we have

$$\begin{aligned} & -L_2\{(\tilde{\nabla}_{(\xi \wedge_g Y)U}\sigma)(\xi, Z) + (\tilde{\nabla}_U\sigma)((\xi \wedge_g Y)\xi, Z) + (\tilde{\nabla}_U\sigma)(\xi, (\xi \wedge_g Y)Z)\} \\ & = R^\perp(\xi, Y)(\tilde{\nabla}_U\sigma)(\xi, Z) - (\tilde{\nabla}_{R(\xi, Y)U}\sigma)(\xi, Z) \\ & \quad - (\tilde{\nabla}_U\sigma)(R(\xi, Y)\xi, Z) - (\tilde{\nabla}_U\sigma)(\xi, R(\xi, Y)Z). \end{aligned} \quad (44)$$

Now we will calculate them separately. In view of (15), (21), (28) and (29), we can derive

$$\begin{aligned} (\tilde{\nabla}_{(\xi \wedge_g Y)U}\sigma)(\xi, Z) & = -\sigma(\nabla_{(\xi \wedge_g Y)U}\xi, Z) \\ & = \sigma(\phi^2(\xi \wedge_g Y)U + \phi h(\xi \wedge_g Y)U, Z) \\ & = -\sigma(g(Y, U)\xi - \eta(U)Y, Z) - \eta(U)\sigma(\phi hY, Z) \\ & = \eta(U)\{\sigma(Y, Z) - \sigma(\phi hY, Z)\}. \end{aligned} \quad (45)$$

In the same way,

$$\begin{aligned} (\tilde{\nabla}_U\sigma)((\xi \wedge_g Y)\xi, Z) & = (\tilde{\nabla}_U\sigma)(\eta(Y)\xi - Y, Z) \\ & = -\sigma(\nabla_U\eta(Y)\xi, Z) - (\tilde{\nabla}_U\sigma)(Y, Z) \\ & = -\sigma(U[\eta(Y)]\xi + \eta(Y)\nabla_U\xi, Z) - (\tilde{\nabla}_U\sigma)(Y, Z) \\ & = \eta(Y)\sigma(\phi^2U + \phi hU, Z) - (\tilde{\nabla}_U\sigma)(Y, Z) \\ & = \eta(Y)\{\sigma(\phi hU, Z) - \sigma(U, Z)\} - (\tilde{\nabla}_U\sigma)(Y, Z), \end{aligned} \quad (46)$$

$$\begin{aligned} (\tilde{\nabla}_U\sigma)(\xi, (\xi \wedge_g Y)Z) & = -\sigma(\nabla_U\xi, (\xi \wedge_g Y)Z) \\ & = \sigma(\phi^2U + \phi hU, g(Y, Z)\xi - \eta(Z)Y) \\ & = \eta(Z)\{\sigma(U, Y) - \sigma(\phi hU, Y)\}. \end{aligned} \quad (47)$$

For the right hand side of (44), by view of Proposition 2.1, (15) and (17), we obtain

$$\begin{aligned} R^\perp(\xi, Y)(\tilde{\nabla}_U\sigma)(\xi, Z) & = R^\perp(\xi, Y)\{\nabla_U^\perp\sigma(\xi, Z) - \sigma(\nabla_U\xi, Z) - \sigma(\xi, \nabla_UZ)\} \\ & = -R^\perp(\xi, Y)\sigma(\nabla_U\xi, Z) = R^\perp(\xi, Y)\sigma(\phi^2U + \phi hU, Z) \\ & = R^\perp(\xi, Y)\{\sigma(\phi hU, Z) - \sigma(U, Z)\}. \end{aligned} \quad (48)$$

Also, making use of (3) and (26), we derive

$$\begin{aligned} (\tilde{\nabla}_{R(\xi, Y)U}\sigma)(\xi, Z) & = -\sigma(\nabla_{R(\xi, Y)U}\xi, Z) = \sigma(\phi^2R(\xi, Y)U + \phi hR(\xi, Y)U, Z) \\ & = -\sigma(-\kappa\eta(U)Y - \mu\eta(U)hY - v\eta(U)\phi hY, Z) \\ & \quad + \sigma(-\kappa\eta(U)\phi hY - \mu\eta(U)\phi h^2Y - v\eta(U)\phi h\phi hY, Z) \\ & = \eta(U)\{\kappa\sigma(Y, Z) + \mu\sigma(hY, Z) + v\phi\sigma(hY, Z) \\ & \quad - \kappa\phi\sigma(hY, Z) + (\kappa + 1)\mu\phi\sigma(Y, Z) + v(\kappa + 1)\sigma(Y, Z)\}, \end{aligned} \quad (49)$$

$$\begin{aligned} (\tilde{\nabla}_U\sigma)(R(\xi, Y)\xi, Z) & = (\tilde{\nabla}_U\sigma)(\kappa[\eta(Y)\xi - Y] - \mu hY - v\phi hY, Z). \end{aligned} \quad (50)$$

Finally,

$$\begin{aligned} (\tilde{\nabla}_U\sigma)(\xi, R(\xi, Y)Z) & = -\sigma(\nabla_U\xi, R(\xi, Y)Z) = \sigma(\phi^2U + \phi hU, R(\xi, Y)Z) \\ & = -\sigma(-\kappa\eta(Z)Y - \mu\eta(Z)hY - v\eta(Z)\phi hY, U) \\ & \quad + \sigma(-\kappa\eta(Z)\phi hY - \mu\eta(Z)\phi h^2Y - v\eta(Z)\phi h\phi hY, U) \\ & = \eta(Z)\{\kappa\sigma(Y, U) + \mu\sigma(hY, U) + v\phi\sigma(hY, U) \\ & \quad - \kappa\sigma(Y, \phi hU) + \mu(\kappa + 1)\phi\sigma(U, Y) \\ & \quad - v(\kappa + 1)\sigma(U, Y)\}. \end{aligned} \quad (51)$$

Consequently, the values of (45–51) are put in (44), we have

$$\begin{aligned}
 & L_2\{\eta(U)\sigma(Y, Z) - \eta(U)\phi\sigma(hY, Z) - \eta(Y)\sigma(U, Z) \\
 & \quad + \eta(Y)\phi\sigma(hU, Z) + \eta(Z)\sigma(U, Y) - \eta(Z)\phi\sigma(hU, Y) \\
 & \quad - (\tilde{\nabla}_U\sigma)(Y, Z)\} \\
 & = R^\perp(\xi, Y)\sigma(U, Z) - R^\perp(\xi, Y)\phi\sigma(hU, Z) \\
 & \quad + \eta(U)\{\kappa\sigma(Y, Z) + \mu\sigma(hY, Z) + \nu\phi\sigma(hY, Z) \\
 & \quad - \kappa\phi\sigma(hY, Z) + \mu(\kappa + 1)\phi\sigma(Y, Z) \\
 & \quad + \nu(\kappa + 1)\sigma(Y, Z)\} + \eta(Z)\{\kappa\sigma(Y, U) + \mu\sigma(hY, U) \\
 & \quad + \nu\phi\sigma(hY, U) - \kappa\phi\sigma(Y, hU) + \mu(\kappa + 1)\phi\sigma(U, Y) \\
 & \quad - \nu(\kappa + 1)\sigma(U, Y)\} \\
 & \quad + (\tilde{\nabla}_U\sigma)(\kappa[\eta(Y)\xi - Y] - \mu hY - \nu\phi hY, Z).
 \end{aligned} \tag{52}$$

Taking $Z = \xi$ in (52) and taking into account Proposition 2.1, (52) reduce to

$$\begin{aligned}
 & L_2\{\sigma(U, Y) - \phi\sigma(hU, Y) - (\tilde{\nabla}_U\sigma)(Y, \xi)\} = \kappa\sigma(Y, U) + \mu\sigma(hY, U) \\
 & \quad + \nu\phi\sigma(hY, U) - \kappa\phi\sigma(Y, hU) + \mu(\kappa + 1)\phi\sigma(U, Y) \\
 & \quad - \nu(\kappa + 1)\sigma(U, Y) \\
 & \quad + (\tilde{\nabla}_U\sigma)(\kappa[\eta(Y)\xi - Y] - \mu hY - \nu\phi hY, \xi),
 \end{aligned} \tag{53}$$

where, by direct calculations, one can easily see that

$$\begin{aligned}
 & (\tilde{\nabla}_U\sigma)(\kappa[\eta(Y)\xi - Y] - \mu hY - \nu\phi hY, \xi) \\
 & = -\sigma(\nabla_U\xi, \kappa[\eta(Y)\xi - Y] - \mu hY - \nu\phi hY) \\
 & = \sigma(\phi^2U + \phi hU, \kappa[\eta(Y)\xi - Y] - \mu hY - \nu\phi hY) \\
 & = -\sigma(U, \kappa[\eta(Y)\xi - Y] - \mu hY - \nu\phi hY) \\
 & \quad + \sigma(\phi hU, \kappa[\eta(Y)\xi - Y] - \mu hY - \nu\phi hY) \\
 & = \kappa\sigma(Y, U) + \mu\sigma(hY, U) + \nu\phi\sigma(U, hY) - \kappa\phi\sigma(Y, hU) \\
 & \quad + \mu(\kappa + 1)\phi\sigma(U, Y) - \nu(\kappa + 1)\sigma(U, Y)
 \end{aligned} \tag{54}$$

and

$$\begin{aligned}
 & (\tilde{\nabla}_U\sigma)(Y, \xi) = -\sigma(\nabla_U\xi, Y) = \sigma(\phi^2U + \phi hU, Y) \\
 & = \phi\sigma(hU, Y) - \sigma(U, Y).
 \end{aligned} \tag{55}$$

If (54) and (55) are put in (53), we reach at

$$\begin{aligned}
 & [L_2 - \kappa + (\nu - \mu\phi)(\kappa + 1)]\sigma(U, Y) - [(L_2 - \kappa)\phi + (\mu + \phi\nu)]\sigma(U, hY) \\
 & = 0.
 \end{aligned} \tag{56}$$

Substituting hY instead of Y in (56), by virtue of Proposition 2.1 and (6), we can easily see that

$$\begin{aligned}
 & (\kappa + 1)[(L_2 - \kappa)\phi + (\mu + \phi\nu)]\sigma(U, Y) - [L_2 - \kappa + (\nu - \mu\phi)(\kappa + 1)]\sigma(U, hY) \\
 & = 0.
 \end{aligned} \tag{57}$$

From common solutions of (56) and (57) provided $\kappa \neq 0$, we can infer

$$\begin{aligned}
 & [(L_2 - \kappa)^2 - (\kappa + 1)(\nu^2 - \mu^2)]\sigma(U, Y) \\
 & + 2\mu\nu(\kappa + 1)\phi\sigma(U, Y) = 0.
 \end{aligned} \tag{58}$$

This implies that M is either totally geodesic or (42) is satisfied. Thus, the proof is completed. $\square$

From Theorem 2.6, we have the following corollary.

Corollary 2.7 Let M be an invariant submanifold of an almost Kenmotsu (κ, μ, ν) -space $\tilde{M}^{2n+1}(\phi, \eta, \xi, g)$. M is 2-semiparallel submanifold if and only if M is totally geodesic submanifold provided

$$\kappa^2 - (\kappa + 1)(\nu^2 - \mu^2) \neq 0, \text{ or } \mu\nu(\kappa + 1) \neq 0.$$

Theorem 2.8 Let M be an invariant submanifold of an almost Kenmotsu (κ, μ, ν) -space $\tilde{M}^{2n+1}(\phi, \eta, \xi, g)$. If M is a 2-generalized Ricci pseudoparallel submanifold of $\tilde{M}^{2n+1}(\phi, \eta, \xi, g)$, then M is either totally geodesic or the function L_4 satisfies the condition

$$L_4 = \frac{1}{2n} \left(1 \mp \frac{1}{k} \sqrt{(\kappa + 1)(\nu^2 - \mu^2)} \right), \quad \mu \cdot \nu \cdot (\kappa + 1) = 0. \quad (59)$$

Proof Let M be an invariant 2-generalized Ricci pseudoparallel submanifold of an almost Kenmotsu $\tilde{M}^{2n+1}(\phi, \eta, \xi, g)$ -space. It follows that

$$L_4 Q(S, \tilde{\nabla}\sigma)(U, V, Z; X, Y) = (\tilde{R}(X, Y) \cdot \tilde{\nabla}\sigma)(U, V, Z), \quad (60)$$

for all $X, Y, Z, U, V \in \Gamma(TM)$, that is,

$$\begin{aligned} & -L_4 \{ (\tilde{\nabla}_{(X \wedge_S Y)U} \sigma)(V, Z) + (\tilde{\nabla}_U \sigma)((X \wedge_S Y)V, Z) \\ & + (\tilde{\nabla}_U \sigma)(V, (X \wedge_S Y)Z) \} = R(X, Y)(\tilde{\nabla}_U \sigma)(V, Z) \\ & - (\tilde{\nabla}_{R(X, Y)U} \sigma)(V, Z) - (\tilde{\nabla}_U \sigma)(R(X, Y)V, Z) \\ & - (\tilde{\nabla}_U \sigma)(V, R(X, Y)Z). \end{aligned}$$

This equality reduces for $X = V = \xi$

$$\begin{aligned} & -L_4 \{ (\tilde{\nabla}_{(\xi \wedge_S Y)U} \sigma)(\xi, Z) + (\tilde{\nabla}_U \sigma)((\xi \wedge_S Y)\xi, Z) \\ & + (\tilde{\nabla}_U \sigma)(\xi, (\xi \wedge_S Y)Z) \} = R(\xi, Y)(\tilde{\nabla}_U \sigma)(\xi, Z) \\ & - (\tilde{\nabla}_{R(\xi, Y)U} \sigma)(\xi, Z) - (\tilde{\nabla}_U \sigma)(R(\xi, Y)\xi, Z) \\ & - (\tilde{\nabla}_U \sigma)(\xi, R(\xi, Y)Z). \end{aligned} \quad (61)$$

Now, we will calculate these expressions separately. Firstly, making use of (6), (10), (15) and (29), we have

$$\begin{aligned} (\tilde{\nabla}_{(\xi \wedge_S Y)U} \sigma)(\xi, Z) &= -\sigma(\nabla_{(\xi \wedge_S Y)U} \xi, Z) = \sigma(\phi^2(\xi \wedge_S Y)U, Z) \\ &+ \sigma(\phi h(\xi \wedge_S Y)U, Z) \\ &= -\sigma(S(Y, U)\xi - S(\xi, U)Y, Z) \\ &+ \sigma(\phi h(S(Y, U)\xi - S(\xi, U)Y), Z) \\ &= 2n\kappa\eta(U)\{\sigma(Y, Z) - \phi\sigma(hY, Z)\}, \end{aligned} \quad (62)$$

$$\begin{aligned} (\tilde{\nabla}_U \sigma)((\xi \wedge_S Y)\xi, Z) &= (\tilde{\nabla}_U \sigma)(S(Y, \xi)\xi - S(\xi, \xi)Y, Z) \\ &= 2n\{(\tilde{\nabla}_U \sigma)(\kappa\eta(Y)\xi, Z) - (\tilde{\nabla}_U \sigma)(\kappa Y, Z)\} \\ &= 2n\{-\sigma(\nabla_U \kappa\eta(Y)\xi, Z) - (\tilde{\nabla}_U \sigma)(\kappa Y, Z)\} \\ &= 2n\{-\sigma(U\eta[\kappa(Y)]\xi + \kappa\eta(Y)\nabla_U \xi, Z) \\ &\quad - (\tilde{\nabla}_U \sigma)(\kappa Y, Z)\} \\ &= 2n\{\kappa\eta(Y)\sigma(\phi^2 U + \phi hU, Z) - (\tilde{\nabla}_U \sigma)(\kappa Y, Z)\} \\ &= 2n\eta(Y)\kappa\{\phi\sigma(hY, U) - \sigma(U, Z)\} \\ &\quad - 2n(\tilde{\nabla}_U \sigma)(\kappa Y, Z), \end{aligned} \quad (63)$$

$$\begin{aligned} (\tilde{\nabla}_U \sigma)(\xi, S(Y, Z)\xi - S(\xi, Z)Y) &= (\tilde{\nabla}_U \sigma)(\xi, S(Y, Z)\xi) \\ &\quad - 2n(\tilde{\nabla}_U \sigma)(\kappa\eta(Z)Y, \xi) \\ &= -2n(\tilde{\nabla}_U \sigma)(\kappa\eta(Z)Y, \xi) \\ &= 2n\kappa\sigma(\nabla_U \xi, \eta(Z)Y) \\ &= -2n\kappa\sigma(\phi^2 U + \phi hU, \eta(Z)Y) \\ &= 2n\kappa\eta(Z)\{\sigma(U, Y) - \sigma(\phi hU, Y)\}. \end{aligned} \quad (64)$$

Next, let's calculate the right side of the equality.

$$\begin{aligned} R^\perp(\xi, Y)(\tilde{\nabla}_U \sigma)(\xi, Z) &= -R^\perp(\xi, Y)\sigma(\nabla_U \xi, Z) = R^\perp(\xi, Y)\sigma(\phi^2 U + \phi h U, Z) \\ &= R^\perp(\xi, Y)\{\phi\sigma(hU, Z) - \sigma(U, Z)\}. \end{aligned} \quad (65)$$

Furthermore, by virtue of Proposition 2.1, (6) and (11), we observe

$$\begin{aligned} (\tilde{\nabla}_{R(\xi, Y)U} \sigma)(\xi, Z) &= -\sigma(\nabla_{R(\xi, Y)U} \xi, Z) = \sigma(\phi^2 R(\xi, Y)U, Z) \\ &\quad + \sigma(\phi h R(\xi, Y)U, Z) = -\sigma(R(\xi, Y)U, Z) \\ &\quad + \sigma(\phi h R(\xi, Y)U, Z) \\ &= \eta(U)\{\kappa\sigma(Y, Z) + \mu\sigma(hY, Z) + \nu\sigma(\phi h U, Z) \\ &\quad - \kappa\sigma(\phi h Y, Z) - \mu\sigma(\phi h^2 Y, Z) - \nu\sigma(\phi h \phi h Y, Z)\} \\ &= \eta(U)\{\kappa\sigma(Y, Z) + \mu\sigma(hY, Z) + \nu\sigma(\phi h Y, Z) \\ &\quad - \kappa\sigma(\phi h Y, Z) + \mu(\kappa + 1)\phi\sigma(Y, Z) \\ &\quad + \nu(\kappa + 1)\sigma(Y, Z)\}, \end{aligned} \quad (66)$$

$$\begin{aligned} (\tilde{\nabla}_U \sigma)(R(\xi, Y)\xi, Z) \\ = (\tilde{\nabla}_U \sigma)(\kappa[\eta(Y)\xi - Y] - \mu h Y - \nu \phi h Y, Z). \end{aligned} \quad (67)$$

Finally,

$$\begin{aligned} (\tilde{\nabla}_U \sigma)(\xi, R(\xi, Y)Z) &= -\sigma(\nabla_U \xi, R(\xi, Y)Z) \\ &= \sigma(\phi^2 U + \phi h U, R(\xi, Y)Z) \\ &= -\sigma(U, R(\xi, Y)Z) + \sigma(\phi h U, R(\xi, Y)Z) \\ &= \eta(Z)\{\sigma(U, Y) + \mu\sigma(U, hY) + \nu\phi\sigma(U, hY) \\ &\quad - \kappa\phi\sigma(hU, Y) - \mu\phi\sigma(hU, hY) - \nu\phi^2\sigma(hY, hU)\} \\ &= \eta(Z)\{\kappa\sigma(U, Y) + \mu\sigma(hY, U) + \nu\phi\sigma(U, hY) \\ &\quad - \kappa\phi\sigma(hU, Y) + \mu(\kappa + 1)\phi\sigma(U, Y) \\ &\quad - \nu(\kappa + 1)\sigma(U, Y)\} \end{aligned} \quad (68)$$

If (62) and (68) are put in (61), we have

$$\begin{aligned} 2n\kappa L_4\{\eta(U)\sigma(Y, Z) - \eta(U)\phi\sigma(hY, Z) - \eta(Y)\sigma(U, Z) + \eta(Y)\phi\sigma(hU, Z) \\ + \eta(Z)\sigma(U, Y) - \eta(Z)\phi\sigma(hU, Y) - (\tilde{\nabla}_U \sigma)(Y, Z)\} \\ = R^\perp(\xi, Y)\{\sigma(U, Z) - \phi\sigma(hU, Z)\} + \eta(U)\{\kappa\sigma(Y, Z) \\ + \mu\sigma(hY, Z) + \nu\phi\sigma(hY, Z) - \kappa\phi\sigma(hY, Z) \\ + \mu(\kappa + 1)\phi\sigma(Y, Z)\} + \eta(Z)\{\kappa\sigma(U, Y) + \mu\sigma(U, hY) \\ + \nu\phi\sigma(U, hY) - \kappa\phi\sigma(hU, Y) + \mu(\kappa + 1)\phi\sigma(U, Y) \\ - \nu(\kappa + 1)\sigma(U, Y)\} \\ + (\tilde{\nabla}_U \sigma)(\kappa[\eta(Y)\xi - Y] - \mu h Y - \nu \phi h Y, Z). \end{aligned} \quad (69)$$

Taking $Z = \xi$ in (69), we arrive at

$$\begin{aligned} 2n\kappa L_4\{\sigma(U, Y) - \phi\sigma(hU, Y) - (\tilde{\nabla}_U \sigma)(Y, \xi)\} &= \kappa\sigma(U, Y) + \mu\sigma(U, hY) \\ &\quad + \phi\sigma(U, hY) - \kappa\phi\sigma(hU, Y) + \mu(\kappa + 1)\sigma(U, Y) \\ &\quad + (\tilde{\nabla}_U \sigma)(\kappa[\eta(Y)\xi - Y] - \mu h Y - \nu \phi h Y, \xi) \\ &\quad - \nu(\kappa + 1)\sigma(U, Y), \end{aligned} \quad (70)$$

where

$$\begin{aligned} (\tilde{\nabla}_U \sigma)(\kappa[\eta(Y)\xi - Y] - \mu h Y - \nu \phi h Y, \xi) \\ = -\sigma(\nabla_U \xi, \kappa[\eta(Y)\xi - Y] - \mu h Y - \nu \phi h Y) \\ = \sigma(\phi^2 U + \phi h U, \kappa[\eta(Y)\xi - Y] - \mu h Y - \nu \phi h Y) \end{aligned}$$

$$= \kappa\sigma(U, Y) + \mu\sigma(U, hY) + \nu\phi\sigma(U, hY) - \kappa\phi\sigma(Y, hU) \\ + \mu(\kappa + 1)\phi\sigma(U, Y) - \nu(\kappa + 1)\sigma(U, Y) \quad (71)$$

and

$$(\tilde{\nabla}_U\sigma)(Y, \xi) = -\sigma(\nabla_U\xi, Y) = \sigma(\phi^2U + \phi hU, Y) \\ = \phi\sigma(hU, Y) - \sigma(U, Y). \quad (72)$$

From (69), (70) and (71), we conclude

$$2n\kappa L_4\{\sigma(U, Y) - \phi\sigma(hU, Y)\} = \kappa\sigma(U, Y) + \mu\sigma(U, hY) + \nu\phi\sigma(U, hY) \\ - \kappa\phi\sigma(hU, Y) + \mu(\kappa + 1)\phi\sigma(U, Y) \\ - \nu(\kappa + 1)\sigma(U, Y),$$

that is,

$$[2n\kappa L_4 - \kappa + \nu(\kappa + 1)]\sigma(U, Y) = \mu(\kappa + 1)\phi\sigma(U, Y) + \mu\sigma(U, hY) \\ + [2n\kappa L_4 + \nu - \kappa]\phi\sigma(U, hY). \quad (73)$$

Substituting hY for Y in (73) and again considering (6) and (11), we conclude

$$[2n\kappa L_4 - \kappa + \nu(\kappa + 1)]\sigma(U, hY) = -(\kappa + 1)[2n\kappa L_4 + \nu - \kappa]\phi\sigma(U, Y) \\ + \mu(\kappa + 1)\phi\sigma(U, hY) - \mu(\kappa + 1)\sigma(U, Y). \quad (74)$$

From (73) and (74), we observe

$$[\kappa^2(2nL_4 - 1)^2 - (\kappa + 1)(\nu^2 - \mu^2)]\sigma(U, Y) + 2\mu\nu(\kappa + 1)\phi\sigma(U, Y) = 0.$$

Since σ and $\phi\sigma$ are orthogonal, the last equality implies that σ is either identically zero or (59) is satisfied. $\square$

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References

- Kim, T.W.; Pak, H.K.: Canonical foliations of certain classes of almost contact metric structures. *Acta Math. Sin. English Ser.* **21**(4), 841–856 (2005)
- Koufogiorgos, T.; Markellos, M.; Papantoniou, V.J.: The harmonicity of the Reeb vector fields on contact 3-manifolds. *Pacific J. Math.* **234**(2), 325–344 (2008)
- Blair, D.E.; Koufogiorgos, T.; Papantoniou, B.J.: Contact metric manifolds satisfying a nullity condition. *Israel J. Math.* **91**(1–3), 189–214 (1995)
- Ozturk, H., Aktan, N., Murathan, C.: Almost α -Cosymplectic (κ, μ, ν) -Spaces. [arXiv:1007.0527v1](https://arxiv.org/abs/1007.0527v1).
- Carriazo, A.; Martin-Molina, V.: Almost cosymplectic and almost Kenmotsu (κ, μ, ν) -spaces. *Mediterr. J. Math.* **10**, 1551–1571 (2013)
- Dacko, P.; Olszak, Z.: On almost cosymplectic (κ, μ, ν) -spaces. *Banach Center Publ.* **69**(1), 211–220 (2005)
- Dacko, P.; Olszak, Z.: On almost cosymplectic $(-1, \mu, \phi)$ -spaces. *Central Eur. J. Math. CEJM* **2**, 318–330 (2005)
- Ateken, M.; Yildirim, .; Dirik, S.: Pseudoparallel invariant submanifolds of $(LCS)_n$ -manifolds. *Korean J. Math.* **28**(2), 275–284 (2020)
- Ateken, M.; Uygun, P.: Characterizations for totally geodesic submanifolds of (κ, μ) -paracontact metric manifolds. *Korean J. Math.* **28**(3), 555–571 (2021)
- Bejancu, A.; Papaghuic, N.: Semi-invariant submanifolds of a Sasakian manifold. *Annal. Alexandru Ioan Cuza Univ. Iași (New Ser.) Math.* **27**, 163–170 (1981)
- Chinea, D.; Prestelo, P.S.: Invariant submanifolds of a trans-Sasakian manifold. *Publ. Math. Debrecen.* **38**(1–2), 103–109 (1991)
- De, A.: A note on the paper on invariant submanifolds of LP-Sasakian manifolds. *Extracta Math.* **28**(1), 33–36 (2013)

13. De, U.C.; Majhi, P.: On invariant submanifolds of Kenmotsu manifolds. *J. Geom.* **106**, 109–122 (2015)
14. Guojing, Z.; Jiangu, W.: Invariant submanifolds and modes of nonlinear autonomous systems. *Appl. Math. Mech.* **19**(7), 687–693 (1998)
15. Hu, C.; Wang, Y.: A note invariant submanifolds of trans-Sasakian manifolds. *Int. Electron. J. Geom.* **9**(2), 27–35 (2016)
16. Sarkar, A.; Sen, M.: On invariant submanifolds of LP-Sasakian manifolds. *Extracta Math.* **27**(1), 145–154 (2012)
17. Kowalczyk, D.: On some subclass of semisymmetric manifolds. *Soochow J. Math.* **27**, 445–461 (2001)
18. Rovenski, V.; Patra, D.S.: On non-gradient (m, ρ) -quasi-Einstein contact metric manifolds. *J. Geometry* **112**, 14 (2021)
19. Nagaraja, H.G.; Kumari, D.: Submanifolds of $N(\kappa)$ -contact metric manifolds. *Palestine J. Math.* **10**(2), 777–783 (2021)

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