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A note on quasi-bi-slant submanifolds of Sasakian manifolds

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Abstract The object of the present paper is to study the notion of quasi-bi-slant submanifolds of almost contact metric manifolds as a generalization of slant, semi-slant, hemi-slant, bi-slant, and quasi-hemi-slant submanifolds. We study and characterize quasi-bi-slant submanifolds of Sasakian manifolds and provide non-trivial examples to signify that the structure presented in this paper is valid. Furthermore, the integrability of distributions and geometry of foliations are researched. Moreover, we characterize quasi-bi-slant submanifolds with parallel canonical structures.

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1 Introduction

Presently, the theory of submanifolds has gained prominence in computer design, image processing, economic modeling, as well as in mathematical physics and mechanics. These extensive applications of this topic make it an active and interesting field of research for geometers. The notion of the geometry of submanifolds begins with the idea of the extrinsic geometry of the surface and it is developed for ambient space with time.

First, the notion of slant submanifolds of almost Hermitian manifolds was studied by Chen [7] as a natural generalization of holomorphic immersions and totally real immersions. Chen collected many consequent results on slant submanifolds in his book [5]. Several examples of slant submanifolds in complex Euclidean spaces C^2 and C^4 are given by Chen and Tazawa [8, 9]. On the other hand, using this notion Lotta introduced slant immersion of a Riemannian manifold into almost contact metric manifold in [15, 16]. Bejancu and Papaghiuc studied the semi-invariant submanifolds of Sasakian manifold [1]. In the course of time, this interesting subject has been studied broadly by several geometers [7, 11, 13, 14, 20, 22].

Cabrerizo and his co-authors [4] studied slant submanifolds of Sasakian manifold and have given several examples of such submanifolds. Semi-slant and bi-slant submanifolds of an almost contact metric manifold were studied by Cabrerizo et al. [3]. Chen et al. [22] investigated bi-slant submanifolds in Kaehler manifolds. Primarily, the hemi-slant submanifolds were known as anti-slant submanifolds. Later, Sahin [21] named these submanifolds as hemi-slant submanifolds. Hemi-slant submanifolds are one of the classes of bi-slant submanifolds. Recently, pseudo-slant submanifolds of Sasakian manifolds have been studied by Khan and Khan [12]. In [18, 19], we have studied and characterized quasi-hemi-slant submanifolds of Sasakian and cosymplectic manifolds and we have given several examples of such immersions.

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Inspired from the good and interesting ideas of the above studies, we introduce the notion of quasi-bi-slant submanifolds of Sasakian manifolds as a generalization of slant, semi-slant, hemi-slant, and quasi-hemi-slant submanifolds, in which the tangent bundle consists of one invariant and two slant distributions and the characteristic vector field.

The present paper is organized as follows: In Sect. 2, we mention the basic definition and some properties of Sasakian manifolds. In Sect. 3, we define the notion of quasi-bi-slant submanifolds, giving a non-trivial example and obtain some basic results for the next sections. Section 4 deals with necessary and sufficient conditions for the integrability of distributions. We also investigate the geometry of foliations in this section. In the last section, we characterize quasi-bi-slant submanifolds with parallel canonical structures.

2 Preliminaries

We consider $\widehat{M}$ is a $(2n + 1)$ -dimensional almost contact manifold [10] which carries a tensor field ϕ of the tangent space, 1-form η , and characteristic vector field ξ satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad (2.1)$$

where $I : T\widehat{M} \rightarrow T\widehat{M}$ is the identity map. We have from definition $\phi\xi = 0$, $\eta \circ \phi = 0$ and $\text{rank}(\phi) = 2n$.

Since any almost contact manifold $(\widehat{M}, \phi, \xi, \eta)$ admits a Riemannian metric g , such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (2.2)$$

for any vector fields $X, Y \in \Gamma(T\widehat{M})$, where $\Gamma(T\widehat{M})$ represents the Lie algebra of vector fields on $\widehat{M}$. The manifold $\widehat{M}$ together with the structure (ϕ, ξ, η, g) is called an almost contact metric manifold.

From the immediate consequence of (2.2), we have

$$\eta(X) = g(X, \xi) \quad \text{and} \quad g(\phi X, Y) + g(X, \phi Y) = 0, \quad (2.3)$$

for all vector fields $X, Y \in \Gamma(T\widehat{M})$.

An almost contact structure (ϕ, ξ, η) is said to be normal [2] if the almost complex structure J on the product manifold $\widehat{M} \times R$ is given by

$$J \left(U, f \frac{d}{dt} \right) = \left(\phi U - f\xi, \eta(U) \frac{d}{dt} \right),$$

where $J^2 = -I$ and f is the differentiable function on $\widehat{M} \times R$ has no torsion, i.e., J is integrable. The condition for normality in terms of ϕ , ξ , and η is $[\phi, \phi] + 2d\eta \otimes \xi = 0$ on $\widehat{M}$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ . A Sasakian manifold [4] is a normal contact metric manifold and every Sasakian manifold is a K -contact manifold. It is easy to show that an almost contact metric manifold is a Sasakian manifold if and only if

$$(\widehat{\nabla}_X \phi)Y = g(X, Y)\xi - \eta(Y)X \quad (2.4)$$

for all vector fields $X, Y \in \Gamma(T\widehat{M})$.

Also, in a Sasakian manifold, we have

$$\widehat{\nabla}_X \xi = -\phi X.$$

Let M be a Riemannian manifold isometrically immersed in $\widehat{M}$ and the induced Riemannian metric on M is denoted by the same symbol g throughout this paper. Let A and h denote the shape operator and second fundamental form, respectively, of immersion of M into $\widehat{M}$. The Gauss and Weingarten formulas are given by [6]

$$\widehat{\nabla}_X Y = \nabla_X Y + h(X, Y) \quad (2.5)$$

and

$$\widehat{\nabla}_X V = -A_V X + \nabla_X^\perp V, \quad (2.6)$$

for any vector fields $X, Y \in \Gamma(TM)$ and V on $\Gamma(T^\perp M)$, where ∇ is the induced connection on M and $\nabla^\perp$ represents the connection on the normal bundle $T^\perp M$ of M and A_V is the shape operator of M with respect

to normal vector $V \in \Gamma(T^\perp M)$. Moreover, A_V and the second fundamental form $h : TM \otimes TM \longrightarrow T^\perp M$ of M into $\widehat{M}$ are related by

$$g(h(X, Y), V) = g(A_V X, Y), \quad (2.7)$$

for any vector fields $X, Y \in \Gamma(TM)$ and V on $\Gamma(T^\perp M)$.

The mean curvature vector is defined by

$$H = \frac{1}{n} \text{trace}(h) = \frac{1}{n} \sum_{i=1}^n h(e_i, e_i), \quad (2.8)$$

where n denotes the dimension of submanifold M and $\{e_1, e_2, \dots, e_n\}$ is the local orthonormal basis of tangent space at each point of M .

A submanifold M of Sasakian manifold $\widehat{M}$ is said to be totally umbilical if

$$h(X, Y) = g(X, Y)H, \quad (2.9)$$

where H is the mean curvature vector. If $h(X, Y) = 0$ for all $X, Y \in \Gamma(TM)$, then M is said to be totally geodesic, and if $H = 0$, then M is said to be a minimal submanifold.

3 Quasi-bi-slant submanifolds of Sasakian manifolds

In the present section of the paper, we define the notion of quasi-bi-slant submanifolds of almost contact metric manifolds, providing a non-trivial example, and obtain some related results for later use.

Definition 3.1 A quasi-bi-slant submanifold M of almost contact metric manifold $\widehat{M}$ is a submanifold that admits three orthogonal complementary distributions D , D_1 , and D_2 , at the point $p \in M$, such that

(1) TM admits the orthogonal direct decomposition

$$TM = D \oplus D_1 \oplus D_2 \oplus \langle \xi \rangle,$$

(2) The distribution D is invariant, i.e., $\phi D = D$,

(3) $\phi(D_1) \perp D_2$ and $\phi(D_2) \perp D_1$,

(4) The distributions D_1, D_2 are slant with slant angles θ_1 and θ_2 , respectively.

Let the dimension of distributions D, D_1 and D_2 are be m_1, m_2 and m_3 , respectively. We easily observe the following cases:

- (a) If $m_1 \neq 0, m_2 = 0$ and $m_3 = 0$, then M is an invariant submanifold.
- (b) If $m_1 = m_2 = 0$ and $\theta_2 = \frac{\pi}{2}$, then M is an anti-invariant submanifold.
- (c) If $m_1 \neq 0, m_2 = 0$ and $\theta_2 = \frac{\pi}{2}$, then M is a semi-invariant submanifold.
- (d) If $m_1 = m_2 = 0$ and $0 < \theta_2 < \frac{\pi}{2}$, then M is a slant submanifold with slant angles θ_2 .
- (e) If $m_1 \neq 0, m_2 \neq 0, m_3 \neq 0, \theta_1 = 0$ and $0 < \theta_2 < \frac{\pi}{2}$, then M is a semi-slant submanifold.
- (f) If $m_1 \neq 0, m_2 \neq 0, m_3 \neq 0, \theta_1 = \frac{\pi}{2}$ and $0 < \theta_2 < \frac{\pi}{2}$, then M is a hemi-slant submanifold.
- (g) If $m_1 \neq 0, m_2 \neq 0, m_3 \neq 0$ and $0 < \theta_1, \theta_2 < \frac{\pi}{2}$, then M is a bi-slant submanifold.
- (h) If $m_1 \neq 0, m_2 \neq 0, m_3 \neq 0, 0 < \theta_1 < \frac{\pi}{2}$ and $\theta_2 = \frac{\pi}{2}$, then M is a quasi-hemi-slant submanifold.
- (i) If $m_1 \neq 0, m_2 \neq 0, m_3 \neq 0$ and $\theta_1, \theta_2 \neq 0, \frac{\pi}{2}$, then M is proper quasi-bi-slant submanifold.

This means that a quasi-bi-slant submanifold is a generalization of invariant, anti-invariant, semi-invariant, slant, semi-slant, hemi-slant, bi-slant, and quasi-hemi-slant submanifolds.

Remark 3.2 It is clear from Definition 3.1 that if $D \neq \{0\}, D_1 \neq \{0\}$ and $D_2 \neq \{0\}$, then $\dim D \geq 2, \dim D_1 \geq 2$ and $\dim D_2 \geq 2$. Therefore, for a proper quasi-bi-slant submanifold M of an almost contact metric manifold $\widehat{M}$, $\dim M \geq 7$ and $\dim \widehat{M} \geq 11$.

Remark 3.3 The above definition can be generalized by taking $TM = D \oplus D_{\theta_1} \oplus D_{\theta_2} \cdots \oplus D_{\theta_k}$. Hence, we can define multi-slant submanifolds, quasi-multi-slant submanifolds, etc.

Remark 3.4 In this paper, we assume that M is a proper quasi-bi-slant submanifold of Sasakian manifold $\widehat{M}$.

Example 3.5 [4] Let $\mathbb{R}^{2k+1} = \{(x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_k, z) : x_i, y_i, z \in \mathbb{R}, i = 1, 2, \dots, k\}$. Consider $\mathbb{R}^{2k+1}$ with the following structure:

$$\begin{aligned}\phi\left(\sum_{i=1}^k\left(X^i\frac{\partial}{\partial x_i}+Y^i\frac{\partial}{\partial y_i}\right)+Z\frac{\partial}{\partial z}\right) &= \sum_{i=1}^k Y^i\frac{\partial}{\partial x_i}-\sum_{i=1}^k X^i\frac{\partial}{\partial y_i}+\sum_{i=1}^k Y^i y_i\frac{\partial}{\partial z}, \\ g &= (\eta \otimes \eta) + \frac{1}{4} \sum_{i=1}^k (dx_i \otimes dx_i + dy_i \otimes dy_i), \\ \eta &= \frac{1}{2} \left(dz - \sum_{i=1}^k y_i dx_i \right), \quad \xi = 2 \frac{\partial}{\partial z}.\end{aligned}$$

Then, $(\mathbb{R}^{2k+1}, \phi, \xi, \eta, g)$ is a Sasakian manifold. The vector fields $e_i = 2 \frac{\partial}{\partial y_i}$, $e_{k+i} = 2(\frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial z})$ and ξ form a ϕ -basis for the almost contact metric structure.

Now, we give an example of proper quasi-bi-slant submanifold in $\mathbb{R}^{13}$.

Example 3.6 Consider a 13-dimensional differentiable manifold

$$\overline{M} = \{(x_1, x_2, \dots, x_6, y_1, y_2, \dots, y_6, z) : x_i, y_i, z \in \mathbb{R}; i = 1, 2, \dots, 6\},$$

with structure (ϕ, ξ, η, g) as defined in Example 3.5. Then, $(\overline{M}, \phi, \xi, \eta, g)$ become a Sasakian manifold of dimension 13. We choose the vector fields

$$e_i = 2 \frac{\partial}{\partial y_i}, \quad e_{6+i} = 2 \left(\frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial z} \right), \quad e_{13} = \xi = 2 \frac{\partial}{\partial z}, \quad \text{for } i = 1, 2, \dots, 6.$$

Then, $\{e_1, e_2, \dots, e_{13}\}$ forms an orthonormal basis at each point of tangent space of $\overline{M}$.

Now, we consider a submanifold M of $\overline{M}$ defined by immersion f as follows:

$$f(u, v, w, r, s, t, q) = (u, w, 0, s, 0, 0, v, r \cos \theta_1, r \sin \theta_1, t \cos \theta_2, 0, t \sin \theta_2, q),$$

where θ_1 and θ_2 are constant.

If we take

$$\begin{aligned}Z_1 &= e_1, \quad Z_2 = e_7, \quad Z_3 = e_2, \\ Z_4 &= \cos \theta_1 e_8 + \sin \theta_1 e_9, \quad Z_5 = e_4, \\ Z_6 &= \cos \theta_2 e_{10} + \sin \theta_2 e_{12}, \quad Z_7 = e_{13}.\end{aligned}$$

Then, the set $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\}$ restricted over M spans the tangent bundle TM of M .

Obviously, we get

$$\begin{aligned}\phi Z_1 &= e_7, \quad \phi Z_2 = -e_1, \quad \phi Z_3 = e_8, \\ \phi Z_4 &= -(\cos \theta_1 e_2 + \sin \theta_1 e_3), \quad \phi Z_5 = e_{10}, \\ \phi Z_6 &= -(\cos \theta_2 e_4 + \sin \theta_2 e_6), \quad \phi Z_7 = 0.\end{aligned}$$

Now, let us put $D = \text{Span}\{Z_1, Z_2\}$, $D_1 = \text{Span}\{Z_3, Z_4\}$, $D_2 = \text{Span}\{Z_5, Z_6\}$.

Then, one can easily check that the distribution D is invariant and the distributions D_1, D_2 are slant with slant angles θ_1 and θ_2 , respectively. Also, we observe that D, D_1 and D_2 , satisfy the definition of quasi-bi-slant submanifold of Sasakian manifold. Hence, submanifold M defined by f is a proper quasi-bi-slant submanifold of $\overline{M}$ with θ_1, θ_2 as its bi-slant angles.

Let M be a quasi-bi-slant submanifold of a Sasakian manifold $\widehat{M}$. We denote the projections of $X \in \Gamma(TM)$ on the distributions D, D_1 and D_2 by P, Q , and R , respectively. Then, for any $X \in \Gamma(TM)$, we can write

$$X = PX + QX + RX + \eta(X)\xi. \quad (3.1)$$

Now, put

$$\phi X = TX + NX, \quad (3.2)$$

where TX and NX are tangential and normal components of ϕX on M . Making use of (3.1) and (3.2), we obtain

$$\phi X = TPX + NPX + TQX + NQX + TRX + NRX. \quad (3.3)$$

Since $\phi D = D$, we have $NPX = 0$. Therefore, we get

$$\phi X = TPX + TQX + TRX + NQX + NRX. \quad (3.4)$$

Thus, for any $X \in \Gamma(TM)$, it is easy to see that

$$TX = TPX + TQX + TRX$$

and

$$NX = NQX + NRX.$$

Thus, from (3.4), we have the following decomposition:

$$\phi(TM) = D \oplus TD_1 \oplus TD_2 \oplus ND_1 \oplus ND_2, \quad (3.5)$$

where ‘ $\oplus$ ’ denotes orthogonal direct sum.

Since $ND_1 \subset \Gamma(T^\perp M)$ and $ND_2 \subset \Gamma(T^\perp M)$, we have

$$T^\perp M = ND_1 \oplus ND_2 \oplus \mu, \quad (3.6)$$

where μ is the orthogonal complement of $ND_1 \oplus ND_2$ in $\Gamma(T^\perp M)$ and it is invariant with respect to ϕ . For any non-zero vector field $V \in \Gamma(T^\perp M)$, we have

$$\phi V = tV + nV, \quad (3.7)$$

where $tV \in \Gamma(TM)$ and $nV \in \Gamma(T^\perp M)$.

Lemma 3.7 *Let M be a quasi-bi-slant submanifold of Sasakian manifold $\widehat{M}$. Then, we obtain*

$$TD = D, TD_1 \subset D_1, TD_2 \subset D_2, tND_1 \subset D_1, tND_2 \subset D_2. \quad (3.8)$$

Lemma 3.8 *Let M be a submanifold of Sasakian manifold $\widehat{M}$. Then, for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, we have*

$$\begin{aligned} \nabla_X TY - A_{NY}X - T\nabla_X Y - th(X, Y) &= g(X, Y)\xi - \eta(Y)X \\ h(X, TY) + \nabla_X^\perp NY - N(\nabla_X Y) - nh(X, Y) &= 0, \\ \nabla_X tV - t\nabla_X^\perp V &= A_{nV}X - TA_V X \end{aligned}$$

and

$$\nabla_X^\perp nV - n\nabla_X^\perp V = -h(X, tV) - NA_V X.$$

Proof Since $\widehat{M}$ is a Sasakian manifold, we have

$$\widehat{\nabla}_X \phi Y - \phi \widehat{\nabla}_X Y = g(X, Y)\xi - \eta(Y)X.$$

By making use of Eqs. (2.5), (2.6), (3.2), and (3.7), we obtain

$$\begin{aligned} \nabla_X TY - A_{NY}X - T\nabla_X Y - th(X, Y) \\ + h(X, TY) + \nabla_X^\perp NY - N(\nabla_X Y) - nh(X, Y) &= g(X, Y)\xi - \eta(Y)X. \end{aligned}$$

On comparing tangential and normal components, we have the first two results.

In a similar way, we have the last two. $\square$

Taking into account Eqs. (3.2) and (3.7), and using the fact that $\phi^2 = -I + \eta \otimes \xi$, we have the followings:

Lemma 3.9 *Let M be a quasi-bi-slant submanifold of a Sasakian manifold $\widehat{M}$. Then, the endomorphisms T and N (resp., t and n) in the tangent bundle (resp., in the normal bundle) of M , satisfy the following identities:*

- (i) $T^2 + tN = -I + \eta \otimes \xi$ on TM ,
- (ii) $NT + nN = 0$ on TM ,
- (iii) $Nt + n^2 = -I$ on $(T^\perp M)$,
- (iv) $Tt + tn = 0$ on $(T^\perp M)$,

where I is the identity operator.

Lemma 3.10 Let M be a quasi-bi-slant submanifold of a Sasakian manifold $\widehat{M}$. Then

- (1) $T^2X = -(\cos^2 \theta_i)X$,
- (2) $g(TX, TY) = (\cos^2 \theta_i)g(X, Y)$,
- (3) $g(NX, NY) = (\sin^2 \theta_i)g(X, Y)$,

for any $X, Y \in D_i$, for $i = 1, 2$.

Proof The proof follows using similar steps as in Proposition 2.8 of [17]. $\square$

Lemma 3.11 Let M be a quasi-bi-slant submanifold of a Sasakian manifold $\widehat{M}$. Then

$$h(X, \xi) = -NX \quad \text{and} \quad \nabla_X \xi = -TX \quad \text{for all } X \in \Gamma(TM).$$

Lemma 3.12 Let M be a quasi-bi-slant submanifold of a Sasakian manifold $\widehat{M}$. Then

- (i) $g(\widehat{\nabla}_X Y, \xi) = g(TX, Y)$;
- (ii) $g([X, Y], \xi) = 2g(TX, Y)$

for all $X, Y \in \Gamma(D \oplus D_1 \oplus D_2)$.

Proof Let $X, Y \in \Gamma(D \oplus D_1 \oplus D_2)$, and consider

$$\widehat{\nabla}_X \{g(Y, \xi)\} = g(\widehat{\nabla}_X Y, \xi) + g(Y, \widehat{\nabla}_X \xi).$$

Since Y is orthogonal to ξ , therefore from above equation, we have

$$g(\widehat{\nabla}_X Y, \xi) = -g(Y, \widehat{\nabla}_X \xi).$$

Using $\widehat{M}$ is a Sasakian manifold and above equation, we have

$$g(\widehat{\nabla}_X Y, \xi) = g(TX, Y). \quad (3.9)$$

Now

$$g([X, Y], \xi) = g(\widehat{\nabla}_X Y, \xi) - g(\widehat{\nabla}_Y X, \xi). \quad (3.10)$$

Using (3.9) and (3.10), we have

$$g([X, Y], \xi) = 2g(TX, Y) \quad \text{for all } X, Y \in \Gamma(D \oplus D_1 \oplus D_2).$$

This completes the proof. $\square$

4 Integrability and totally geodesic foliations

In this section, we examine the integrability conditions for invariant distribution D , and slant distributions D_1 and D_2 .

Proposition 4.1 Let M be a proper quasi-bi-slant submanifold of a Sasakian manifold $\widehat{M}$. Then, the invariant distribution D and slant distribution D_i ; ($i = 1, 2$) are not integrable.

Proof The proof follows using Lemma 3.12(ii). $\square$

Theorem 4.2 Let M be a proper quasi-bi-slant submanifold of a Sasakian manifold $\widehat{M}$. Then, the distribution $D \oplus \langle \xi \rangle$ is integrable if and only if

$$g(\nabla_X TY - \nabla_Y TX, TQZ + TRZ) = g(h(Y, TX) - h(X, TY), NQZ + NRZ), \quad (4.1)$$

for any $X, Y \in \Gamma(D \oplus \langle \xi \rangle)$ and $Z \in \Gamma(D_1 \oplus D_2)$.

Proof The distribution $D \oplus \langle \xi \rangle$ is integrable on M if and only if $g([X, Y], Z) = 0$ for any $X, Y \in \Gamma(D \oplus \langle \xi \rangle)$ and $Z \in \Gamma(D_1 \oplus D_2)$.

Now, for any $X, Y \in \Gamma(D \oplus \langle \xi \rangle)$ and $Z = QZ + RZ \in \Gamma(D_1 \oplus D_2)$, using (2.2), we have

$$g([X, Y], Z) = g(\phi \widehat{\nabla}_X Y, \phi Z) + \eta(\widehat{\nabla}_X Y)\eta(Z) - g(\phi \widehat{\nabla}_Y X, \phi Z) - \eta(\widehat{\nabla}_Y X)\eta(Z),$$

Using (2.4), (3.2), and $NY = 0$ for any $Y \in (D \oplus \langle \xi \rangle)$, we obtain

$$\begin{aligned} g([X, Y], Z) &= g(\widehat{\nabla}_X \phi Y, \phi Z) - g(\widehat{\nabla}_Y \phi X, \phi Z), \\ &= g(\widehat{\nabla}_X TY, \phi Z) - g(\widehat{\nabla}_Y TX, \phi Z). \end{aligned}$$

By making use of (2.5) and (3.4) in the above equation, we get

$$\begin{aligned} g([X, Y], Z) &= g(\nabla_X TY - \nabla_Y TX, TQZ + TRZ) \\ &\quad + g(h(X, TY) - h(Y, TX), NQZ + NRZ). \end{aligned}$$

This completes the proof. $\square$

Theorem 4.3 *Let M be a proper quasi-bi-slant submanifold of a Sasakian manifold $\widehat{M}$. Then, the distribution $D_1 \oplus \langle \xi \rangle$ is integrable if and only if*

$$\begin{aligned} g(A_{NW}Z - A_{NZ}W, TX) &= g(A_{NTW}Z - A_{NTZ}W, X) \\ &\quad + g(\nabla_Z^\perp NW - \nabla_W^\perp NZ, NRX) \end{aligned} \quad (4.2)$$

for any $Z, W \in \Gamma(D_1 \oplus \langle \xi \rangle)$ and $X \in \Gamma(D \oplus D_2)$.

Proof The distribution $D_1 \oplus \langle \xi \rangle$ is integrable on M if and only if $g([Z, W], X) = 0$, for any $Z, W \in \Gamma(D_1 \oplus \langle \xi \rangle)$ and $X \in \Gamma(D \oplus D_2)$.

Now, for any $Z, W \in \Gamma(D_1 \oplus \langle \xi \rangle)$ and $X = PX + RX \in \Gamma(D \oplus D_2)$, using (2.2), we have

$$g([Z, W], X) = g(\phi \widehat{\nabla}_Z W, \phi X) + \eta(\widehat{\nabla}_Z W)\eta(X) - g(\phi \widehat{\nabla}_W Z, \phi X) - \eta(\widehat{\nabla}_W Z)\eta(X).$$

Using (2.2), (2.4), 2.5, and (3.2), we obtain

$$\begin{aligned} g([Z, W], X) &= g(\widehat{\nabla}_Z TW, \phi X) + g(\widehat{\nabla}_Z NW, \phi X) \\ &\quad - g(\widehat{\nabla}_W TZ, \phi X) - g(\widehat{\nabla}_W NZ, \phi X). \\ &= -g(\widehat{\nabla}_Z \phi TW, X) + g(\widehat{\nabla}_Z NW, \phi X) \\ &\quad + g(\widehat{\nabla}_W \phi TZ, X) - g(\widehat{\nabla}_W NZ, \phi X). \end{aligned}$$

Then, from (2.6), (3.2) and Lemma 3.10, we have

$$\begin{aligned} g([Z, W], X) &= \cos^2 \theta_1 g([Z, W], X) - g(A_{NW}Z - A_{NZ}W, \phi X) \\ &\quad + g(A_{NTW}Z - A_{NTZ}W, X) + g(\nabla_Z^\perp NW - \nabla_W^\perp NZ, \phi X), \end{aligned}$$

which leads to

$$\begin{aligned} \sin^2 \theta_1 g([Z, W], X) &= g(A_{NTW}Z - A_{NTZ}W, X) + g(\nabla_Z^\perp NW - \nabla_W^\perp NZ, NRX) \\ &\quad - g(A_{NW}Z - A_{NZ}W, TX). \end{aligned}$$

This completes the proof. $\square$

In a similar way, we obtain the following case for the distribution $D_2 \oplus \langle \xi \rangle$:

Theorem 4.4 *Let M be a quasi-bi-slant submanifold of a Sasakian manifold $\widehat{M}$. Then, the slant distribution $D_2 \oplus \langle \xi \rangle$ is integrable if and only if*

$$\begin{aligned} g(A_{NW}Z - A_{NZ}W, TX) &= g(A_{NTW}Z - A_{NTZ}W, X) \\ &\quad + g(\nabla_Z^\perp NW - \nabla_W^\perp NZ, NQX) \end{aligned} \quad (4.3)$$

for any $Z, W \in \Gamma(D_2 \oplus \langle \xi \rangle)$ and $X \in \Gamma(D \oplus D_1)$.

From Theorems 4.3 and 4.4, we have the following sufficient conditions for the distribution $D_i \oplus \langle \xi \rangle$; ($i = 1, 2$) to be integrable.

Theorem 4.5 *Let M be a proper quasi-bi-slant submanifold of a Sasakian manifold $\widehat{M}$. If*

$$\begin{aligned}\nabla_Z^\perp NW - \nabla_W^\perp NZ &\in ND_i \oplus \mu, \\ A_{NTW}Z - A_{NTZ}W &\in D_i \quad \text{and} \\ A_{NW}Z - A_{NZ}W &\in D_i\end{aligned}\quad (4.4)$$

for any $Z, W \in \Gamma(D_i \oplus \langle \xi \rangle)$, then the slant distribution $D_i \oplus \langle \xi \rangle$ is integrable, for $i = 1, 2$.

We now investigate the geometry of leaves of invariant and slant distributions.

Proposition 4.6 *Let M be a proper quasi-bi-slant submanifold of a Sasakian manifold $\widehat{M}$. Then, the invariant distribution D and slant distribution D_i ; ($i = 1, 2$) do not define a totally geodesic foliation on M .*

Proof The proof directly follows from Lemma 3.12(i). $\square$

Theorem 4.7 *Let M be a proper quasi-bi-slant submanifold of a Sasakian manifold $\widehat{M}$. Then, the invariant distribution $D \oplus \langle \xi \rangle$ defines a totally geodesic foliation on M if and only if*

$$\begin{aligned}g(\nabla_X TY, TQZ + TRZ) &= -g(h(X, TY), NQZ + NRZ) \quad \text{and} \\ g(\nabla_X TY, tU) &= -g(h(X, TY), nU)\end{aligned}\quad (4.5)$$

for any $X, Y \in \Gamma(D \oplus \langle \xi \rangle)$, $Z \in \Gamma(D_1 \oplus D_2)$ and $U \in \Gamma(T^\perp M)$.

Proof The distribution $D \oplus \langle \xi \rangle$ defines a totally geodesic foliation on M if and only if $g(\widehat{\nabla}_X Y, Z) = 0$ and $g(\widehat{\nabla}_X Y, U) = 0$ for any $X, Y \in \Gamma(D \oplus \langle \xi \rangle)$, $Z \in \Gamma(D_1 \oplus D_2)$, and $U \in \Gamma(T^\perp M)$.

Now, for any $X, Y \in \Gamma(D \oplus \langle \xi \rangle)$, $Z = QZ + RZ \in \Gamma(D_1 \oplus D_2)$ and using (2.2), (2.5), (3.2), and $NY = 0$, we have

$$\begin{aligned}g(\widehat{\nabla}_X Y, Z) &= g(\widehat{\nabla}_X TY, \phi Z), \\ &= g(\nabla_X TY, TQZ + TRZ) + g(h(X, TY), NQZ + NRZ),\end{aligned}$$

which gives first part of (4.5).

Now, for any $U \in \Gamma(T^\perp M)$ and $X, Y \in \Gamma(D \oplus \langle \xi \rangle)$, we have

$$\begin{aligned}g(\widehat{\nabla}_X Y, U) &= g(\widehat{\nabla}_X TY, \phi U) \\ &= g(\nabla_X TY, tU) + g(h(X, TY), nU),\end{aligned}$$

which gives second part of (4.5). $\square$

Theorem 4.8 *Let M be a proper quasi-bi-slant submanifold of a Sasakian manifold $\widehat{M}$. Then, the slant distribution $D_1 \oplus \langle \xi \rangle$ defines a totally geodesic foliation on M if and only if*

$$\begin{aligned}g(\nabla_X^\perp NY, NRZ) &= g(A_{NY}X, TPZ + TRZ) - g(A_{NTY}X, Z) \quad \text{and} \\ g(A_{NY}X, tV) &= g(\nabla_X^\perp NY, nV) - g(\nabla_X^\perp NTY, V)\end{aligned}\quad (4.6)$$

for any $X, Y \in \Gamma(D_1 \oplus \langle \xi \rangle)$, $Z \in \Gamma(D \oplus D_2)$ and $V \in \Gamma(T^\perp M)$.

Proof For any $X, Y \in \Gamma(D_1 \oplus \langle \xi \rangle)$, $Z = PZ + RZ \in \Gamma(D \oplus D_2)$ and using (2.2), (2.4), and (3.2), we have

$$\begin{aligned}g(\widehat{\nabla}_X Y, Z) &= g(\widehat{\nabla}_X \phi Y, \phi Z) = g(\widehat{\nabla}_X TY, \phi Z) + g(\widehat{\nabla}_X NY, \phi Z) \\ &= -g(\widehat{\nabla}_X T^2 Y, Z) - g(\widehat{\nabla}_X NTY, Z) + g(\widehat{\nabla}_X NY, \phi Z).\end{aligned}$$

Then, using (2.6), (3.2), and Lemma 3.10, and the fact that $NPZ = 0$, we have

$$\begin{aligned}g(\widehat{\nabla}_X Y, Z) &= \cos^2 \theta_1 g(\widehat{\nabla}_X Y, Z) + g(A_{NTY}X, Z) \\ &\quad - g(A_{NY}X, TPZ + TRZ) + g(\nabla_X^\perp NY, NRZ), \\ \sin^2 \theta_1 g(\widehat{\nabla}_X Y, Z) &= g(A_{NTY}X, Z) - g(A_{NY}X, TPZ + TRZ) \\ &\quad + g(\nabla_X^\perp NY, NRZ).\end{aligned}\quad (4.7)$$

Similarly, we get

$$\sin^2 \theta_1 g(\widehat{\nabla}_X Y, V) = -g(\nabla_X^\perp N T Y, V) - g(A_{N Y} X, t V) + g(\nabla_X^\perp N Y, n V). \quad (4.8)$$

Thus, from (4.7) and (4.8), we have the assertions. $\square$

In a similar way, we obtain the following case for the distribution $D_2 \oplus \langle \xi \rangle$:

Theorem 4.9 *Let M be a proper quasi-bi-slant submanifold of a Sasakian manifold $\widehat{M}$. Then, the distribution $D_2 \oplus \langle \xi \rangle$ defines a totally geodesic foliation on M if and only if*

$$\begin{aligned} g(\nabla_X^\perp N Y, N Q Z) &= g(A_{N Y} X, T P Z + T Q Z) - g(A_{N T Y} X, Z) \quad \text{and} \\ g(A_{N Y} X, t V) &= g(\nabla_X^\perp N Y, n V) - g(\nabla_X^\perp N T Y, V) \end{aligned} \quad (4.9)$$

for any $X, Y \in \Gamma(D_2 \oplus \langle \xi \rangle)$, $Z \in \Gamma(D \oplus D_1)$ and $V \in \Gamma(T^\perp M)$.

5 Quasi-bi-slant submanifolds with parallel canonical structures

In this section, we obtain some results for the quasi-bi-slant submanifolds with parallel canonical structures. Let M be a proper quasi-bi-slant submanifold of a Sasakian manifold $\widehat{M}$. Now, we define the covariant derivative of tangential and normal components of (3.2) and (3.7) as follows:

$$(\widehat{\nabla}_X T)Y = \nabla_X T Y - T \nabla_X Y, \quad (5.1)$$

$$(\widehat{\nabla}_X N)Y = \nabla_X^\perp N Y - N \nabla_X Y, \quad (5.2)$$

$$(\widehat{\nabla}_X t)V = \nabla_X t V - t \nabla_X^\perp V \quad (5.3)$$

and

$$(\widehat{\nabla}_X n)V = \nabla_X^\perp n V - n \nabla_X^\perp V \quad (5.4)$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$.

Then, the endomorphism T (resp., N) and the endomorphism t (resp., n) are parallel if $\widehat{\nabla} T \equiv 0$ (resp., $\widehat{\nabla} N \equiv 0$) and $\widehat{\nabla} t \equiv 0$ (resp., $\widehat{\nabla} n \equiv 0$), respectively.

Making use of Lemma 3.8 and (5.1)–(5.4), we have the following lemma:

Lemma 5.1 *Let M be a quasi-bi-slant submanifold of a Sasakian manifold $\widehat{M}$. Then, we obtain*

$$(\widehat{\nabla}_X T)Y = A_{N Y} X + t h(X, Y) + g(X, Y)\xi - \eta(Y)X, \quad (5.5)$$

$$(\widehat{\nabla}_X N)Y = n h(X, Y) - h(X, T Y), \quad (5.6)$$

$$(\widehat{\nabla}_X t)V = A_{n V} X - T A_V X \quad (5.7)$$

and

$$(\widehat{\nabla}_X n)V = -h(X, t V) - N A_V X \quad (5.8)$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$.

Proposition 5.2 *Let M be a quasi-bi-slant submanifold of a Sasakian manifold $\widehat{M}$. Then, T is parallel if and only if the shape operator satisfies*

$$A_{N Y} Z - A_{N Z} Y = \eta(Y)Z - \eta(Z)Y \quad (5.9)$$

for any $Y, Z \in \Gamma(TM)$.

Proof For any $X, Y, Z \in \Gamma(TM)$, from (5.5), we have

$$\begin{aligned} g((\widehat{\nabla}_X T)Y, Z) &= g(A_{NY}X + th(X, Y) + g(X, Y)\xi - \eta(Y)X, Z) \\ &= g(A_{NY}X, Z) - g(h(X, Y), \phi Z) + g(X, Y)\eta(Z) - g(X, Z)\eta(Y) \\ &= g(h(X, Z), NY) - g(h(X, Y), \phi Z) + g(X, Y)\eta(Z) - g(X, Z)\eta(Y) \\ &= g(h(X, Z), NY) - g(h(X, Y), NZ) + g(X, Y)\eta(Z) - g(X, Z)\eta(Y) \\ &= g(A_{NY}Z - A_{NZ}Y + \eta(Z)Y - \eta(Y)Z, X). \end{aligned}$$

From this, we obtain the proposition. $\square$

From (5.6), we have the following proposition:

Proposition 5.3 *Let M be a quasi-bi-slant submanifold of a Sasakian manifold $\widehat{M}$. Then, N is parallel if and only if*

$$nh(X, Y) = h(X, TY) \quad (5.10)$$

or equivalently

$$A_{nV}Y = -A_VTY$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$.

Theorem 5.4 *Let M be a quasi-bi-slant submanifold of a Sasakian manifold $\widehat{M}$. Then, N is parallel if and only if t is parallel.*

Proof For any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, using (2.5), (2.6), and (5.6), we have

$$\begin{aligned} g((\widehat{\nabla}_X N)Y, V) &= g(nh(X, Y) - h(X, TY), V) \\ &= -g(h(X, Y), \phi V) - g(h(X, TY), V) \\ &= -g(h(X, Y), nV) - g(\widehat{\nabla}_X TY, V) \\ &= -g(A_{nV}X, Y) + g(TY, \widehat{\nabla}_X V) \\ &= -g(A_{nV}X, Y) - g(TY, A_VX) \\ &= -g(A_{nV}X, Y) + g(TA_VX, Y) \\ &= -g((\widehat{\nabla}_X t)V, Y). \end{aligned}$$

which gives the assertion. $\square$

6 Example

Now, we give a non-trivial example of proper quasi-bi-slant submanifold in $\mathbb{R}^{11}$.

Consider the Euclidean space $\mathbb{R}^{11}$ with coordinates $(x_1, \dots, x_5, y_1, \dots, y_5, z)$. Define an almost contact structure on $\mathbb{R}^{11}$ as follows:

$$\begin{aligned} \phi \left(\sum_{i=1}^5 \left(X^i \frac{\partial}{\partial x_i} + Y^i \frac{\partial}{\partial y_i} \right) + Z \frac{\partial}{\partial z} \right) &= \sum_{i=1}^5 Y^i \frac{\partial}{\partial x_i} - \sum_{i=1}^5 X^i \frac{\partial}{\partial y_i} + \sum_{i=1}^5 Y^i y_i \frac{\partial}{\partial z}, \\ g &= (\eta \otimes \eta) + \frac{1}{4} \sum_{i=1}^5 (dx_i \otimes dx_i + dy_i \otimes dy_i), \\ \eta &= \frac{1}{2} \left(dz - \sum_{i=1}^5 y_i dx_i \right), \quad \xi = 2 \frac{\partial}{\partial z}. \end{aligned} \quad (6.1)$$

Then, $(\mathbb{R}^{11}, \phi, \xi, \eta, g)$ becomes a Sasakian manifold of dimension 11. We choose the vector fields

$$e_i = 2 \frac{\partial}{\partial y_i}, \quad e_{5+i} = 2 \left(\frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial z} \right), \quad e_{11} = \xi = 2 \frac{\partial}{\partial z}, \quad \text{for } i = 1, 2, \dots, 5.$$

Then, $\{e_1, e_2, \dots, e_{11}\}$ forms an orthonormal basis at each point of tangent space of $\mathbb{R}^{11}$.
Now, we consider a submanifold M of $\mathbb{R}^{11}$ defined by immersion ψ as follows:

$$\psi(u, v, w, r, s, t, q) = \left(u, \frac{w}{2}, 0, \frac{s}{2}, 0, v, \frac{\sqrt{3}}{2}r, \frac{r}{2}, \frac{t}{2}, \frac{t}{2}, q \right).$$

If we take

$$\begin{aligned} X_1 &= e_1, & X_2 &= e_6, & X_3 &= \frac{1}{2}e_2, \\ X_4 &= \frac{1}{2}(\sqrt{3}e_7 + e_8), & X_5 &= \frac{1}{2}e_4, \\ X_6 &= \frac{1}{2}(e_9 + e_{10}), & X_7 &= e_{11}. \end{aligned}$$

Then, the set $\{X_1, X_2, X_3, X_4, X_5, X_6, X_7\}$ restricted over M spans the tangent bundle TM of M .
Obviously, we get

$$\begin{aligned} \phi X_1 &= e_6, & \phi X_2 &= -e_1, & \phi X_3 &= \frac{1}{2}e_7, \\ \phi X_4 &= -\frac{1}{2}(\sqrt{3}e_2 + e_3), & \phi X_5 &= \frac{1}{2}e_9, \\ \phi X_6 &= -\frac{1}{2}(e_4 + e_5), & \phi X_7 &= 0. \end{aligned}$$

Now, let us put $D = \text{Span}\{X_1, X_2\}$, $D_1 = \text{Span}\{X_3, X_4\}$, $D_2 = \text{Span}\{X_5, X_6\}$.
Now, for any $X \in TM$, we have

$$\begin{aligned} X &= a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 + a_5X_5 + a_6X_6 + a_7X_7 \\ &= a_1e_1 + a_2e_6 + \frac{a_3}{2}e_2 + \frac{a_4}{2}(\sqrt{3}e_7 + e_8) + \frac{a_5}{2}e_4 + \frac{a_6}{2}(e_9 + e_{10}) + a_7e_{11}, \end{aligned}$$

where $a_1, a_2, \dots, a_7$ are C^∞ -functions on M .

Therefore, $PX = a_1e_1 + a_2e_6$, $QX = \frac{a_3}{2}e_2 + \frac{a_4}{2}(\sqrt{3}e_7 + e_8)$, and $RX = \frac{a_5}{2}e_4 + \frac{a_6}{2}(e_9 + e_{10})$.

By the use of (6.1), we have

$$\begin{aligned} \phi X &= a_1e_6 - a_2e_1 + \frac{a_3}{2}e_7 - \frac{a_4}{2}(\sqrt{3}e_2 + e_3) + \frac{a_5}{2}e_9 - \frac{a_6}{2}(e_4 + e_5) \\ &= a_1e_6 - a_2e_1 + \frac{a_3}{4\sqrt{3}}(\sqrt{3}e_7 + e_8) + \frac{a_3}{4\sqrt{3}}(\sqrt{3}e_7 - e_8) - \frac{a_4}{2}(\sqrt{3}e_2 + e_3) \\ &\quad + \frac{a_5}{4}(e_9 + e_{10}) + \frac{a_5}{4}(e_9 - e_{10}) - \frac{a_6}{2}(e_4 + e_5). \end{aligned}$$

Therefore, we have

$$\begin{aligned} TX &= a_1e_6 - a_2e_1 + \frac{a_3}{4\sqrt{3}}(\sqrt{3}e_7 + e_8) - \frac{\sqrt{3}a_4}{2}e_2 + \frac{a_5}{4}(e_9 + e_{10}) - \frac{a_6}{2}e_4, \\ NX &= \frac{a_3}{4\sqrt{3}}(\sqrt{3}e_7 - e_8) - \frac{a_4}{2}e_3 + \frac{a_5}{4}(e_9 - e_{10}) - \frac{a_6}{2}e_5. \end{aligned}$$

Now, using above relations, we have

$$\begin{aligned} TPX &= a_1e_6 - a_2e_1, & TQX &= \frac{a_3}{4\sqrt{3}}(\sqrt{3}e_7 + e_8) - \frac{\sqrt{3}a_4}{2}e_2, & TRX &= \frac{a_5}{4}(e_9 + e_{10}) - \frac{a_6}{2}e_4, \\ NPX &= 0, & NQX &= \frac{a_3}{4\sqrt{3}}(\sqrt{3}e_7 - e_8) - \frac{a_4}{2}e_3, & NRX &= \frac{a_5}{4}(e_9 - e_{10}) - \frac{a_6}{2}e_5. \end{aligned}$$

If we take $Z_1 = \frac{1}{2}(\sqrt{3}e_7 - e_8)$, $Z_2 = \frac{1}{2}e_3$, $Z_3 = \frac{1}{2}(e_9 - e_{10})$, and $Z_4 = \frac{1}{2}e_5$. Then

$$T^\perp M = \langle Z_1, Z_2 \rangle \oplus \langle Z_3, Z_4 \rangle.$$

Here, in this example, $\mu = \{0\}$.

Now, for any $V \in T^\perp M$, we have $V = b_1 Z_1 + b_2 Z_2 + b_3 Z_3 + b_4 Z_4$, where b_1, b_2, b_3, b_4 are C^∞ -functions on M . Therefore

$$V = \frac{b_1}{2}(\sqrt{3}e_7 - e_8) + \frac{b_2}{2}e_3 + \frac{b_3}{2}(e_9 - e_{10}) + \frac{b_4}{2}e_5, \quad (6.2)$$

Using (6.1), we get

$$\begin{aligned} \phi V &= -\frac{b_1}{2}(\sqrt{3}e_2 - e_3) + \frac{b_2}{2}e_8 - \frac{b_3}{2}(e_4 - e_5) + \frac{b_4}{2}e_{10} \\ &= -\frac{b_1}{2}(\sqrt{3}e_2 - e_3) + \frac{b_2}{4}(\sqrt{3}e_7 + e_8) - \frac{b_2}{4}(\sqrt{3}e_7 - e_8) \\ &\quad - \frac{b_3}{2}(e_4 - e_5) + \frac{b_4}{4}(e_9 + e_{10}) - \frac{b_4}{4}(e_9 - e_{10}). \end{aligned}$$

Therefore, we have

$$\begin{aligned} {}^t V &= -\frac{b_1}{2}\sqrt{3}e_2 + \frac{b_2}{4}(\sqrt{3}e_7 + e_8) - \frac{b_3}{2}e_4 + \frac{b_4}{4}(e_9 + e_{10}), \\ {}^n V &= \frac{b_1}{2}e_3 - \frac{b_2}{4}(\sqrt{3}e_7 - e_8) + \frac{b_3}{2}e_5 - \frac{b_4}{4}(e_9 - e_{10}). \end{aligned}$$

Using the property of Lie bracket, we have

$$\begin{aligned} [e_i, e_{5+i}] &= e_{11} \quad \text{for } i = 1, 2, \dots, 5, \\ \text{otherwise } [e_i, e_j] &= 0. \end{aligned} \quad (6.3)$$

Let $\widehat{\nabla}$ be the Levi-Civita connection of Riemannian metric g which is given by the Koszul's formula defined by

$$2g(\widehat{\nabla}_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \quad (6.4)$$

Using Koszul's formula, we can easily obtain non-zero components of Levi-Civita connection $\widehat{\nabla}$ as follows:

$$\begin{aligned} \widehat{\nabla}_{e_i} e_{11} &= -\frac{1}{2}e_{5+i}, \quad \widehat{\nabla}_{e_{5+i}} e_{11} = \frac{1}{2}e_i, \\ \widehat{\nabla}_{e_i} e_{5+i} &= \frac{1}{2}e_{11}, \quad \widehat{\nabla}_{e_{5+i}} e_i = -\frac{1}{2}e_{11}, \\ \widehat{\nabla}_{e_{11}} e_i &= -\frac{1}{2}e_{5+i}, \quad \widehat{\nabla}_{e_{11}} e_{5+i} = \frac{1}{2}e_i, \end{aligned} \quad (6.5)$$

for $i = 1, 2, \dots, 5$.

By the use of (2.5) and (6.5), we can easily obtain non-zero components of second fundamental form h as follows:

$$\begin{aligned} h(e_2, e_{11}) &= h(e_{11}, e_2) = -\frac{1}{2\sqrt{3}}Z_1, \\ h(e_4, e_{11}) &= h(e_{11}, e_4) = -\frac{1}{2}Z_3, \\ h(e_8, e_{11}) &= h(e_{11}, e_8) = Z_2, \\ h(e_{10}, e_{11}) &= h(e_{11}, e_{10}) = Z_4. \end{aligned} \quad (6.6)$$

Now, for any $X, Y \in TM$, we have

$$\begin{aligned} X &= a_1 X_1 + a_2 X_2 + a_3 X_3 + a_4 X_4 + a_5 X_5 + a_6 X_6 + a_7 X_7 \\ &= a_1 e_1 + a_2 e_6 + \frac{a_3}{2} e_2 + \frac{a_4}{2} (\sqrt{3} e_7 + e_8) + \frac{a_5}{2} e_4 + \frac{a_6}{2} (e_9 + e_{10}) + a_7 e_{11}, \\ Y &= c_1 e_1 + c_2 e_6 + \frac{c_3}{2} e_2 + \frac{c_4}{2} (\sqrt{3} e_7 + e_8) + \frac{c_5}{2} e_4 + \frac{c_6}{2} (e_9 + e_{10}) + c_7 e_{11}, \end{aligned} \quad (6.7)$$

where $a_1, a_2, \dots, a_7, c_1, \dots, c_7$ are C^∞ -functions on M .

Therefore, by the use of (6.6) and (6.7), we have

$$\begin{aligned} h(X, Y) &= -\frac{1}{4\sqrt{3}}(a_3 c_7 + a_7 c_3) Z_1 + \frac{1}{2}(a_4 c_7 + a_7 c_4) Z_2 \\ &\quad - \frac{1}{4}(a_5 c_7 + a_7 c_5) Z_3 + \frac{1}{2}(a_6 c_7 + a_7 c_6) Z_4. \end{aligned} \quad (6.8)$$

Using (2.7), (6.2), and (6.8), we can find shape operator A as follows:

$$\begin{aligned} A_V X &= -\frac{1}{\sqrt{3}} a_7 b_1 X_3 + \frac{1}{8} a_7 b_2 X_4 - \frac{1}{2} a_7 b_3 X_5 + \frac{1}{4} a_7 b_4 X_6 \\ &\quad - \left(\frac{1}{4\sqrt{3}} a_3 b_1 - \frac{1}{8} a_4 b_2 + \frac{1}{8} a_5 b_3 - \frac{1}{8} a_6 b_4 \right) X_7. \end{aligned}$$

Also one can easily check that the distribution D is invariant, the distribution D_1 is slant with slant angle $\frac{\pi}{6}$, and the distribution D_2 is also slant with slant angle $\frac{\pi}{4}$. Also, the distributions D , D_1 and D_2 satisfy the definition of quasi-bi-slant submanifold of Sasakian manifold. Hence, the submanifold M defined by ψ is a proper quasi-bi-slant submanifold of $\mathbb{R}^{11}$ with $\frac{\pi}{6}, \frac{\pi}{4}$ as its bi-slant angles.

Clearly, the distributions D , D_1 , and D_2 are not integrable. Hence, the given example verifies Proposition 4.1.

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