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Lightlike hypersurfaces in spaces with concircular fields

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Abstract Lightlike hypersurfaces in semi-Riemannian manifolds admitting concircular vector fields are investigated. We prove that such hypersurfaces are generally products of lightlike curves and warped product manifolds. In special cases, we show that these hypersurfaces are totally geodesic or totally screen geodesic provided such concircular fields belong to their normal or transversal bundles. A number of examples are furnished, where possible, to illustrate the main concepts.

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1 Introduction

A vector field $\bar{U}$ on a Riemannian manifold $\bar{M}$ is said to be a concircular vector field if, for any $\bar{X}$ tangent to $\bar{M}$, one has $\bar{\nabla}_{\bar{X}}\bar{U} = \rho\bar{X}$, where ρ is a smooth function on $\bar{M}$, and $\bar{\nabla}$ denotes the Levi-Civita connection of $\bar{M}$ [10]. In literature, such vector fields are also known as geodesic fields since integral curves of such vector fields are geodesics [17]. $\bar{U}$ is called a concurrent vector field if $\rho = 1$ [3,5,6]. The notion of concircular vector fields can, naturally, be extended to semi-Riemannian manifolds. In that line, Chen, in [5], has proved that a Lorentzian manifold is a generalised Robertson–Walker spacetime if and only if it admits a timelike concircular vector field. On the other hand, concircular vector fields are also important in the study of concircular mappings, that is; conformal mappings preserving geodesic circles [19]. In physics, particularly in general relativity, concircular vector fields have interesting applications. For instance, it has been noted that trajectories of timelike concircular fields in the de-Sitter model determine the world lines of receding or colliding galaxies satisfying the Weyl hypothesis [18].

A (semi-)Riemannian manifold $(\bar{M}, \bar{g})$ is called a Ricci soliton if there exists a smooth vector field $\bar{V}$ on $\bar{M}$ such that $\mathcal{L}_{\bar{V}}\bar{g} + 2\bar{Ric} = 2\lambda\bar{g}$, where λ is a real constant, $\bar{Ric}$ is the Ricci tensor of $\bar{M}$ and $\mathcal{L}_{\bar{V}}$ is the Lie derivative operator. We often denote a Ricci soliton as $(\bar{M}, \bar{g}, \bar{V}, \lambda)$. The smooth vector field $\bar{V}$ is called a potential field for the Ricci soliton. Furthermore, a Ricci soliton $(\bar{M}, \bar{g}, \bar{V}, \lambda)$ is said to be steady, shrinking or expanding if $\lambda = 0$, $\lambda > 0$ or $\lambda < 0$, respectively. A Ricci soliton is said to be a gradient Ricci soliton if there exists a smooth function μ on $\bar{M}$ such that $\bar{V} = \bar{\nabla}\mu$. Furthermore, μ is called the potential function of the Ricci soliton. It follows that $\bar{Ric} + \bar{\nabla}^2\mu = \lambda\bar{g}$, where $\bar{\nabla}^2\mu$ denotes the Hessian of μ . Einstein metrics are some of the trivial examples of Ricci solitons.

We study the geometry of lightlike hypersurfaces of semi-Riemannian manifolds admitting a concircular vector field $\bar{U}$. To the best of our understanding, this is the first initiative in this direction. The study of lightlike hypersurfaces, and submanifolds in general, was initiated Duggal and Bejancu [8], and later by Duggal and Sahin [9]. These two books contain a collection of many interesting results on such submanifolds, and have,

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further, motivated other scholars to take an active role in the study of lightlike geometry. For instance see [1, 11–13], and many other references cited therein.

The paper is arranged as follows: in Sect. 2, we quote some basic notions on lightlike hypersurfaces needed in the rest of the paper. In Sect. 3, lightlike hypersurfaces in semi-Riemannian manifolds with concircular vector fields are introduced. It is shown that such hypersurfaces are totally geodesic (resp., totally screen geodesic) provided $\bar{U}$ belongs to their normal bundle (resp., transversal bundle). In Sect. 4, we study the geometry of hypersurfaces in space forms with concircular vector fields, by showing that the ambient manifolds are always flat whenever $\bar{U}$ belongs to the normal or the transversal bundle of the hypersurface. Finally, as an application, in Sect. 5, we characterise lightlike hypersurfaces, whose screen distributions admit a Ricci soliton induced from the screen component of $\bar{U}$.

2 Preliminaries

Let $(\bar{M}, \bar{g})$ be an $(n + 2)$ -dimensional semi-Riemannian manifold, where $\bar{g}$ is a metric tensor of index $0 < \nu < n + 2$. Suppose that (M, g) is a hypersurface of $\bar{M}$, such that $g = \bar{g}|_M$. If g degenerates on M , then there exists a nonzero vector field ξ on M such that $g(\xi, \nu) = 0$, for any ν tangent to M . Moreover, the radical space (also known as the null space) [16, p. 53] of $T_p M$, at each point $p \in M$, is a subspace $\text{Rad } T_p M$ defined by $\text{Rad } T_p M = \{\xi \in T_p M : g_p(\xi, \nu) = 0, \forall \nu \in T_p M\}$, whose dimension is called the nullity degree of g and M is called a lightlike hypersurface of $\bar{M}$. It follows from the above discussion that the normal space $T_p M^\perp$ of M is also lightlike and satisfies $\text{Rad } T_p M = T_p M \cap T_p M^\perp$. Since M is a hypersurface of $\bar{M}$, it follows that $\dim T_p M^\perp = 1$, which implies that $\dim \text{Rad } T_p M = 1$ and $\text{Rad } T_p M = T_p M^\perp$. We often call $\text{Rad } TM$ a radical distribution of M . It is now clear that for a lightlike hypersurface M , TM and $TM^\perp$ have a nontrivial intersection and their sum is not the whole of tangent bundle space $T\bar{M}$. This anomaly implies that a vector of $T_p \bar{M}$ cannot be decomposed uniquely into a component tangent to $T_p M$ and a component of $T_p M^\perp$. Moreover, the standard text-book definitions of the second fundamental form and the Gauss–Weingarten formulas are not applicable in the lightlike case.

As a remedy to this anomaly, Duggal–Bejancu [8] (also see Duggal Bejancu [9]) introduced an approach to lightlike geometry, which we have followed in the present paper. More precisely, their approach consists of fixing, on the lightlike hypersurface, a geometric data formed by a lightlike section and a screen distribution. By a screen distribution of M , we mean a complementary subbundle of $TM^\perp$ in TM , which is a rank n nondegenerate distribution over M . A downside to this approach is that there are infinitely many possibilities of choices for such a screen distribution provided the hypersurface M is paracompact, but each of them is canonically isomorphic to the factor vector bundle $TM/ TM^\perp$ studied in detail by Kupeli [14]. Furthermore, we denote by $S(TM)$ the screen distribution over M . With such notation, we have the decomposition $TM = S(TM) \oplus TM^\perp$, where $\oplus$ denotes the orthogonal direct sum. It is known [8] (also see [9]) that for a lightlike hypersurface equipped with a screen distribution, there exists a unique rank one vector subbundle $\text{ltr}(TM)$ of TM over M , called a lightlike transversal vector bundle, such that for any nonzero section ξ of $TM^\perp$ on a coordinate neighbourhood $\mathcal{U} \subset M$, there exists a unique section N of $\text{ltr}(TM)$ on $\mathcal{U}$ satisfying $\bar{g}(\xi, N) = 1$, and $\bar{g}(N, N) = \bar{g}(N, Z) = 0$, for any Z tangent to $S(TM)$. It then follows that

$$T\bar{M}|_M = S(TM) \oplus \{TM^\perp \oplus \text{ltr}(TM)\} = TM \oplus \text{ltr}(TM), \quad (2.1)$$

where $\oplus$ denote the direct (nonorthogonal) sum. Throughout this paper, we denote by $\Gamma(\Xi)$ the set of smooth sections of a vector bundle Ξ over M , and $F(M)$ the set of smooth functions on M .

Let M be a lightlike hypersurface of $\bar{M}$, and let $\bar{\nabla}$ denote the Levi-Civita connection on $\bar{M}$. Then, the Gauss–Weingarten equations of M are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \quad (2.2)$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N, \quad \forall X, Y \in \Gamma(TM), \quad (2.3)$$

where, $\nabla_X Y$ and $A_N X$ belongs to $\Gamma(TM)$, while $B(X, Y)$ and $\tau(X)$ belongs to $F(M)$. Note that, B is a symmetric tensor field of type $(0, 2)$, called the local second fundamental form of M , τ is a differential 1-form, A_N is a tensor field of type $(1, 1)$, called the shape operator of M and ∇ is a torsion-free linear connection on M . Let P be the projection morphism of TM onto $S(TM)$. Then, we have

$$\nabla_X PY = \nabla_X^* PY + C(X, PY)\xi, \quad (2.4)$$

$$\nabla_X \xi = -A_\xi^* X - \tau(X)\xi, \quad \forall X, Y \in \Gamma(TM). \quad (2.5)$$

Here, $\nabla_X^* PY$ belongs to $S(TM)$ and $A_\xi^* X$ belongs to $S(TM)$, while $C(X, PY)$ belongs to $F(M)$. Unlike B , C is generally not symmetric and called the local second fundamental of $S(TM)$. In addition, A_ξ^* is a $(1,1)$ tensor, called the screen shape operator. On the other hand, ∇^* is the Levi-Civita connection on $S(TM)$. The local second fundamental forms B and C are not related by means of g . In fact, by (2.2)–(2.5), we have

$$g(A_\xi^* X, PY) = B(X, PY), \quad \bar{g}(A_\xi^* X, N) = 0, \quad (2.6)$$

$$g(A_N X, PY) = C(X, PY), \quad \bar{g}(A_N X, N) = 0, \quad (2.7)$$

for any X and Y tangent to M . It follows from the second equality in (2.7) that A_N is screen-valued. In addition, from $\bar{g}(\nabla_X \xi, \xi) = 0$, (2.2) leads to

$$B(X, \xi) = 0, \quad (2.8)$$

for any X tangent to M . As A_ξ^* is screen-valued, (2.6) and (2.8) implies that

$$A_\xi^* \xi = 0. \quad (2.9)$$

The induced linear connection ∇ is not a metric connection on M . In fact, let θ be a 1-form given by $\theta(X) = \bar{g}(X, N)$, then

$$(\nabla_X g)(Y, Z) = B(X, Z)\theta(Y) + B(X, Y)\theta(Z), \quad (2.10)$$

for any X, Y and Z tangent to M .

Let (M, g) be a lightlike hypersurface of a semi-Riemannian manifold $\bar{M}$. Duggal and Bejancu [8, p. 94] have showed that the curvature tensor, $\bar{R}$, of $\bar{M}$ satisfies the following relations:

$$\bar{g}(\bar{R}(X, Y)Z, \xi) = (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + \tau(X)B(Y, Z) - \tau(Y)B(X, Z), \quad (2.11)$$

$$\bar{g}(\bar{R}(X, Y)PZ, N) = (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) - \tau(X)C(Y, PZ) + \tau(Y)C(X, PZ), \quad (2.12)$$

where ∇B and ∇C are given by

$$(\nabla_X B)(Y, Z) = X \cdot B(Y, Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z), \quad (2.13)$$

$$(\nabla_X C)(Y, PZ) = X \cdot C(Y, PZ) - C(\nabla_X Y, PZ) - C(Y, \nabla_X^* PZ), \quad (2.14)$$

for any X, Y and Z tangent to M .

Definition 2.1 Let (M, g) be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. A point $p \in M$ is said to be umbilic [8, p. 106] if $B_p(X_p, Y_p) = kg_p(X_p, Y_p)$, for $X_p, Y_p \in M$, where B is the local second fundamental form defined around p and $k \in \mathbb{R}$. One says that M is totally umbilic if any point of M is umbilic. It is easy to see that M is totally umbilic if and only if, locally, on each $\mathcal{U}$ there exists a smooth function ϕ such that $B(X, Y) = \phi g(X, Y)$, for any X and Y tangent to M . In case $\phi = 0$ (resp., $\phi \neq 0$) on $\mathcal{U}$ one says M is totally geodesic (resp., properly totally umbilic).

Definition 2.2 Let (M, g) be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. $S(TM)$ is totally umbilic [8] if on any coordinate neighbourhood $\mathcal{U}$ there exists a smooth function ψ such that $C(X, PY) = \psi g(X, PY)$, for any X and Y tangent to M . In case $\psi = 0$ (resp., $\psi \neq 0$) on $\mathcal{U}$ one says $S(TM)$ is totally geodesic (resp., properly totally umbilic). We call M a totally screen umbilic lightlike hypersurface if its screen distribution is totally umbilic.

Definition 2.3 A lightlike hypersurface (M, g) of a semi-Riemannian manifold is called screen locally conformal [9] if the shape operators A_N and A_ξ^* of M and $S(TM)$, respectively, are related by $A_N = \varphi A_\xi^*$, where φ is a non-vanishing smooth function on a neighbourhood $\mathcal{U}$ of M . We say that M is screen homothetic if φ is nonzero constant.

We windup this section by giving an example of a lightlike hypersurface satisfying all the above three definitions above.

Example 2.4 Consider the future-directed null cone Λ_0^3 of R_1^4 with signature $(-, +, +, +)$ with respect to a canonical basis $\{\partial_t, \partial_1, \partial_2, \partial_3\}$, and given by

$$\Lambda_0^3 = \left\{ (t, x^1, x^2, x^3) : t = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} =: r > 0 \right\}.$$

By a direct calculation, we have $\text{Rad } TM = \text{Span}\{\xi\}$, where

$$\xi = \partial_t + \frac{x^1}{r}\partial_1 + \frac{x^2}{r}\partial_2 + \frac{x^3}{r}\partial_3. \quad (2.15)$$

On the other hand, $S(TM) = \text{Span}\{X, Y\}$, where

$$X = X^1\partial_1 + X^2\partial_2 + X^3\partial_3, \quad Y = Y^1\partial_1 + Y^2\partial_2 + Y^3\partial_3, \quad (2.16)$$

such that

$$\sum_{i=1}^3 x^i X^i = \sum_{i=1}^3 x^i Y^i = 0. \quad (2.17)$$

Using (2.15), (2.16) and (2.17),

$$\bar{\nabla}_\xi \xi = 0, \quad \bar{\nabla}_X \xi = \frac{1}{r}X, \quad (2.18)$$

for all X tangent to $S(T\Lambda_0^3)$, where $\bar{\nabla}$ is the connection on R_1^4 . It follows from (2.2), (2.5) and (2.18), that

$$A_\xi^* \xi = 0, \quad \tau(\xi) = 0, \quad A_\xi^* X = -\frac{1}{r}X, \quad \tau(X) = 0, \quad (2.19)$$

for any X tangent to $S(T\Lambda_0^3)$. Thus, ξ is a geodesic lightlike vector field and Λ_0^3 is properly totally umbilic with $\phi = -\frac{1}{r}$. Next, $\text{ltr}(TM)$ is spanned by N , where

$$N = \frac{1}{2} \left\{ -\partial_t + \frac{x^1}{r}\partial_1 + \frac{x^2}{r}\partial_2 + \frac{x^3}{r}\partial_3 \right\}. \quad (2.20)$$

From (2.15), (2.16), (2.17) and (2.20), we find

$$\bar{\nabla}_\xi N = 0, \quad \bar{\nabla}_X N = \frac{1}{2r}X, \quad (2.21)$$

for all X tangent to $S(T\Lambda_0^3)$. Hence, using (2.3) and (2.21), we have

$$A_N \xi = 0, \quad A_N X = -\frac{1}{2r}X. \quad (2.22)$$

It follows from (2.22) that $S(T\Lambda_0)$ is totally umbilic with $\psi = -\frac{1}{2r}$. We also note from (2.19) and (2.22) that Λ_0 is screen conformal, with $\varphi = \frac{1}{2}$, since $A_N = \frac{1}{2}A_\xi^*$.

3 Concircular vector fields and hypersurfaces

Let $(\bar{M}, \bar{g})$ be a semi-Riemannian manifold. A nonzero vector field $\bar{U}$ tangent to $\bar{M}$ is called a concircular [6] (also see [4] and [5]) vector field if, for any $\bar{X}$ tangent to $\bar{M}$, one has

$$\bar{\nabla}_{\bar{X}} \bar{U} = \rho \bar{X}, \quad (3.1)$$

where $\rho : \bar{M} \rightarrow \mathbb{R}$ is a smooth function. $\bar{U}$ is called a concurrent vector field if $\rho = 1$. A direct calculation, while using (3.1), leads to

$$\bar{R}(\bar{X}, \bar{Y})\bar{U} = \bar{\nabla}_{\bar{X}} \bar{\nabla}_{\bar{Y}} \bar{U} - \bar{\nabla}_{\bar{Y}} \bar{\nabla}_{\bar{X}} \bar{U} - \bar{\nabla}_{[\bar{X}, \bar{Y}]} \bar{U} = (\bar{X}\rho)\bar{Y} - (\bar{Y}\rho)\bar{X}, \quad (3.2)$$

for any $\bar{X}$ and $\bar{Y}$ tangent to $\bar{M}$.

Example 3.1 Let $\bar{M} = (\mathbb{R}_q^{n+2}, \bar{g})$ be a pseudo-Euclidean space of signature ν , with $0 < \nu < n + 2$, that is, $\bar{g}(x, x) = -(x^1)^2 - \dots - (x^\nu)^2 + (x^{\nu+1})^2 + \dots + (x^{n+2})^2$. The position vector field $\bar{U}(x) = \sum_{i=1}^{n+2} x^i \partial_i|_x$ satisfies $\bar{\nabla}_{\bar{X}} \bar{U} = \bar{X}$, for each $\bar{X}$ tangent to $\bar{M}$. It follows that $\bar{U}$ is concircular with $\rho = 1$. In particular, $\bar{U}$ is a concurrent vector field on $\bar{U}$.

Example 3.2 Consider (F, g_F) to be an $(n + 1)$ -dimensional, connected, Riemannian manifold, $(I, -dt^2)$ an open interval of $\mathbb{R}$ with its usual metric reversed, and $f > 0$ a smooth function on I . A Generalised Robertson–Walker (GRW) spacetime with base $(I, -dt^2)$, and fibre (F, g_F) and warping function f is the product manifold $\bar{M}(\varepsilon, f) = I \times_f F$ endowed with the Lorentz metric $\bar{g} = -\pi_I^* dt^2 + (f \circ \pi_I)^2 \pi_F^* g_F \equiv -dt^2 + f^2(t)g_F$, where π_I and π_F are the natural projections of $I \times F$ onto I and F , respectively, and ε the constant sectional curvature of F . The timelike vector field $\bar{U} = f\partial_t$ satisfy $\bar{\nabla}_{\bar{X}} \bar{U} = f'\bar{X}$, for any $\bar{X}$ tangent to $\bar{M}$. Therefore, $\bar{U}$ is concircular with $\rho = f'$.

Next, define a 1-form η on $\bar{M}$ by $\eta(\bar{X}) = \bar{g}(\bar{X}, \bar{U})$, for any $\bar{X}$ tangent to $\bar{M}$. As $\bar{U}$ is a nonzero vector field, we see that η is never identically zero on a semi-Riemannian manifold $\bar{M}$. Furthermore, let us set $f := \frac{1}{2}\eta(\bar{U})$. Then, using (3.1), we have

$$\bar{X}f = \rho\eta(\bar{X}), \quad (3.3)$$

for any $\bar{X}$ tangent to $\bar{M}$. In view of (3.3), the following hold.

Proposition 3.3 Let $(\bar{M}, \bar{g})$ be a semi-Riemannian manifold, with a nonzero concircular vector field $\bar{U}$. Then, f is constant if and only if $\rho = 0$.

Remark 3.4 Note that when $\bar{U}$ is a lightlike vector field of $\bar{M}$, one gets $f = \frac{1}{2}\eta(\bar{U}) = 0$, and hence $\rho = 0$.

Corollary 3.5 There does not exist any semi-Riemannian manifold with a lightlike concurrent vector field.

Corollary 3.6 There does not exist any lightlike hypersurface (M, g) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ such that the lightlike vector fields ξ or N are concurrent vector fields in $\bar{M}$.

Let (M, g) be a lightlike hypersurface of a semi-Riemannian manifold $\bar{M}$, admitting a nonzero concircular vector field $\bar{U}$. Next, define two smooth functions $\alpha, \beta : \bar{M} \rightarrow \mathbb{R}$ by $\alpha = \bar{g}(N, \bar{U}) = \eta(N)$ and $\beta = \bar{g}(\xi, \bar{U}) = \eta(\xi)$. Then, in view of (2.1), $\bar{U}$ can be written as

$$\bar{U} = U^* + \alpha\xi + \beta N, \quad (3.4)$$

where U^* is the component of $\bar{U}$ tangent to $S(TM)$.

Proposition 3.7 Let $\bar{U}$ be a concircular vector field on $(\bar{M}, \bar{g})$, and (M, g) a lightlike hypersurface of $\bar{M}$. Then, for any X tangent to M , the following holds:

$$\nabla_X^* U^* = \rho PX + \alpha A_\xi^* X + \beta A_N X, \quad (3.5)$$

$$X\alpha = \alpha\tau(X) - C(X, U^*) + \rho\theta(X), \quad (3.6)$$

$$X\beta = -\beta\tau(X) - B(X, U^*), \quad (3.7)$$

$$\bar{g}(\bar{\nabla}_N U^*, PX) = -\alpha\bar{g}(\bar{\nabla}_N \xi, PX) - \beta\bar{g}(\bar{\nabla}_N N, PX), \quad (3.8)$$

$$N\alpha = -\bar{g}(\bar{\nabla}_N U^*, N) - \alpha\bar{g}(\bar{\nabla}_N \xi, N), \quad (3.9)$$

$$N\beta = \rho - \bar{g}(\bar{\nabla}_N U^*, \xi) - \beta\bar{g}(\bar{\nabla}_N N, \xi). \quad (3.10)$$

Proof Applying $\bar{\nabla}_X$ to (3.4) and considering (3.1), we have

$$\rho X = \bar{\nabla}_X U^* + (X\alpha)\xi + \alpha \bar{\nabla}_X \xi + (X\beta)N + \beta \bar{\nabla}_X N, \quad (3.11)$$

for any X tangent to M . Then, using (2.2)–(2.5) and (3.11), we have

$$\rho X = \nabla_X^* U^* - \alpha A_\xi^* X - \beta A_N X + \{C(X, U^*) + X\alpha - \alpha\tau(X)\}\xi + \{B(X, U^*) + X\beta + \beta\tau(X)\}N. \quad (3.12)$$

As A_ξ^* and A_N are screen-valued, we obtain (3.5), (3.6) and (3.7) by comparing components in (3.12). On the other hand, replacing $\bar{X}$ with N in (3.1) and using (3.4), we have

$$\bar{\nabla}_N U^* + (N\alpha)\xi + \alpha \bar{\nabla}_N \xi + (N\beta)N + \beta \bar{\nabla}_N N = \rho N. \quad (3.13)$$

Then, (3.8), (3.9) and (3.10) follows from (3.13) by taking the inner product with PX , N and ξ , respectively. $\square$

Corollary 3.8 *For any X tangent to M , the following holds:*

(1) *If $\bar{U}$ is tangent to $TM^\perp$, then α satisfies*

$$\alpha A_\xi^* X = 0, \quad X\alpha - \alpha\tau(X) = 0, \quad (3.14)$$

$$N\alpha - \eta(\bar{\nabla}_N N) = 0, \quad \eta(\bar{\nabla}_N PX) = 0. \quad (3.15)$$

(2) *If $\bar{U}$ is tangent to $\text{ltr}(TM)$, then β satisfies*

$$\beta A_N X = 0, \quad X\beta + \beta\tau(X) = 0, \quad (3.16)$$

$$N\beta - \eta(\bar{\nabla}_N \xi) = 0, \quad \eta(\bar{\nabla}_N PX) = 0. \quad (3.17)$$

(3) *If $\bar{U}$ is tangent to $S(TM)$, then the following hold:*

$$\nabla_X^* U^* = \rho PX, \quad \rho = C(\xi, U^*) = \bar{g}(\bar{\nabla}_N U^*, \xi), \quad (3.18)$$

$$C(PX, U^*) = B(X, U^*) = 0,$$

$$\bar{g}(\bar{\nabla}_N U^*, PX) = \bar{g}(\bar{\nabla}_N U^*, N) = 0, \quad (3.19)$$

and for any $\bar{X}$ orthogonal to U^ , we have*

$$(\bar{X}\rho)f = 0. \quad (3.20)$$

(4) *If $\bar{U}$ is tangent to $S(TM)^\perp$, then*

$$\beta A_N X + \alpha A_\xi^* X + \rho PX = 0, \quad X\beta + \beta\tau(X) = 0,$$

$$\rho = \xi\alpha - \alpha\tau(\xi) = N\beta + \beta\bar{g}(\bar{\nabla}_N N, \xi),$$

$$\eta(\bar{\nabla}_N PX) = 0, \quad N\alpha - \alpha\bar{g}(\bar{\nabla}_N N, \xi) = 0. \quad (3.21)$$

Proof When $\bar{U}$ belongs to $TM^\perp$, we have $U^* = 0$, $\beta = 0$ and $\bar{U} = \alpha\xi$. Moreover, $f = 0$ and by Proposition 3.3 we have $\rho = 0$. Hence, (3.5), (3.6), (3.8) and (3.9) of Proposition 3.7 reduces to $\alpha A_\xi^* X = 0$, $X\alpha = \alpha\tau(X)$, $\bar{g}(\bar{\nabla}_N \xi, PX) = 0$ and $N\alpha = -\alpha\bar{g}(\bar{\nabla}_N \xi, N)$, for any X tangent to M . Next, if $\bar{U}$ belongs to $\text{ltr}(TM)$, we have $U^* = 0$, $\alpha = 0$ and $\bar{U} = \beta N$, where $\beta \neq 0$. In addition, in this case, $f = 0$ and by Proposition 3.3, we have $\rho = 0$. Thus, (3.5), (3.7), (3.8) and (3.10) gives $\beta A_N X = 0$, $X\beta = -\beta\tau(X)$, $\bar{g}(\bar{\nabla}_N N, PX) = 0$ and $N\beta = -\beta\bar{g}(\bar{\nabla}_N N, \xi)$, for any X tangent to M . When $\bar{U}$ belongs to $S(TM)$, we see that $\alpha = \beta = 0$ and $\bar{U} = U^*$. It follows from (3.5), (3.6), (3.7), (3.8)–(3.10) that

$$\nabla_X^* U^* = \rho PX, \quad C(X, U^*) = \rho\theta(X), \quad B(X, U^*) = 0,$$

$$\bar{g}(\bar{\nabla}_N U^*, PX) = \bar{g}(\bar{\nabla}_N U^*, N) = 0, \quad \rho = \bar{g}(\bar{\nabla}_N U^*, \xi),$$

for any X tangent to M . Now, using (3.2) with $\bar{Y} = \bar{U} = U^*$, we have

$$\bar{g}(\bar{R}(\bar{X}, U^*)U^*, U^*) = (\bar{X}\rho)g(U^*, U^*) = 2(\bar{X}\rho)f = 0,$$

for any $\bar{X}$ orthogonal to U^* , which completes part three. Finally, if $\bar{U}$ belongs to $S(TM)^\perp$, we see that $U^* = 0$. Thus, from (3.5), (3.6), (3.8)–(3.10), we get

$$\begin{aligned}\beta A_N X + \alpha A_\xi^* X + \rho P X &= 0, & X\alpha - \alpha\tau(X) &= \rho\theta(X), \\ X\beta &= -\beta\tau(X), & \alpha\bar{g}(\bar{\nabla}_N \xi, PX) + \beta\bar{g}(\bar{\nabla}_N N, PX) &= 0, \\ N\alpha &= -\alpha\bar{g}(\bar{\nabla}_N \xi, N), & N\beta &= \rho - \beta\bar{g}(\bar{\nabla}_N N, \xi),\end{aligned}$$

which completes the proof. $\square$

Theorem 3.9 *Let (M, g) be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$, admitting a nonzero concircular vector field $\bar{U}$.*

- (1) *If $\bar{U}$ belongs to $TM^\perp$ then, M is totally geodesic and the induced linear connection ∇ on M is a metric connection.*
- (2) *If $\bar{U}$ belongs to $\text{ltr}(TM)$, then M is totally screen geodesic, and locally a product manifold $\ell_\xi \times M^*$, where ℓ_ξ is a lightlike curve tangent to $TM^\perp$ and M^* is a leaf of $S(TM)$.*
- (3) *If $\bar{U}$ belongs to $S(TM)^\perp := TM^\perp \oplus \text{ltr}(TM)$, then M is one of the following:*
 - (a) *a lightlike hypersurface of the first category above;*
 - (b) *$A_N \xi = 0$ and M is locally a product manifold $\ell_\xi \times M^*$, where ℓ_ξ is a lightlike curve tangent to $TM^\perp$ and M^* is a leaf of $S(TM)$.*

In all the above cases, M admits an induced Ricci tensor.

Proof Suppose that $\bar{U}$ belongs to $TM^\perp$, then the first relation in (3.14) gives $A_\xi^* = 0$, and hence, by Definition 2.1, M is totally geodesic. Now, from (2.10) we see that $\nabla g = 0$, showing that ∇ is a metric connection. Next, if $\bar{U}$ belongs to $\text{ltr}(TM)$, the first relation in (3.16) gives $A_N = 0$. That is, by Definition 2.2, M is totally screen geodesic. Moreover, from (2.4), we see that $S(TM)$ is parallel with respect to ∇ , and hence an integrable distribution on M . It follows that M is locally a product manifold $\ell_\xi \times M^*$, where ℓ_ξ is a lightlike curve tangent to $TM^\perp$ and M^* is a leaf of $S(TM)$. Finally, when $\bar{U}$ belongs to $S(TM)^\perp$, we take $X = \xi$ in the first relation of (3.21) and using (2.9), to have $\beta A_N \xi = 0$. It follows that either $\beta = 0$ or $A_N \xi = 0$. In the first case, we note that $\bar{U}$ belongs to $TM^\perp$ and M lies in the first category of the theorem. On the other hand, the second case implies that $\beta \neq 0$ and so $A_N + \frac{\alpha}{\beta} A_\xi^* = \frac{\rho}{\beta} P$, which in turn means that A_N is symmetric on $S(TM)$ since A_ξ^* is symmetric. This means that $S(TM)$ is an integrable distribution over M . As before, M is locally a product manifold $\ell_\xi \times M^*$, where ℓ_ξ is a lightlike curve tangent to $TM^\perp$ and M^* is a leaf of $S(TM)$. Now, in view of the second relations in (3.14), (3.16) and (3.21), we note that $d\tau = 0$, and hence, by [9, Theorem 2.4.1, p. 68], M admits an induced Ricci tensor. $\square$

Corollary 3.10 *There does not exist any proper totally umbilic or proper totally screen umbilic lightlike hypersurface such that $\bar{U}$ is tangent to $TM^\perp$ or $\text{ltr}(TM)$.*

Theorem 3.11 *There does not exist any proper totally umbilic or proper totally screen umbilic lightlike hypersurface such that $\bar{U}$ is tangent to $S(TM)$.*

Proof Suppose that M is totally umbilic and $\bar{U}$ belongs to $S(TM)$, then Corollary 3.8, relation (3.19) and Definition 2.1 implies that $B(X, U^*) = \phi g(X, U^*) = 0$, for any X tangent to $S(TM)$. Since $S(TM)$ is nondegenerate, we have $\phi = 0$, that is M is totally geodesic. In a similar way, if $S(TM)$ is totally umbilic, we have $C(PX, U^*) = \psi g(PX, U^*) = 0$, which gives $\psi = 0$. $\square$

Theorem 3.12 *Any totally screen umbilic or screen conformal lightlike hypersurface such that $\bar{U}$ is tangent to $S(TM)$ is locally a triple product $\ell_\xi \times l_{U^*} \times M'$, where ℓ_ξ is a lightlike curve tangent to $TM^\perp$, l_{U^*} is a curve of the line bundle $\langle U^* \rangle$ spanned by U^* and M' is an $(n - 1)$ -dimensional leaf of the orthogonal complement of $\langle U^* \rangle$ in $S(TM)$.*

Proof When M is either screen totally umbilic or screen conformal, (3.18), Definitions (2.2), (2.3) and relation (2.8) leads to $\rho = C(\xi, U^*) = \psi g(\xi, U^*) = 0$ and $\rho = C(\xi, U^*) = \varphi B(\xi, U^*) = 0$, respectively. That is, $\bar{U} = U^*$ is a parallel vector field on $S(TM)$, i.e. $\nabla_X^* U^* = 0$, for any X tangent to $S(TM)$. Moreover, $S(TM)$ is an integrable distribution over M , and therefore M is a product manifold $\ell_\xi \times M^*$. Here, ℓ_ξ is a lightlike curve tangent to $TM^\perp$ and M^* is a leaf of $S(TM)$. Next, a direct calculation shows that $g(U^*, \nabla_{X'}^* Y') = 0$, for any X', Y' tangent to M^* and orthogonal to U^* . Thus, if $\mathcal{D}$ is the complementary distribution of the line

bundle $\langle U^* \rangle$ spanned by U^* in M^* , then $\mathcal{D}$ is parallel and therefore, an integrable distribution over $S(TM)$. Hence, by de-Ram's decomposition theorem in [7], each leaf M^* of $S(TM)$ is locally a product manifold $l_{U^*} \times M'$, where l_{U^*} is a curve of $\langle U^* \rangle$ and M' is an $(n-1)$ -dimensional leaf of $\mathcal{D}$. $\square$

We can generalise Theorem 3.12 to the following result.

Theorem 3.13 *Let (M, g) be a screen integrable lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$, admitting a concircular vector field $\bar{U}$ tangent to a semi-Riemannian screen distribution $S(TM)$. Then, M is locally a triple product manifold $\ell_\xi \times I \times_{\gamma(t)} M'$, where ℓ_ξ is a lightlike curve tangent to $TM^\perp$, $\gamma(t)$ is a nonzero smooth function on an open interval I with arc length t , $\partial_t = E_0$ is a unit vector field satisfying $\bar{U} = U^* = \gamma E_0$, and M' is a semi-Riemannian $(n-1)$ -manifold.*

Proof The integrability of $S(TM)$ implies that C is symmetric on $S(TM)$, hence M is a product manifold $\ell_\xi \times M^*$, where ℓ_ξ is a lightlike curve tangent to $TM^\perp$ and M^* is a leaf of $S(TM)$. Note that, in general $C(\xi, \cdot)$ is nonzero, although $S(TM)$ is integrable. As $\bar{U}$ belongs to $S(TM)$, (3.18) and (3.20) of Corollary (3.8) and (3.3) gives

$$\nabla_E^* U^* = \rho E, \quad \rho = C(\xi, U^*), \quad \xi \rho = N\rho = 0, \quad \xi f = Nf = 0, \quad (3.22)$$

for any E tangent to $S(TM)$ and $f = (1/2)g(U^*, U^*)$, which turns out to be a nonzero smooth function on the leaves M^* . Now, the rest of the proof will follow similar steps as in the proof of [5, Theorem 1]. In that line, let E_0 be a unit vector field parallel to U^* , i.e. there exist a nonzero smooth function γ , on M^* , such that

$$U^* = \gamma E_0. \quad (3.23)$$

We may now extend E_0 to a quasi-orthonormal basis $\{E_0, E_i\}$ of M^* , where $i = 1, 2, \dots, n-1$. Note from (3.20) of Corollary (3.8) that

$$E_i \rho = 0, \quad i = 1, \dots, n-1. \quad (3.24)$$

It follows from (3.22) and (3.24) that the gradient $\nabla^* \rho$ of ρ is collinear with U^* . Next, using (3.22) and (3.23), we derive

$$\rho E_0 = \nabla_{E_0}^* (\gamma E_0) = (E_0 \gamma) E_0 + \gamma \nabla_{E_0}^* E_0. \quad (3.25)$$

It then follows from (3.25) that

$$E_0 \gamma = \rho \quad \text{and} \quad \nabla_{E_0}^* E_0 = 0. \quad (3.26)$$

From the second relation in (3.26), we note that the integral curves of E_0 are geodesics in the leaves M^* . It follows that the line bundle $\langle E_0 \rangle$ spanned by E_0 is a totally geodesic foliation. Next, consider the distribution $\mathcal{D}$ spanned by $E_i, i = 1, \dots, n-1$. Taking $E = E_i$ in (3.22), and then consider (3.23), we have

$$E_i \gamma = 0 \quad \text{and} \quad \gamma \nabla_{E_i}^* E_0 = \rho E_i. \quad (3.27)$$

We see from (3.27) that $\mathcal{D}$ is integrable, whose leaves are totally umbilic in M^* , with umbilicity factor $-\frac{\rho}{\gamma}$. From (3.24) and the first relation of (3.27) the mean curvature vector $-\frac{\rho}{\gamma} E_0$ of leaves of $\mathcal{D}$, as hypersurfaces in M^* , is parallel in $\langle E_0 \rangle$ (the normal space of leaves of $\mathcal{D}$ in M^*). Therefore, $\mathcal{D}$ is a spherical distribution. By [16, Proposition 35, p. 206] (or [4, Theorem 4.4, p. 90]), the leaf M^* is a semi-Riemannian warped product $I \times_{\beta(t)} M'$, where $\beta(t)$ is a function on I , $\partial_t = E_0$ and M' (which is a leaf of $\mathcal{D}$) is an $(n-1)$ -dimensional semi-Riemannian manifold. Moreover, for any E tangent to M^* and orthogonal to E_0 , the sectional curvature κ of M^* satisfy

$$\kappa(E_0, E) = -\beta''(t)/\beta(t). \quad (3.28)$$

By a direct calculation, while considering (3.22), we have

$$R^*(E_i, U^*)U^* = (E_i \rho)U^* - (U^* \rho)E_i. \quad (3.29)$$

In view of (3.29) and (3.23), we calculate

$$\gamma^2 \kappa(E_0, E) = -\gamma E_0 \rho = -\gamma \rho'(t), \quad (3.30)$$

for any E orthogonal to E_0 . Thus, by (3.28), (3.30) and the first relation in (3.26), we have

$$\frac{\beta''(t)}{\beta(t)} = \frac{\rho'(t)}{\gamma(t)} = \frac{\gamma''(t)}{\gamma(t)}. \quad (3.31)$$

Therefore, in view of (3.31), we may take $\beta(t) = \gamma(t)$ and hence M^* is a warped product $I \times_{\gamma(t)} M'$. Moreover, by a direct calculation, we have $\nabla_E^* \gamma \partial_t = \gamma'(t)E$ for any E tangent to $I \times_{\gamma(t)} M'$. $\square$

4 Space forms and concircular vector fields

Let $(\bar{M}, \bar{g})$ be a semi-Riemannian manifold. We say that $\bar{M}$ is a semi-Riemannian space form [16, p. 80] if the sectional curvature $\bar{\kappa}$ of $\bar{M}$ is constant, say $\bar{\kappa} = c$. We denote such a space form by $\bar{M}(c)$. Moreover, $\bar{R}$ satisfies the relation

$$\bar{R}(\bar{X}, \bar{Y})\bar{Z} = c\{\bar{g}(\bar{Y}, \bar{Z})\bar{X} - \bar{g}(\bar{X}, \bar{Z})\bar{Y}\}, \quad (4.1)$$

for any $\bar{X}, \bar{Y}$ and $\bar{Z}$ tangent to $\bar{M}$. Basic examples of space forms include the semi-Euclidean space $\mathbb{R}_q^{n+2}$ with $c = 0$, the De Sitter space $\mathbb{S}_q^{n+2}$ with $c > 0$ and the semi-Hyperbolic space $\mathbb{H}_q^{n+2}$ with $c < 0$.

Next, in view of (3.2) and (4.1), we have the following result.

Proposition 4.1 *Let $(\bar{M}(c), \bar{g})$ be a semi-Riemannian space form of dimension $(n + 2) > 3$, admitting a nonzero concircular vector field. Then, for any $\bar{X}$ tangent to $\bar{M}$, we have*

$$\bar{X}\rho + c\eta(\bar{X}) = 0. \quad (4.2)$$

Suppose that (M, g) is a lightlike hypersurface of $\bar{M}(c)$ such that;

(1) $\bar{U}$ is tangent to $TM^\perp$, then $c = 0$ and, for any X, Y and Z tangent to M , we have

$$\alpha(\nabla_X C)(Y, PZ) - (X\alpha)C(Y, PZ) = \alpha(\nabla_Y C)(X, PZ) - (Y\alpha)C(X, PZ). \quad (4.3)$$

(2) $\bar{U}$ is tangent to $\text{ltr}(TM)$, then $c = 0$ and, for any X, Y and Z tangent to M , we have

$$\beta(\nabla_X B)(Y, Z) - (X\beta)B(Y, Z) = \beta(\nabla_Y B)(X, Z) - (Y\beta)B(X, Z). \quad (4.4)$$

(3) $\bar{U}$ belongs to $S(TM)$ or $S(TM)^\perp$, then c, f and ρ satisfies

$$\bar{U}\rho + 2cf = 0, \quad \bar{U}f - 2\rho f = 0, \quad \bar{U}^\perp \rho = \bar{U}^\perp f = 0. \quad (4.5)$$

Proof As $\bar{M}$ is a space form, (3.2) and (4.1) gives

$$c\{\bar{g}(\bar{Y}, \bar{U})\bar{X} - \bar{g}(\bar{X}, \bar{U})\bar{Y}\} = (\bar{X}\rho)\bar{Y} - (\bar{Y}\rho)\bar{X}, \quad (4.6)$$

for any $\bar{X}$ and $\bar{Y}$ tangent to $\bar{M}$. It follows from (4.6) that

$$\{\bar{Y}\rho + c\eta(\bar{Y})\}\bar{X} = \{\bar{X}\rho + c\eta(\bar{X})\}\bar{Y}. \quad (4.7)$$

As $\dim \bar{M} > 3$, relation (4.7) leads to $\bar{X}\rho + c\eta(\bar{X}) = 0$, which proves (4.2).

Next, assume that $\bar{U}$ belongs to either $TM^\perp$ or $\text{ltr}(TM)$. Then, we have $f = (1/2)\bar{g}(\xi, \xi) = 0$ and $f = (1/2)\bar{g}(N, N) = 0$, respectively. Hence, by Proposition 3.3 we have $\rho = 0$, and therefore, (4.2) reduces to $c\eta(\bar{X}) = 0$. This gives $c = 0$ since $\bar{g}$ is nondegenerate. Finally, the case $\bar{U}$ belongs to $S(TM)$ or $S(TM)^\perp$ follows easily from (3.8) and (4.2). $\square$

Corollary 4.2 *There does not exist any lightlike hypersurface (M, g) of a positively or negatively curved semi-Riemannian space form $\bar{M}(c)$, admitting a concircular vector field $\bar{U}$ belonging either to $TM^\perp$ or $\text{ltr}(TM)$.*

Corollary 4.3 *There does not exist any lightlike hypersurface (M, g) of a positively or negatively curved semi-Riemannian space form $\bar{M}(c)$, admitting a concircular vector field $\bar{U}$ such that ρ is a constant function.*

Corollary 4.4 (1) If $\bar{U}$ belongs to $TM^\perp$ and M is totally screen umbilic, then

$$X\left(\frac{\psi}{\alpha}\right) = 0, \quad N\alpha - \alpha\bar{g}(\bar{\nabla}_N N, \xi) = 0;$$

(2) If $\bar{U}$ belongs to $\text{ltr}(TM)$ and M is totally umbilic, then

$$X\left(\frac{\phi}{\beta}\right) - \theta(X)\frac{\phi^2}{\beta} = 0, \quad N\beta + \beta\bar{g}(\bar{\nabla}_N N, \xi) = 0,$$

for any X tangent to M .

Proof Let M be totally screen umbilic, then using Definition 2.2, relations (2.14) (2.10) and (2.12), we derive

$$(\nabla_X C)(Y, PZ) = (X\psi)g(Y, PZ) + \psi B(X, PZ)\theta(Y), \quad (4.8)$$

for any X, Y and Z tangent to M . Replacing (4.8) in (4.3) and noting that M is totally geodesic whenever $\bar{U}$ belongs to $TM^\perp$ (see Theorem 3.9), we have

$$\{\alpha X\psi - \psi X\alpha\}g(Y, PZ) = \{Y\psi - \psi Y\alpha\}g(X, PZ). \quad (4.9)$$

As $\dim S(TM) > 1$, relation (4.9) gives $\alpha X\psi - \psi X\alpha = 0$, for any X tangent to M . Hence, we have $X\left(\frac{\psi}{\alpha}\right) = 0$. In addition, the second relation follows from (3.15). In the same way, when $\bar{U}$ belongs to $\text{ltr}(TM)$ and M is totally umbilic, Definition 2.1, relations (2.11), (2.10), (2.12), (2.13) and (4.4) gives

$$\{\beta X\phi - \phi X\beta - \theta(X)\beta\phi^2\}g(PY, PZ) = \{\beta Y\phi - \phi Y\beta - \theta(Y)\beta\phi^2\}g(PX, PZ). \quad (4.10)$$

It follows from (4.10) and the fact $\dim S(TM) > 1$ that $\beta X\phi - \phi X\beta - \theta(X)\beta\phi^2 = 0$, for any X tangent to M . Thus, we get $X\left(\frac{\phi}{\beta}\right) - \theta(X)\frac{\phi^2}{\beta} = 0$. Moreover, the second relation follows from (3.17), which completes the proof. $\square$

Remark 4.5 When $\bar{M}$ is a space of constant curvature c , we can give explicit expressions for the functions ρ and γ of Theorem 3.13. From (4.5), with $\bar{U} = U^* = \gamma E_0$ and $2f = g(U^*, U^*) = \gamma^2 g(E_0, E_0) = \epsilon_0 \gamma^2$, where $\epsilon_0 = g(E_0, E_0)$, we get $E_0 \rho + \epsilon_0 c \gamma = 0$ and $E_0 \gamma - \rho = 0$. There are two possible values of ϵ_0 , that is $\epsilon_0 = \pm 1$. Thus, if $\epsilon_0 = 1$ then the above relations lead to the system of ODEs: $\rho'(t) + c\gamma(t) = 0$ and $\gamma'(t) - \rho(t) = 0$. On the other hand, for $\epsilon_0 = -1$, we have the system $\rho'(t) - c\gamma(t) = 0$ and $\gamma'(t) - \rho(t) = 0$. Solving the above systems of ODEs for $c = 0$, $c < 0$ and $c > 0$ gives the functions ρ and γ .

Finally, suppose that $\bar{U}$ belongs to $S(TM)^\perp$ such that $\beta \neq 0$. We know from Theorem 3.9 that $S(TM)$ is integrable. One may be interested in the geometry of a leaf M^* of $S(TM)$, as an isometric immersion in $\bar{M} = \mathbb{R}_k^{n+2}$. In this regard, we may consider the vector field $\bar{U} = \alpha\xi + \beta N$, which is everywhere normal to M^* . We define the $\bar{U}$ -shape operator $A_{\bar{U}}$ of M^* as a submanifold of $\mathbb{R}_k^{n+2}$. As ξ and N are linearly independent, we express $A_{\bar{U}}$, using (2.3) and (2.5), in terms of A_ξ^* and A_N as

$$A_{\bar{U}}X = \alpha A_\xi^* X + \beta A_N X, \quad (4.11)$$

for any X tangent to M^* . Thus, using (3.21) and (4.11), we have

$$A_{\bar{U}}X = \alpha A_\xi^* X + \beta A_N X = -\rho X. \quad (4.12)$$

It follows, from (4.12), that M^* is a partially umbilic immersion in $\mathbb{R}_k^{n+2}$ (see [15] for details). Moreover, by a direct calculation, while considering $\nabla_X^* \xi = -\tau(X)\xi$ and $\nabla_X^t N = \tau(X)N$, we have

$$\nabla_X^\perp \bar{U} = \{X\alpha - \alpha\tau(X)\}\xi + \{X\beta + \beta\tau(X)\}N = 0, \quad (4.13)$$

for all X tangent to M^* , where $\nabla^\perp$ is the normal connection of M^* . Therefore, using (4.12), (4.13) and Section 0.9 of [15, p. 37], we have the following result:

Theorem 4.6 Let $\bar{U}$ belong to $S(TM)^\perp$, such that $\pi : M_j^* \rightarrow \mathbb{R}_k^{n+2}$ is an isometric immersion of a leaf M^* of $S(TM)$, with $A_{\bar{U}}$ as its $\bar{U}$ -shape operator. Then, with reference to (4.12) and (4.13), $\pi(M^*)$ is contained:

- (1) inside $\mathbb{S}_k^{n+1}\left(-\frac{\sqrt{2\alpha\beta}}{\rho}\right)$ if $\alpha\beta > 0$ and
- (2) inside $\mathbb{H}_{k-1}^{n+1}\left(\frac{\sqrt{-2\alpha\beta}}{\rho}\right)$ if $\alpha\beta < 0$.

5 U^* as a Ricci soliton potential on leaves of $S(TM)$

Let (M, g) be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$, admitting a concircular vector field $\bar{U}$. In this section, we consider a screen integrable lightlike hypersurface M such that (M^*, g^*, U^*, λ) is a Ricci soliton. Here, M^* is a leaf of $S(TM)$ and $g^* = g|_{S(TM)}$. In this case, we have

$$\mathcal{L}_{U^*}g^* + 2Ric^* = 2\lambda g^*, \quad (5.1)$$

where $\mathcal{L}_{U^*}$ is the Lie derivative operator, Ric^* is the Ricci tensor of M^* and λ is a real constant.

Proposition 5.1 (M^*, g^*, U^*, λ) is a Ricci soliton if and only if

$$Ric^*(X, Y) = (\lambda - \rho)g^*(X, Y) - \alpha B(X, Y) - \beta C(X, Y), \quad (5.2)$$

for any X, Y tangent to M^* .

Proof As the connection ∇^* on leaves is a metric connection, one has

$$(\mathcal{L}_{U^*}g^*)(X, Y) = g^*(\nabla_X^*U^*, Y) + g^*(X, \nabla_Y^*U^*), \quad (5.3)$$

for each X, Y tangent to M^* . It then follows from (3.5) and (5.3) that

$$(\mathcal{L}_{U^*}g^*)(X, Y) = 2\rho g^*(X, Y) + 2\alpha B(X, Y) + 2\beta C(X, Y). \quad (5.4)$$

Then, (5.2) follows easily from (5.4) and (5.1). $\square$

Corollary 5.2 If $\alpha B + \beta C = \varrho g^*$, for some smooth function ϱ , then the scalar curvature of M^* is $n(\lambda - \rho - \varrho)$.

Theorem 5.3 Let $\bar{U}$ be a concircular vector field on a Lorentzian manifold $(\bar{M}, \bar{g})$, and let (M^{n+1}, g) , where $n \geq 3$, be a lightlike hypersurface of $\bar{M}$. Suppose that (M^*, g^*, U^*, λ) is a Ricci soliton on M such that $\alpha B + \beta C = \varrho g^*$, for some smooth function ϱ . Then, the following are all true:

- (1) $\rho + \varrho$ is constant function, and given by $\rho + \varrho = \lambda$;
- (2) $M = \ell_\xi \times I \times_{(\lambda t + c_0)} M'$, where ℓ_ξ is a lightlike curve tangent to $TM^\perp$, I is an open interval with arc length t , c_0 is a constant and M' is an Einstein $(n-1)$ -dimensional manifold whose Ricci tensor satisfies $Ric' = (n-1)\lambda^2 g'$, g' being the metric of M' .

Proof As $S(TM)$ is integrable, we already know that M is locally a product manifold $\ell_\xi \times M^*$, where ℓ_ξ is a lightlike curve tangent to $TM^\perp$ and M^* is a Riemannian leaf of $S(TM)$. Moreover, from (5.2) and the assumption $\alpha B + \beta C = \varrho g^*$, we see that $Ric^*(X, Y) = (\lambda - \rho - \varrho)g^*(X, Y)$, which shows that M^* is Einstein manifold. Now, since M^* is Einstein and $n \geq 3$, it follows that M^* has constant scalar curvature [6]. Thus, $\lambda - \rho - \varrho$ is a constant. Hence, $\rho + \varrho$ is also constant. Moreover, by (3.5), we have

$$\nabla_X^*U^* = (\rho + \varrho)X, \quad (5.5)$$

for each X tangent to M^* . By a direct calculation, we have

$$R^*(X, U^*)U^* = (X(\rho + \varrho))U^* - (U^*(\rho + \varrho))X = 0. \quad (5.6)$$

From (5.6), we derive

$$Ric^*(U^*, U^*) = 0. \quad (5.7)$$

It then follows from (5.7) and (5.2) that

$$(\lambda - \rho - \varrho)g(U^*, U^*) = 0,$$

which gives $\rho + \varrho = \lambda$, which proves (1). Next, as U^* is a concircular vector field on M^* , see (5.5), we have by Theorem 3.13 that M^* is a warped product manifold, in which $\gamma(t) = \lambda t + c_0$, for some constant c_0 (in particular see (3.26)). Finally, it follows from (9.109) of [2, p. 267] that M' is an Einstein manifold with $Ric' = (n-1)\lambda^2 g'$, where g' is the metric of M' . $\square$

Concluding remarks

Concircular vector fields on (semi-)Riemannian manifolds have attracted the attention of many researchers, due to their numerous applications. For instance, in general relativity, it has been noted that trajectories of timelike concircular fields in the de-Sitter model determine the world lines of receding or colliding galaxies satisfying the Weyl hypothesis (see [18] for details). Furthermore, concircular vector fields are useful in the characterisation of (Ricci) solitons [6], as well as providing a simpler way of characterising generalised Robertson–Walker spacetimes [5]. Using the above literature, we have proved that lightlike hypersurfaces admitting concircular vector fields are generally totally geodesic or products of lightlike curves and warped product manifolds. In addition, following Theorem 5.3, we mention that this paper opens a way to further research in lightlike hypersurfaces whose screen distributions admit (Ricci) solitons.

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Declarations

Conflict of interest The author has no conflict of interests to declare that are relevant to the content of this article.

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