Rachid Ech-chaouy · Abdelouahab Idelhadj · Rachid Tribak

On strongly coseparable modules

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Abstract A module M is called $\mathfrak{s}$ -coseparable if for every nonzero submodule U of M such that M/U is finitely generated, there exists a nonzero direct summand V of M such that $V \subseteq U$ and M/V is finitely generated. It is shown that every non-finitely generated free module is $\mathfrak{s}$ -coseparable but a finitely generated free module is not, in general, $\mathfrak{s}$ -coseparable. We prove that the class of $\mathfrak{s}$ -coseparable modules over a right noetherian ring is closed under finite direct sums. We show that the class of commutative rings R for which every cyclic R -module is $\mathfrak{s}$ -coseparable is exactly that of von Neumann regular rings. Some examples of modules M for which every direct summand of M is $\mathfrak{s}$ -coseparable are provided.

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1 Introduction

Throughout this paper, all rings are associative with an identity element and R denotes such a ring. All modules considered are unital right R -modules, unless otherwise specified. Let M be an R -module. The injective hull of M will be denoted by $E(M)$. The notation $N \subseteq M$ (respectively, $N \leq M$) means that N is a subset (respectively, a submodule) of M . By $\mathbb{N}$ and $\mathbb{Z}$ we denote the set of natural numbers and the ring of integers, respectively.

A module M is called *coseparable* if every submodule U of M with M/U finitely generated contains a direct summand V of M such that M/V is finitely generated. This concept was initially appeared in 1968 [9] for abelian groups. Then the concept was generalized to modules in 1978 [12] by Hiremath and was further studied by other authors (see e.g., [8, 19]). In this paper, we focus on a subclass of coseparable modules. A module M is said to be *strongly coseparable* (or $\mathfrak{s}$ -coseparable) if for every nonzero submodule U of M with the property that M/U is finitely generated, there exists a nonzero direct summand U' of M such that $U' \subseteq U$ and M/U' is finitely generated.

In Sect. 2, we study some basic properties of $\mathfrak{s}$ -coseparable modules. We begin by giving many examples belonging to this class of modules. In particular, we show that the class of $\mathfrak{s}$ -coseparable modules is a proper

R. Ech-chaouy · A. Idelhadj
Department of Mathematics, Faculty of Sciences, Abdelmalek Essaâdi University, B.P. 2121, Tetouan, Morocco
E-mail: echaouirachid@yahoo.fr

A. Idelhadj
E-mail: idelhadj@gmail.com

R. Tribak (✉)
Centre Régional des Métiers de l'Éducation et de la Formation (CRMEF-TTH)-Tanger, Avenue My Abdelaziz, B.P. 3117, Souani, Tangier, Morocco
E-mail: tribak12@yahoo.com

subclass of that of coseparable modules. But we prove that for a non-finitely generated module M , M is coseparable if and only if M is $\mathfrak{s}$ -coseparable. It is shown that a direct summand of an $\mathfrak{s}$ -coseparable module is not, in general, $\mathfrak{s}$ -coseparable. On the other hand, we show that the class of finitely generated $\mathfrak{s}$ -coseparable modules is closed under direct summands. Moreover, we prove that the class of $\mathfrak{s}$ -coseparable modules over a right noetherian ring is closed under finite direct sums.

Section 3 is devoted to shed some light on a proper subclass of that of $\mathfrak{s}$ -coseparable modules. A module M is called *completely $\mathfrak{s}$ -coseparable* if every direct summand of M is $\mathfrak{s}$ -coseparable. Some examples of completely $\mathfrak{s}$ -coseparable modules are provided.

In Sect. 4, we characterize several classes of rings in terms of $\mathfrak{s}$ -coseparable modules. The class of rings R for which every finitely generated projective R -module is $\mathfrak{s}$ -coseparable, is shown to be exactly that of the weakly regular semiprimitive rings; while the class of commutative rings R for which every cyclic R -module is $\mathfrak{s}$ -coseparable is characterized as that of the von Neumann regular rings.

2 Some examples and properties of strongly coseparable modules

Recall that a module M is called *coseparable* if for every submodule U of M such that M/U is finitely generated, there exists a direct summand V of M such that $V \subseteq U$ and M/V is finitely generated (see e.g., [9, 12, 19]). It is clear that a finitely generated module is always coseparable. In this paper, we introduce and study a subclass of coseparable modules as follows:

Definition 2.1 A module M is said to be *strongly coseparable* (or $\mathfrak{s}$ -coseparable for short) if for every nonzero submodule U of M with the property that M/U is finitely generated, there exists a nonzero direct summand U' of M such that $U' \subseteq U$ and M/U' is finitely generated.

A submodule of a module M is called *cofinite* if M/N is finitely generated.

Remark 2.2 It is easily seen that a finitely generated module is $\mathfrak{s}$ -coseparable if and only if every nonzero submodule of M contains a nonzero direct summand. A submodule N of a module M is said to be small in M , written $N << M$, provided $N + X \neq M$ for any proper submodule X of M .

It is easily seen that any semisimple module is $\mathfrak{s}$ -coseparable. In addition, any module M with $\text{Rad}(M) = M$ is $\mathfrak{s}$ -coseparable. In this section, we present more examples of $\mathfrak{s}$ -coseparable modules. Recall that a module M is called *regular* if every finitely generated submodule of M is a direct summand.

Example 2.3 Let $M = x_1 R + \cdots + x_n R$ be a finitely generated regular module. Let U be a nonzero cofinite submodule of M . Since $\text{Rad}(M) = 0$, there exists a proper submodule N of M such that $M = U + N$. For each $i \in \{1, \dots, n\}$, there exist $u_i \in U$ and $y_i \in N$ such that $x_i = u_i + y_i$. Let U' be the submodule of U generated by $\{u_1, \dots, u_n\}$. Since $M = U' + N$ and $N \neq M$, we have $U' \neq 0$. Moreover, U' is a direct summand of M as M is regular. Clearly, M/U' is finitely generated. Therefore, M is $\mathfrak{s}$ -coseparable.

In particular, the right R -module R_R is $\mathfrak{s}$ -coseparable for every von Neumann regular ring R .

A module M is said to be *A-projective* if for every submodule X of A , any homomorphism $\varphi : M \rightarrow A/X$ can be lifted to a homomorphism $\psi : M \rightarrow A$.

Example 2.4 (i) Let a module $M = M_1 \oplus M_2$ be a direct sum of a finitely generated submodule M_1 and a semisimple submodule M_2 which is not finitely generated. Assume that M_2 is M_1 -projective (e.g., M_2 can be projective). We want to prove that M is $\mathfrak{s}$ -coseparable. To do this, let U be a cofinite submodule of M . Then $M = U + K$ for some finitely generated submodule K of M . Note that $K \subseteq M_1 \oplus K_2$ for some finitely generated direct summand K_2 of M_2 . Hence, $M = U + (M_1 \oplus K_2)$. Moreover, we have $M = L_2 \oplus (M_1 \oplus K_2)$ for some submodule L_2 of M_2 . Using [6, 4.3], it follows that L_2 is $(M_1 \oplus K_2)$ -projective. Therefore, $M = U' \oplus (M_1 \oplus K_2)$ for some submodule U' of U by [6, 4.12]. Clearly, $U' \neq 0$ as M is not finitely generated. In addition, M/U' is finitely generated. Consequently, M is an $\mathfrak{s}$ -coseparable module.

(ii) Let T be a commutative ring and let $\mathfrak{m}$ be a maximal ideal of T . Consider the ring $R = T \times T/\mathfrak{m}$. Taking (i) into account, it follows that for any finitely generated R -module M_1 , the R -module $M_1 \oplus (0 \oplus T/\mathfrak{m})^{(\mathbb{N})}$ is $\mathfrak{s}$ -coseparable.

In the following proposition, we determine the structure of the indecomposable $\mathfrak{s}$ -coseparable modules.

Proposition 2.5 *Let M be an indecomposable module. Then the following statements are equivalent:*

- (i) M is an $\mathfrak{s}$ -coseparable module;
- (ii) $\text{Rad}(M) = M$ or M is a simple module.

Proof (i) $\Rightarrow$ (ii) Assume that $\text{Rad}(M) \neq M$. Then M contains a maximal submodule N . Suppose that $N \neq 0$. Since M is $\mathfrak{s}$ -coseparable, there exists a nonzero direct summand N' of M such that $N' \subseteq N$. Hence, $N' = M$ as M is indecomposable. This contradicts the fact that $N \neq M$. Therefore, $N = 0$ and hence M is a simple module.

(ii) $\Rightarrow$ (i) This is immediate. $\square$

It is easily seen that every $\mathfrak{s}$ -coseparable module is coseparable. But the converse is not true, in general, as illustrated in the next example.

Example 2.6 Let M be an indecomposable finitely generated module which is not simple (e.g., we can take $M = R$ where R is any commutative integral domain which is not a field or $M = \mathbb{Z}/p^n\mathbb{Z}$ for some prime number p and an integer $n \geq 2$). Then M is not $\mathfrak{s}$ -coseparable by Proposition 2.5. On the other hand, M is coseparable since M is finitely generated.

A module M is called *weakly regular* (or an I_0 -module) if every submodule, which is not contained in $\text{Rad}(M)$, contains a nonzero direct summand of M (see, e.g., [1, 16]). A ring R is called *weakly regular* (or an I_0 -ring as in [16, p. 131]) if the right R -module R_R is weakly regular. Equivalently, if the left R -module ${}_R R$ is weakly regular. Many examples of weakly regular rings are given in [16].

The following characterization provides a rich source of examples of $\mathfrak{s}$ -coseparable modules (see [8]).

Theorem 2.7 *The following conditions are equivalent for an R -module M :*

- (i) M is $\mathfrak{s}$ -coseparable;
- (ii) Either (a) M is coseparable and not finitely generated, or (b) $\text{Rad}(M) = 0$ and M is a finitely generated module which is weakly regular.

Proof (i) $\Rightarrow$ (ii) Clearly, M is coseparable. Assume that M is finitely generated. Suppose that $\text{Rad}(M) \neq 0$. By hypothesis, $\text{Rad}(M)$ contains a nonzero direct summand U of M . But $\text{Rad}(M) \ll M$. Therefore, $U \ll M$ and hence $U = 0$, a contradiction. It follows that $\text{Rad}(M) = 0$. It is clear that M is weakly regular.

(a) $\Rightarrow$ (i) Let U be a nonzero cofinite submodule of M . As M is coseparable, U contains a direct summand U' of M such that M/U' is finitely generated. Thus $U' \neq 0$ since M is not finitely generated. It follows that M is $\mathfrak{s}$ -coseparable.

(b) $\Rightarrow$ (i) This is obvious. $\square$

Following [6, 11.26], a module M is called *refinable* if, for any submodules N and L of M with $N + L = M$, there exists a direct summand N' of M with $N' \subseteq N$ and $N' + L = M$.

Proposition 2.8 *Let M be a refinable module. Assume that $\text{Rad}(M) = 0$ or M is not finitely generated. Then M is $\mathfrak{s}$ -coseparable.*

Proof **Case 1:** Assume that $\text{Rad}(M) = 0$. Let U be a nonzero cofinite submodule of M . Since $\text{Rad}(M) = 0$, U is not small in M . Thus $M = U + N$ for some proper submodule N of M . Note that $N/(U \cap N) \cong M/U$ is finitely generated. Then $N = (U \cap N) + K$ for some finitely generated submodule K of N . Therefore, $M = U + K$. Note that $K \neq M$ since otherwise $N = M$. Since M is a refinable module, there exist a direct summand U' of M such that $U' \subseteq U$ and $M = U' + K$. Notice that $U' \neq 0$ and M/U' is finitely generated. It follows that M is $\mathfrak{s}$ -coseparable.

Case 2: Assume that M is not finitely generated. Then M is $\mathfrak{s}$ -coseparable by [8, Example 2.3] and Theorem 2.7. $\square$

A module M is said to have the *finite exchange property* if for any finite index set I , whenever $M \oplus N = \bigoplus_{i \in I} A_i$ for modules N and A_i , then $M \oplus N = M \oplus (\bigoplus_{i \in I} B_i)$ for submodules $B_i \leq A_i$. A ring R is called an *exchange ring* if the R -module R_R has the finite exchange property. This is equivalent to saying that for any $r \in R$, there is an idempotent e in R with $e \in rR$ and $1 - e \in (1 - r)R$ (see [6, 11.16]).

As an application of the previous proposition, we obtain the following example.

Example 2.9 Let M be a quasi-projective module having the finite exchange property with $\text{Rad}(M) = 0$. For example, we can take the T -module $M = R/\text{Rad}(R)$ where R is an exchange ring and $T = R/\text{Rad}(R)$. Then M is $\mathfrak{s}$ -coseparable by Proposition 2.8 and [6, 11.31]. Using [16, Proposition 28.6], it follows that every finitely generated projective module M with $\text{Rad}(M) = 0$ over an exchange ring is $\mathfrak{s}$ -coseparable.

Recall that a module M is said to be π -projective if for every two submodules U, V of M with $U + V = M$ there exists $f \in \text{End}_R(M)$ such that $\text{Im}(f) \subseteq U$ and $\text{Im}(1 - f) \subseteq V$. It is well known that every quasi-projective module is π -projective (see [18, p. 359]).

Proposition 2.10 *Let M be a π -projective module such that every proper finitely generated submodule is contained in a proper finitely generated direct summand of M . Assume that $\text{Rad}(M) = 0$ or M is not finitely generated. Then M is an $\mathfrak{s}$ -coseparable module.*

Proof **Case 1:** Assume that $\text{Rad}(M) = 0$. Let U be a nonzero cofinite submodule of M . By similar arguments as in the proof of Proposition 2.8, we have $M = U + K$ such that K is a finitely generated proper submodule of M . By hypothesis, there exists a finitely generated proper direct summand L of M such that $K \subseteq L$. Hence, $M = U + L$. Since M is π -projective, there exists a submodule U' contained in U such that $M = U' \oplus L$ by [18, 41.14(3)]. Since $L \neq M$, we have $U' \neq 0$. Hence, M is $\mathfrak{s}$ -coseparable.

Case 2: Assume that M is not finitely generated. Then M is $\mathfrak{s}$ -coseparable by [8, Proposition 2.5] and Theorem 2.7. $\square$

Example 2.11 (i) Let R be a ring and consider the R -module $M = R^{(I)}$, where I is an infinite index set. From Proposition 2.10, it follows that M is $\mathfrak{s}$ -coseparable.

(ii) Let M be a π -projective regular module such that $\text{Rad}(M) = 0$ or M is not finitely generated. Then M is $\mathfrak{s}$ -coseparable by Proposition 2.10.

(iii) Let R be a von Neumann regular ring and let P be a projective R -module. By [16, Proposition 6.7(4)], P is a regular module. Moreover, we have $\text{Rad}(P) = 0$ by [3, Proposition 17.10]. Therefore, P is $\mathfrak{s}$ -coseparable by (ii).

Next, we exhibit some examples to illustrate that a direct summand of an $\mathfrak{s}$ -coseparable module is not, in general, $\mathfrak{s}$ -coseparable.

Example 2.12 Let R be a commutative domain which is not a field.

(i) Assume that $\text{Rad}(R) = 0$ (e.g., we can take $R = \mathbb{Z}$). By the preceding Example, $M = R^{(\mathbb{N})}$ is an $\mathfrak{s}$ -coseparable R -module. However, the right R -module R_R is not $\mathfrak{s}$ -coseparable by Proposition 2.5.

(ii) Let $\mathfrak{m}$ be a maximal ideal of R . Consider the ring $T = R \times R/\mathfrak{m}$. By Example 2.4(ii), the T -module $(R \oplus 0) \oplus (0 \oplus R/\mathfrak{m})^{(\mathbb{N})}$ is $\mathfrak{s}$ -coseparable. On the other hand, the T -module $R \oplus 0$ is not $\mathfrak{s}$ -coseparable by Proposition 2.5.

The following two results should be contrasted with Example 2.12.

Proposition 2.13 *Let M be an $\mathfrak{s}$ -coseparable R -module over a right noetherian ring R . Let N be a fully invariant direct summand of M which is not finitely generated. Then N is $\mathfrak{s}$ -coseparable.*

Proof Note that M is coseparable. Then N is coseparable by [8, Proposition 2.12]. Since N is not finitely generated, it follows that N is $\mathfrak{s}$ -coseparable by Theorem 2.7. $\square$

Combining [8, Proposition 2.16] and Theorem 2.7, we obtain the following result.

Proposition 2.14 *Let M be an R -module and let N be a fully invariant submodule of M . Assume that M is $\mathfrak{s}$ -coseparable and M/N is not finitely generated. Then M/N is $\mathfrak{s}$ -coseparable.*

Next, motivated by Theorem 2.7, we will be concerned with finitely generated $\mathfrak{s}$ -coseparable modules.

Definition 2.15 A module M is called $\mathfrak{fgs}$ -coseparable if M is finitely generated and $\mathfrak{s}$ -coseparable.

Proposition 2.16 (i) *Any finitely generated submodule of an $\mathfrak{fgs}$ -coseparable module M is also an $\mathfrak{fgs}$ -coseparable module.*

(ii) *Any direct summand of an $\mathfrak{fgs}$ -coseparable module M is also an $\mathfrak{fgs}$ -coseparable module.*

Proof (i) Let N be a finitely generated submodule of an $\mathfrak{fgs}$ -coseparable module M . Let U be a nonzero submodule of N . Since M is $\mathfrak{s}$ -coseparable, there exists a nonzero direct summand U' of M such that $U' \subseteq U$. It is clear that U' is a direct summand of N . Thus N is $\mathfrak{fgs}$ -coseparable.

(ii) This follows easily from (i). $\square$

Corollary 2.17 *Let R be a right noetherian ring and let M be an $\mathfrak{s}$ -coseparable R -module. Then every cofinite submodule of M is also $\mathfrak{s}$ -coseparable.*

Proof If M is finitely generated, then the result follows from Proposition 2.16(i). Now assume that M is not finitely generated. Let N be a cofinite submodule of M . Then clearly N is not finitely generated. Note that N is coseparable by [8, Proposition 2.15]. Therefore, N is $\mathfrak{s}$ -coseparable by Theorem 2.7. $\square$

Proposition 2.18 *Let M be an $\mathfrak{s}$ -coseparable module. Then the following hold:*

- (i) *Every indecomposable cofinite submodule of M is a direct summand of M .*
- (ii) *If M is finitely generated, then every indecomposable submodule of M is a simple direct summand of M .*

Proof (i) Let U be a nonzero indecomposable cofinite submodule of M . Since M is $\mathfrak{s}$ -coseparable, there exists a nonzero direct summand U' of M such that $U' \subseteq U$. By modularity, U' is a direct summand of U . But U is indecomposable. Then $U = U'$ and hence U is a direct summand of M .

(ii) Assume that M is $\text{fg}\mathfrak{s}$ -coseparable. Let U be an indecomposable submodule of M . Then U is a direct summand of M by (i). Therefore, U is an $\text{fg}\mathfrak{s}$ -coseparable module by Proposition 2.16(ii). Therefore, M is a simple module by Proposition 2.5. $\square$

Next, we provide some sufficient conditions for an $\text{fg}\mathfrak{s}$ -coseparable module to be semisimple. The following corollary is a direct consequence of Proposition 2.18.

Corollary 2.19 *Let M be a finitely generated module which is a sum of indecomposable submodules. Then M is $\text{fg}\mathfrak{s}$ -coseparable if and only if M is semisimple.*

A module M is said to be nitely cogenerated (or nitely embedded) if there exist nitely many simple modules $S_1, S_2, \dots, S_k$ such that $E(M) \cong E(S_1) \oplus E(S_2) \oplus \dots \oplus E(S_k)$.

Corollary 2.20 *Let M be a finitely generated R -module. Assume that one of the following conditions is fulfilled:*

- (i) *M has either the ascending chain or the descending chain condition on direct summands, or*
- (ii) *M is noetherian or artinian, or*
- (iii) *M is finitely cogenerated, or*
- (iv) *R is a noetherian ring, or*
- (v) *R is a semiperfect ring.*

Then M is an $\text{fg}\mathfrak{s}$ -coseparable module if and only if M is semisimple.

Proof Applying [3, Proposition 10.14] and [18, 21.3(2), 41.6 and 42.6], we conclude that M is a sum of indecomposable submodules. Now the result follows from Corollary 2.19. $\square$

Proposition 2.21 *Let M be a finitely generated R -module with essential socle. Then the following statements are equivalent:*

- (i) *M is an $\mathfrak{s}$ -coseparable module;*
- (ii) *$\text{Rad}(M) = 0$;*
- (iii) *Every simple submodule of M is a direct summand of M .*

Proof (i) $\Rightarrow$ (ii) By Theorem 2.7.

(ii) $\Rightarrow$ (iii) This is immediate.

(iii) $\Rightarrow$ (i) Let U be a nonzero submodule of M . Since $\text{Soc}(M)$ is essential in M , U contains a simple submodule T . By (iii), T is a nonzero direct summand of M . This completes the proof. $\square$

A module M is called *semiartinian* if every nonzero factor module of M has a nonzero socle. A ring R is called a *right semiartinian ring* if the right R -module R_R is semiartinian. The following corollary follows directly from Proposition 2.21.

Corollary 2.22 *Let M be a finitely generated R -module which is a semiartinian module with $\text{Rad}(M) = 0$. Then M is $\text{fg}\mathfrak{s}$ -coseparable.*

Recall that a ring R is called a *right V-ring* if every simple right R -module is an injective module.

Example 2.23 Let R be a right semiartinian right V-ring. Using Proposition 2.21, it follows that every finitely generated R -module is $\text{fg}\mathfrak{s}$ -coseparable.

Next, we focus on the question of when is the direct sum of two $\mathfrak{s}$ -coseparable modules, $\mathfrak{s}$ -coseparable? Combining Theorem 2.7 and [1, Theorem 2.9], we obtain the following proposition.

Proposition 2.24 Any finite direct sum of $\text{fg}\mathfrak{s}$ -coseparable modules is $\text{fg}\mathfrak{s}$ -coseparable.

Proposition 2.25 Let $M = M_1 \oplus M_2$ be a direct sum of $\mathfrak{s}$ -coseparable submodules M_1 and M_2 such that M_2 is finitely generated and semisimple. Then M is $\mathfrak{s}$ -coseparable.

Proof Let U be a nonzero cofinite submodule of M . Then

$$M/U = ((M_1 + U)/U) + ((M_2 + U)/U).$$

Since $(M_2 + U)/U$ is semisimple, it follows that $(M_1 + U)/U$ is a direct summand of M/U . Hence, $M_1/(U \cap M_1)$ is finitely generated.

Case 1: Assume that $U \cap M_1 \neq 0$. Since M_1 is $\mathfrak{s}$ -coseparable, there exists a nonzero direct summand U_1 of M_1 such that $U_1 \subseteq U \cap M_1 \subseteq U$ and M_1/U_1 is finitely generated. This clearly implies that U_1 is a direct summand of M and M/U_1 is finitely generated.

Case 2: Assume that $U \cap M_1 = 0$. Then M_1 is finitely generated. Using Proposition 2.24, it follows that U contains a nonzero cofinite direct summand of M .

Consequently, M is $\mathfrak{s}$ -coseparable. $\square$

Corollary 2.26 Let R be a right noetherian ring. Then any finite direct sum of $\mathfrak{s}$ -coseparable R -modules is $\mathfrak{s}$ -coseparable.

Proof Let $M = M_1 \oplus M_2$ be a direct sum of two $\mathfrak{s}$ -coseparable submodules.

Case 1: Assume that M_1 is finitely generated. Then M_1 is semisimple by Corollary 2.20. Applying Proposition 2.25, we conclude that M is $\mathfrak{s}$ -coseparable.

Case 2: Assume that both M_1 and M_2 are not finitely generated. Then M is $\mathfrak{s}$ -coseparable by [8, Corollary 3.6] and Theorem 2.7. $\square$

A module M is called a *duo module* if for every submodule N of M and any endomorphism f of M , we have $f(N) \subseteq N$.

Proposition 2.27 Let $M = \bigoplus_{i \in I} M_i$ be a direct sum of nonzero submodules M_i ($i \in I$) such that for every submodule N of M , we have $N = \bigoplus_{i \in I} (N \cap M_i)$ (e.g., M is a duo module). Assume that M_i is $\mathfrak{s}$ -coseparable for all $i \in I$. Then M is $\mathfrak{s}$ -coseparable.

Proof **Case 1:** Assume that M is finitely generated. Then I is a finite set and each M_i ($i \in I$) is finitely generated. Using Proposition 2.24, we see that M is $\mathfrak{s}$ -coseparable.

Case 2: Assume that M is not finitely generated. Note that M is coseparable by [8, Proposition 3.16]. Hence, M is $\mathfrak{s}$ -coseparable by Theorem 2.7. $\square$

3 Completely $\mathfrak{s}$ -coseparable modules

Definition 3.1 A module M is called *completely $\mathfrak{s}$ -coseparable* if every direct summand of M is $\mathfrak{s}$ -coseparable.

It is clear that semisimple modules are completely $\mathfrak{s}$ -coseparable. In addition, every module M with $\text{Rad}(M) = M$ is completely $\mathfrak{s}$ -coseparable.

Example 3.2 (i) From Proposition 2.16(ii), it follows that every $\text{fg}\mathfrak{s}$ -coseparable module is completely $\mathfrak{s}$ -coseparable.

(ii) By Example 2.12, the $\mathbb{Z}$ -module $M = \mathbb{Z}^{(\mathbb{N})}$ is $\mathfrak{s}$ -coseparable. On the other hand, M is not completely $\mathfrak{s}$ -coseparable since $\mathbb{Z}$ is not $\mathfrak{s}$ -coseparable by Proposition 2.5.

In the next proposition, we provide more examples of completely $\mathfrak{s}$ -coseparable modules.

Proposition 3.3 Let $M = M_1 \oplus M_2$ be a direct sum of a semisimple submodule M_1 and a submodule M_2 with $\text{Rad}(M_2) = M_2$. Then, M is completely $\mathfrak{s}$ -coseparable.

Proof If $M_2 = 0$, it is clear that M is completely $\mathfrak{s}$ -coseparable. Now assume that $M_2 \neq 0$. Then, M is not finitely generated. Using Theorem 2.7, the proof is completed by showing that M is coseparable. Let U be a nonzero submodule of M with $M/U = ((M_1 + U)/U) + ((M_2 + U)/U)$ finitely generated. Since $(M_1 + U)/U$ is semisimple, it follows that $(M_2 + U)/U$ is a direct summand of M/U . Hence, $(M_2 + U)/U$ is finitely generated. But $\text{Rad}((M_2 + U)/U) = (M_2 + U)/U$. Therefore, $(M_2 + U)/U = 0$. This yields $M_2 \subseteq U$. By modularity, we have $U = (U \cap M_1) \oplus M_2$. Since $U \cap M_1$ is a direct summand of M_1 , it follows that U is a direct summand of M . This completes the proof. $\square$

Recall that a module M is said to have finite uniform dimension if M does not contain an infinite independent set of submodules. Dually, a module M is said to have finite hollow dimension if M does not contain an infinite coindependent family of submodules; that is, for some $n \in \mathbb{N}$, there exists an epimorphism from M to a direct sum of n nonzero modules but no epimorphism from M to a direct sum of more than n nonzero modules (see, for example, [6, p. 47]). Next, we present some special cases where the converse of Proposition 3.3 holds.

Corollary 3.4 *Let R be a right noetherian ring and let M be an R -module with $S = \text{End}_R(M)$. Assume that one of the following conditions is fulfilled:*

- (i) *The ring S is semilocal;*
- (ii) *The module S_S has finite uniform dimension;*
- (iii) *The module M has finite hollow dimension;*
- (iv) *The module M has finite uniform dimension;*
- (v) *The module M is artinian.*

Then, M is completely $\mathfrak{s}$ -coseparable if and only if $M = M_1 \oplus M_2$ is a direct sum of submodules M_1 and M_2 such that $\text{Rad}(M_1) = M_1$ and M_2 is finitely generated and semisimple.

Proof Note that M satisfies ACC on direct summands by [8, Poof of Proposition 3.13].

($\Rightarrow$) By [8, Proposition 3.13], $M = M_1 \oplus M_2$ such that $\text{Rad}(M_1) = M_1$ and M_2 is finitely generated. Since M is completely $\mathfrak{s}$ -coseparable, M_2 is $\mathfrak{s}$ -coseparable. Clearly, M_2 has ACC on direct summands. Hence, M_2 is semisimple by Corollary 2.20.

($\Leftarrow$) By Proposition 3.3. □

Proposition 3.5 *Let R be a semiperfect ring and let M be a projective R -module. Then, M is completely $\mathfrak{s}$ -coseparable if and only if M is semisimple.*

Proof The sufficiency follows by Proposition 3.3. For the necessity, suppose that M is completely $\mathfrak{s}$ -coseparable. By [3, Theorem 27.12], M has a direct decomposition $M = \oplus_{i \in I} M_i$ such that each M_i ($i \in I$) is indecomposable. Therefore, M_i is $\mathfrak{s}$ -coseparable and projective for each $i \in I$. Note that $\text{Rad}(M_i) \neq M_i$ for each $i \in I$ (see [3, Proposition 17.14]). Using Proposition 2.5, it follows that each M_i ($i \in I$) is a simple submodule of M . Therefore, M is semisimple. □

Proposition 3.6 *Let R be a right noetherian ring and let M be an injective R -module. Then, M is completely $\mathfrak{s}$ -coseparable if and only if $M = M_1 \oplus M_2$ is a direct sum of submodules M_1 and M_2 such that $\text{Rad}(M_1) = M_1$ and M_2 is semisimple.*

Proof The sufficiency is obvious. Conversely, suppose that M is completely $\mathfrak{s}$ -coseparable. By [3, Theorem 25.6], $M = \oplus_{i \in I} M_i$ is a direct sum of indecomposable submodules M_i ($i \in I$). It follows that each M_i ($i \in I$) is $\mathfrak{s}$ -coseparable. By Proposition 2.5, for each $i \in I$, we have $\text{Rad}(M_i) = M_i$ or M_i is simple. The result follows. □

A ring R is said to be an *SSI-ring* if every semisimple right R -module is injective.

Proposition 3.7 *Let R be a ring. Then, the following conditions are equivalent:*

- (i) *Every completely $\mathfrak{s}$ -coseparable R -module is injective;*
- (ii) *R is a right noetherian right V-ring.*

Moreover, in this case, every completely $\mathfrak{s}$ -coseparable R -module is semisimple.

Proof (i) $\Rightarrow$ (ii) Let M be a semisimple module. Then, M is completely $\mathfrak{s}$ -coseparable. By hypothesis, M is injective. Hence, R is an SSI-ring. Thus R is a right noetherian right V-ring by [5, Proposition 1].

(ii) $\Rightarrow$ (i) Let M be a completely $\mathfrak{s}$ -coseparable module. By [5, Proposition 1], $\text{Soc}(M)$ is injective. Then, $M = \text{Soc}(M) \oplus N$ for some submodule N of M with $\text{Soc}(N) = 0$. Suppose $N \neq 0$. As R is a right V-ring, N contains a maximal submodule L . Since N is $\mathfrak{s}$ -coseparable, $N = U \oplus K$ is a direct sum of submodules U and K such that $U \subseteq L$ and K is a nonzero finitely generated submodule of M . Note that K is $\mathfrak{s}$ -coseparable. As R is a right noetherian ring, K is semisimple by Corollary 2.20. This implies that $\text{Soc}(N) \neq 0$, a contradiction. It follows that $N = 0$ and $M = \text{Soc}(M)$ is injective. □

Corollary 3.8 *The following assertions are equivalent for a commutative ring R :*

- (i) *Every completely $\mathfrak{s}$ -coseparable R -module is injective;*
- (ii) *R is a semisimple ring.*

Proof This is a consequence of [5, p. 236 Proposition 1 and Corollary] and Proposition 3.7. □

Next, we present a special subclass of completely $\mathfrak{s}$ -coseparable modules. Following [4], a module M is called an *md-module* if every maximal submodule of M is a direct summand of M . This is equivalent to say that every cofinite submodule of M is a direct summand of M by [2, Lemma 2.7]. Therefore, every md-module is $\mathfrak{s}$ -coseparable. Moreover, the class of md-modules is closed under direct summands by [4, Proposition 2.1]. It follows that every md-module is completely $\mathfrak{s}$ -coseparable.

Example 3.9 (i) Let R be a commutative noetherian ring and let $\{U_\lambda\}_{\lambda \in \Lambda}$ be a collection of simple R -modules. Then, $M = \prod_{\lambda \in \Lambda} U_\lambda$ is an md-module by [4, Theorem 6.2]. This implies that M is completely $\mathfrak{s}$ -coseparable.

(ii) Let a module $M = M_1 \oplus M_2$ be a direct sum of a submodule M_1 with $\text{Rad}(M_1) = M_1$ and a semisimple submodule M_2 . From the proof of Proposition 3.3, it follows that M is an md-module.

Example 3.10 Let M be a regular R -module over a right noetherian ring R . Let U be a nonzero submodule of M with M/U finitely generated. Then, $M = U + K$ for some finitely generated submodule K of M . Since R is right noetherian, $U \cap K$ is a finitely generated submodule of M . But M is regular. Then, $U \cap K$ is a direct summand of M . Therefore, $U \cap K$ is a direct summand of K . Hence, $K = (U \cap K) \oplus L$ for some submodule L of K . Clearly, $M = U \oplus L$. Hence, M is an md-module. Therefore, M is completely $\mathfrak{s}$ -coseparable.

4 Rings whose modules are $\mathfrak{s}$ -coseparable

In this section, we characterize some classes of rings via $\mathfrak{s}$ -coseparable modules. A ring R is called *semiprimitive* if $\text{Rad}(R) = 0$.

We begin with the following result which provides a description of all rings R for which the right R -module R_R is $\mathfrak{s}$ -coseparable.

Theorem 4.1 *The following statements are equivalent for a ring R :*

- (i) *The R -module R_R is $\mathfrak{s}$ -coseparable;*
- (ii) *R is a weakly regular semiprimitive ring;*
- (iii) *For every $0 \neq a \in R$, there is $0 \neq x \in R$ with $x = xax$;*
- (iv) *Every (finitely generated) free R -module F is $\mathfrak{s}$ -coseparable;*
- (v) *Every finitely generated projective R -module is $\mathfrak{s}$ -coseparable;*
- (vi) *For every finitely generated projective R -module P , $\text{End}_R(P)$ is a weakly regular semiprimitive ring.*

Proof (i) $\Leftrightarrow$ (ii) This follows from Theorem 2.7.

(ii) $\Leftrightarrow$ (iii) By [16, Remark 15.3(3)].

(i) $\Rightarrow$ (iv) Let F be a free R -module. Then, $\text{Rad}(F) = 0$ by Theorem 2.7. If F is not finitely generated, then F is $\mathfrak{s}$ -coseparable by Example 2.11(i). Now if F is finitely generated, then F is $\mathfrak{s}$ -coseparable by Proposition 2.24.

(iv) $\Rightarrow$ (v) Let P be a finitely generated projective R -module. It is well known that P is isomorphic to a direct summand of a finitely generated free R -module F . Therefore, P is $\mathfrak{s}$ -coseparable by Proposition 2.16.

(v) $\Rightarrow$ (i) This is immediate.

(ii) $\Rightarrow$ (vi) Let P be a finitely generated projective R -module. From [10, Theorem 3.8], it follows that $S = \text{End}_R(P)$ is a weakly regular ring. Moreover, by [3, Corollary 17.12], we have $\text{Rad}(S) = \text{Hom}_R(P, \text{P} \text{Rad}(R)) = 0$. Thus S is a semiprimitive ring.

(vi) $\Rightarrow$ (ii) This follows from the fact that the rings $\text{End}_R(R_R)$ and R are isomorphic. $\square$

The next corollary follows directly from Theorem 4.1.

Corollary 4.2 *For any ring R , the right R -module R_R is $\mathfrak{s}$ -coseparable if and only if it is completely $\mathfrak{s}$ -coseparable.*

Next, we shed some light on the class of rings R for which every cyclic R -module is $\mathfrak{s}$ -coseparable.

Proposition 4.3 *Let R be a ring such that every cyclic R -module is $\mathfrak{s}$ -coseparable. Then, R is a right V-ring.*

Proof Let I be a right ideal of R . By hypothesis, R/I is an $\mathfrak{s}$ -coseparable R -module. Hence, $\text{Rad}(R/I) = 0$ by Theorem 2.7. Therefore, R is a right V-ring by [13, Theorem 3.75]. $\square$

Corollary 4.4 *The following statements are equivalent for a commutative ring R :*

- (i) *Every cyclic R -module is $\mathfrak{s}$ -coseparable;*
- (ii) *R is a von Neumann regular ring.*

Proof (i) $\Rightarrow$ (ii) This follows from Proposition 4.3 and the fact that a commutative ring is von Neumann regular if and only if it is a V-ring (see [13, Corollary 3.73]).

(ii) $\Rightarrow$ (i) Let I be an ideal of R . Note that R/I is a von Neumann regular ring. Then, every cyclic R -submodule of R/I is a direct summand of the R -module R/I . Therefore, R/I is an $\mathfrak{s}$ -coseparable R -module. $\square$

It is shown in Example 2.12 that the class of $\mathfrak{s}$ -coseparable modules is not closed under factor modules. In the same vein, the next example shows that a factor module of a cyclic $\mathfrak{s}$ -coseparable module need not be $\mathfrak{s}$ -coseparable, in general.

Example 4.5 Let R be a commutative weakly regular semiprimitive ring which is not von Neumann regular. An explicit example of R can be found in [16, Example 15.7(9)]. By Theorem 4.1, the R -module R is $\mathfrak{s}$ -coseparable. On the other hand, using Corollary 4.4, it follows that the ring R has a cyclic R -module M which is not $\mathfrak{s}$ -coseparable. It is clear that $M \cong R/I$ for some nonzero ideal I of R .

Recall that a ring R is called *right semiartinian* if every nonzero cyclic right R -module has a nonzero socle.

Corollary 4.6 *Let R be a right semiartinian ring. Then, the following conditions are equivalent:*

- (i) R is a right V-ring;
- (ii) Every finitely generated R -module is $\mathfrak{s}$ -coseparable;
- (iii) Every cyclic R -module is $\mathfrak{s}$ -coseparable.

Proof (i) $\Rightarrow$ (ii) Let M be a finitely generated R -module. Then, $\text{Rad}(M) = 0$ since R is a right V-ring. Moreover, M is a semiartinian R -module by [15, p. 183 Proposition VIII.2.5]. Therefore, M is $\mathfrak{s}$ -coseparable by Corollary 2.22.

(ii) $\Rightarrow$ (iii) This is clear.

(iii) $\Rightarrow$ (i) This follows from Proposition 4.3. $\square$

The next result provides some information about the class of rings R for which every R -module is $\mathfrak{s}$ -coseparable.

Corollary 4.7 *The following statements are equivalent for a ring R :*

- (i) Every R -module is $\mathfrak{s}$ -coseparable;
- (ii) R is a weakly regular semiprimitive ring such that every factor module of an $\mathfrak{s}$ -coseparable module is also $\mathfrak{s}$ -coseparable.

Proof (i) $\Rightarrow$ (ii) By Theorem 4.1.

(ii) $\Rightarrow$ (i) Let M be an R -module. Then, $M \cong F/N$ for some free R -module F and a submodule N of F . By Theorem 4.1, F is $\mathfrak{s}$ -coseparable. By hypothesis, M is also an $\mathfrak{s}$ -coseparable module. $\square$

Recall that a ring R is said to be *right semihereditary* if every finitely generated right ideal of R is projective as a right R -module.

From Proposition 2.5 and [3, Proposition 17.14], it follows that every nonzero indecomposable projective module which is not simple could not be $\mathfrak{s}$ -coseparable. For example, the $\mathbb{Z}$ -module $\mathbb{Z}$ is not $\mathfrak{s}$ -coseparable (see also [8, Example 2.11(i)]). Next, we present some examples of classes of rings R for which every projective R -module is $\mathfrak{s}$ -coseparable.

Proposition 4.8 *Let R be a ring. Suppose that one of the following conditions is satisfied:*

- (i) R is an exchange ring;
- (ii) R is a right semihereditary ring.

Then, every projective R -module is $\mathfrak{s}$ -coseparable if and only if R is a weakly regular semiprimitive ring.

Proof ($\Rightarrow$) This follows from Theorem 4.1.

($\Leftarrow$) By Theorem 4.1, every finitely generated projective module is $\mathfrak{s}$ -coseparable. Now let P be a projective module which is not finitely generated. Applying [14, Corollary 5.5] and [17, Theorem 1], it follows that P is a direct sum of finitely generated submodules. Moreover, we have $\text{Rad}(P) = 0$ since $\text{Rad}(R) = 0$ (see [3, Proposition 17.10]). Therefore, P is $\mathfrak{s}$ -coseparable by Proposition 2.10. $\square$

As an application of Proposition 4.8, we have the following example.

Example 4.9 (i) Let T be an exchange ring and consider the ring $R = T/\text{Rad}(T)$. By [16, Theorem 29.2], R is a semiprimitive exchange ring. Moreover, R is weakly regular by [16, Remark 29.7(1)]. Using Proposition 4.8, it follows that every projective R -module is $\mathfrak{s}$ -coseparable.

(ii) Note that every von Neumann regular ring is a right semihereditary ring which is weakly regular and semiprimitive.

A ring R is called a *right max ring* if every nonzero right R -module contains a maximal submodule. It is well known that right V-rings and right perfect rings are right max rings (see [3, Theorem 28.4]).

Proposition 4.10 *Let R be a ring such that $\text{Rad}(E(S)) \neq E(S)$ for each simple R -module S (for instance, R could be a right max ring). Then, the following statements are equivalent:*

- (i) *Every finitely cogenerated R -module is $\mathfrak{s}$ -coseparable;*
- (ii) *$E(S)$ is $\mathfrak{s}$ -coseparable for every simple R -module S ;*
- (iii) *R is a right V-ring.*

Proof (i) $\Rightarrow$ (ii) This is immediate.

(ii) $\Rightarrow$ (iii) Let S be a simple R -module. Since $E(S)$ is an indecomposable $\mathfrak{s}$ -coseparable module, it follows that $E(S)$ is simple by Proposition 2.5. Thus $S = E(S)$ is injective. Therefore, R is a right V-ring.

(iii) $\Rightarrow$ (i) Since R is a right V-ring, every finitely cogenerated R -module is semisimple. The result follows. $\square$

A ring R is said to be *right small* if the right R -module R_R is a small submodule in its injective hull $E(R_R)$. Note that every commutative domain is a (right) small ring (see [11, Proposition 2.4 and Corollary 2.5]). We conclude this paper by two results devoted to the class of rings R for which every injective R -module is $\mathfrak{s}$ -coseparable. Let us first give an example of a ring belonging to this class.

Example 4.11 Let R be a right small ring. From [11, Proposition 2.4 and Corollary 2.5], it follows that $\text{Rad}(E) = E$ for every injective R -module E . Hence, every injective R -module is $\mathfrak{s}$ -coseparable.

Proposition 4.12 *Let R be a ring and let M be an R -module. Consider the following statements:*

- (i) *R is a right max ring and every R -injective module is $\mathfrak{s}$ -coseparable;*
- (ii) *Every nonzero R -module contains a nonzero finitely generated injective submodule;*
- (iii) *R is a right semiartinian right V-ring.*

Then, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) hold.

Proof (i) $\Rightarrow$ (ii) Let M be a nonzero R -module and let $E(M)$ be the injective hull of M . Since R is a right max ring, $E(M)$ has a maximal submodule L . By hypothesis, $E(M)$ is $\mathfrak{s}$ -coseparable. Then, there exist submodules N and K of $E(M)$ such that $E(M) = N \oplus K$, $N \subseteq L$ and K is a nonzero finitely generated submodule of M . But M is essential in $E(M)$. Therefore, $M \cap K \neq 0$. As K is injective, K is $\mathfrak{s}$ -coseparable. Hence, $M \cap K$ contains a nonzero direct summand U of K . It is clear that U is a finitely generated injective submodule of M .

(ii) $\Rightarrow$ (iii) By [7, Theorem 13]. $\square$

Corollary 4.13 *Let R be a right perfect ring. Then, the following statements are equivalent:*

- (i) *Every injective R -module is $\mathfrak{s}$ -coseparable;*
- (ii) *R is a semisimple ring.*

Proof (i) $\Rightarrow$ (ii) By [3, Theorem 28.4], R is a right max ring and $R/\text{Rad}(R)$ is semisimple. By Proposition 4.12, R is a right V-ring. Hence, $\text{Rad}(R) = 0$. It follows that R is a semisimple ring.

(ii) $\Rightarrow$ (i) This is obvious. $\square$

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