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Some local fixed point theorems and applications to open mapping principles and continuation results

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Abstract The purpose of this article is to present, under weaker assumptions, some local fixed point theorems for Ćirić–Reich–Rus, Chatterjea and Berinde type generalized contractions. Then, as applications we will obtain open mapping theorems and continuation principles for these classes of mappings.

Mathematics Subject Classification 47H10 · 54H25

Introduction

The paper contains local fixed point theorems for three types of generalized contractions: Ćirić–Reich–Rus contractions, Chatterjea contractions and Berinde contractions. As applications, open mapping theorems and continuation principles for these classes of operators are given.

In the first section, we recall some well-known definitions and results, which are used in the main sections. In the next section, we prove local fixed point theorems for the Ćirić–Reich–Rus, Chatterjea and Berinde type generalized contractions, which generalize some local theorems from [9,10,12]. In the last two sections, as applications of the previous results, we present open mapping theorems and continuation principles for the case of the above-mentioned operators, extending results presented in [9,10].

1 Preliminaries

We first recall the definitions of Ćirić–Reich–Rus (see [8,11,17]), Chatterjea (see [7]) and Berinde types of generalized contractions.

Definition 1.1 Let (X, d) be a metric space. We say that an operator $f : X \to X$ is a Ćirić–Reich–Rus contraction if there exist $\alpha, \beta \in (0, 1)$ such that $\alpha + 2\beta < 1$ and for all $x, y \in X$, we have

$$d(f(x), f(y)) \leq \alpha d(x, y) + \beta\, [d(x, f(x)) + d(y, f(y))]\,. \tag{1}$$

Definition 1.2 Let (X, d) be a metric space. An operator $f : X \to X$ is a Chatterjea contraction if there exists $\gamma \in [0, \frac{1}{2})$ such that for all $x, y \in X$, we have

$$d(f(x), f(y)) \leq \gamma\, [d(x, f(y)) + d(y, f(x))]\,. \tag{2}$$

Definition 1.3 Let (X, d) be a metric space. We say that an operator $f : X \to X$ is a Berinde contraction if there exist $\alpha \in (0, 1)$ and $L \geq 0$ such that for all $x, y \in X$ we have

$$d(f(x), f(y)) \leq \alpha d(x, y) + Ld(x, f(y)). \tag{3}$$

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We now recall the notions of Picard and weakly Picard operators.

Definition 1.4 Let (X, d) be a metric space.

a) We say that an operator $f: X \to X$ is a Picard operator if there exists a point $x^* \in X$ such that $\mathrm{Fix}(f) = \{x^*\}$ and the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ converges to x^* for all $x_0 \in X$.
b) An operator $f: X \to X$ is said to be a weakly Picard operator if the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ converges for all $x_0 \in X$ and its limit belongs to the set of fixed points $\mathrm{Fix}(f)$.

It is well known that any Ćirić–Reich–Rus and any Chatterjea contractions are Picard operators (P.o.), while any Berinde contraction is w.P.o. (weakly Picard operator). For other considerations on Picard operators and weakly Picard operators, see [16,18].

We also recall the notion of a field generated by an operator.

Definition 1.5 Let $(X, \|\cdot\|)$ be a normed space. For an operator $f: X \to X$, we define the field generated by f as follows:

$$g: X \to X,\ g(x) = x - f(x).$$

Let (X, d) be a metric space, a point $x_0 \in X$ and a strictly positive r. The set $B(x_0; r) := \{x \in X: d(x_0, x) < r\}$ is the open ball of center x_0 and radius r and also the set $\tilde{B}(x_0; r) := \{x \in X: d(x_0, x) \le r\}$ is the closed ball of center x_0 and radius r.

2 Local fixed point theorems

In this section, we will present some local fixed point theorems for three types of generalized contractions.

The first main result of this section is a local fixed point theorem concerning Ćirić–Reich–Rus type of operators (see [12,13]).

Theorem 2.1 *Let (X, d) be a complete metric space, $x_0 \in X$, a positive number r and let $f: B(x_0; r) \to X$ be a Ćirić–Reich–Rus type contraction. If*

$$d(x_0, f(x_0)) < \frac{1-\alpha-2\beta}{1-\beta} r, \tag{4}$$

then the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ of successive approximations starting from the center of the ball converges to a point x^* which is a fixed point for the Ćirić–Reich–Rus contraction f. Moreover, the fixed point is unique.

Proof Let $0 < s < r$ such that

$$d(x_0, f(x_0)) \le \frac{1-\alpha-2\beta}{1-\beta} s < \frac{1-\alpha-2\beta}{1-\beta} r.$$

The considered sequence $(f^n(x_0))_{n\in\mathbb{N}}$ has the recurrent form $x_{n+1} = f(x_n)$, for all $n \in \mathbb{N}$. Then, since

$$d(x_1, x_2) = d(f(x_0), f(x_1)) \le \alpha d(x_0, x_1) + \beta \left[d(x_0, x_1) + d(x_1, x_2)\right],$$

we have

$$d(x_1, x_2) \le \frac{\alpha+\beta}{1-\beta} d(x_0, x_1).$$

We denote $q := \dfrac{\alpha+\beta}{1-\beta} < 1$. We assume

$$p(n): d(x_{n-1}, x_n) \le q^{n-1} d(x_0, x_1) \tag{5}$$

holds for $n \in \mathbb{N}^*$, $n \ge 2$ and we prove $p(n+1)$ by mathematical induction:

$$\begin{aligned} d(x_n, x_{n+1}) &= d(f(x_{n-1}), f(x_n)) \\ &\le \alpha d(x_{n-1}, x_n) + \beta \left[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right]. \end{aligned}$$

From the above relation, we have

$$d(x_n, x_{n+1}) \le \frac{\alpha+\beta}{1-\beta} d(x_{n-1}, x_n) = q d(x_{n-1}, x_n) \le q^n d(x_0, x_1);$$

thus, our assumption $p(n)$ holds for all $n \in \mathbb{N}^*$, $n \ge 2$.

We continue with proving that the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ is in the closed ball $\tilde{B}(x_0; s)$, for all $n \in \mathbb{N}$. We know that

$$d(x_0, x_1) = d(x_0, f(x_0)) \le \frac{1-\alpha-2\beta}{1-\beta} s = (1-q)s.$$

We consider an arbitrary $n \in \mathbb{N}^*$, $n \ge 2$ and we compute:

$$\begin{aligned} d(x_0, x_n) &\le d(x_0, x_1) + d(x_1, x_2) + \cdots + d(x_{n-1}, x_n) \\ &\le d(x_0, x_1) + q d(x_0, x_1) + \cdots + q^{n-1} d(x_0, x_1) \\ &= d(x_0, x_1)(1 + q + \cdots + q^{n-1}) \\ &= \frac{1-q^n}{1-q} d(x_0, x_1) \le \frac{1}{1-q} d(x_0, x_1) \le s, \end{aligned}$$

which proves that all elements of the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ are still in the closed ball $\tilde{B}(x_0; s)$.

Next, we prove that the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ is Cauchy in $\tilde{B}(x_0; s)$.
For $n \in \mathbb{N}$ and $p \in \mathbb{N}^*$ we have

$$\begin{aligned} d(x_n, x_{n+p}) &\le d(x_n, x_{n+1}) + \cdots + d(x_{n+p-1}, x_{n+p}) \\ &\le q^n d(x_0, x_1) + \cdots + q^{n+p-1} d(x_0, x_1) \\ &= q^n d(x_0, x_1) \left(1 + q + \cdots + q^{p-1}\right) \\ &= q^n \frac{1-q^p}{1-q} d(x_0, x_1) \\ &\le \frac{q^n}{1-q} d(x_0, x_1) \longrightarrow 0 \text{ (as } n, p \longrightarrow \infty). \end{aligned}$$

Thus, we obtain that the sequence is Cauchy. By the completeness of the metric space, we also get that $(f^n(x_0))_{n\in\mathbb{N}}$ is convergent to a point $x^* \in \tilde{B}(x_0; s)$.

We now prove that x^* is a fixed point. We have

$$\begin{aligned} d(x^*, f(x^*)) &\le d(x^*, x_{n+1}) + d(x_{n+1}, f(x^*)) \\ &\le d(x^*, x_{n+1}) + \alpha d(x_n, x^*) + \beta \left[d(x_n, x_{n+1}) + d(x^*, f(x^*))\right], \end{aligned}$$

implying

$$(1-\beta) d(x^*, f(x^*)) \le d(x^*, x_{n+1}) + \alpha d(x_n, x^*) + \beta d(x_n, x_{n+1}).$$

Taking $n \longrightarrow \infty$ in the above inequality we obtain

$$d(x^*, f(x^*)) = 0,$$

which proves that x^* is a fixed point for f.

Lastly, we will show by contradiction that the fixed point x^* is unique in the open ball $B(x_0; r)$. We assume there exists another fixed point $y^* \in B(x_0; r)$ such that $y^* \ne x^*$. Then

$$\begin{aligned} d(x^*, y^*) = d(f(x^*), f(y^*)) &\le \alpha d(x^*, y^*) + \beta \left[d(x^*, f(x^*)) + d(y^*, f(y^*))\right] \\ &\le \alpha d(x^*, y^*), \end{aligned}$$

which implies that $\alpha \ge 1$, contradicting the hypothesis and proving that x^* is the unique fixed point. □

In our next result we will consider Chatterjea generalized contractions and we will prove another local fixed point result for this class of mappings.

Theorem 2.2 *Let (X, d) be a complete metric space, $x_0 \in X$, a positive number r and let $f\colon B(x_0; r) \to X$ be a Chatterjea type contraction. If*

$$d(x_0, f(x_0)) < \frac{1-2\gamma}{1-\gamma} r,$$

then the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ of successive approximations starting from the center of the ball converges to a point x^* which is a fixed point of f. Moreover, the fixed point is unique.

Proof Let $0 < s < r$ such that

$$d(x_0, f(x_0)) \leq \frac{1-2\gamma}{1-\gamma} s < \frac{1-2\gamma}{1-\gamma} r.$$

The considered sequence $(f^n(x_0))_{n\in\mathbb{N}}$ has the recurrent form $x_{n+1} = f(x_n)$ for all $n \in \mathbb{N}$. We will compute the following distance:

$$d(x_1, x_2) = d(f(x_0), f(x_1)) \leq \gamma\, [d(x_0, x_2) + d(x_1, x_1)] = \gamma d(x_0, x_2),$$

implying

$$d(x_1, x_2) \leq \frac{\gamma}{1-\gamma} d(x_0, x_1).$$

We now denote $q := \dfrac{\gamma}{1-\gamma} < 1$ and assume

$$p(n)\colon d(x_{n-1}, x_n) \leq q^{n-1} d(x_0, x_1) \tag{6}$$

true for $n \in \mathbb{N}^*$, $n \geq 2$. We compute $p(n+1)$

$$\begin{aligned} d(x_n, x_{n+1}) &= d(f(x_{n-1}), f(x_n)) \\ &\leq \gamma d(x_{n-1}, x_{n+1}) \\ &\leq \gamma d(x_{n-1}, x_n) + \gamma d(x_n, x_{n+1}). \end{aligned}$$

This implies

$$d(x_n, x_{n+1}) \leq \frac{\gamma}{1-\gamma} d(x_{n-1}, x_n) = q d(x_{n-1}, x_n) \leq q^n d(x_0, x_1),$$

proving $p(n+1)$, so by mathematical induction $p(n)$ holds for all $n \in \mathbb{N}^*$, $n \geq 2$.

We will now show that the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ is in the closed ball $\tilde{B}(x_0; s)$, for all $n \in \mathbb{N}$. From the hypothesis, the following relation is known:

$$d(x_0, x_1) = d(x_0, f(x_0)) \leq \frac{1-2\gamma}{1-\gamma} s = (1-q)r.$$

We consider an arbitrary $n \in \mathbb{N}^*$, $n \geq 2$ and we compute

$$\begin{aligned} d(x_0, x_n) &\leq d(x_0, x_1) + d(x_1, x_2) + \cdots + d(x_{n-1}, x_n) \\ &\leq d(x_0, x_1) + q d(x_0, x_1) + \cdots + q^{n-1} d(x_0, x_1) \\ &= d(x_0, x_1)(1 + q + \cdots + q^{n-1}) \\ &\leq \frac{1-q^n}{1-q} d(x_0, x_1) \leq \frac{1}{1-q} d(x_0, x_1) \leq s. \end{aligned}$$

This proves that all the elements of the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ are in the closed ball $\tilde{B}(x_0; s)$.

We now prove that the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ is Cauchy in $\tilde{B}(x_0; s)$. Letting $n \in \mathbb{N}$ and $p \in \mathbb{N}^*$ we obtain

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + \cdots + d(x_{n+p-1}, x_{n+p}) \\ &\leq q^n d(x_0, x_1) + \cdots + q^{n+p-1} d(x_0, x_1) \\ &= q^n d(x_0, x_1) \left(1 + q + \cdots + q^{p-1}\right) \end{aligned}$$

$$= q^n \frac{1-q^p}{1-q} d(x_0, x_1)$$
$$\leq \frac{q^n}{1-q} d(x_0, x_1) \longrightarrow 0 \text{ (as } n \text{ and } p \longrightarrow \infty).$$

Therefore, the sequence is Cauchy and considering the completeness of the metric space, the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ is convergent to a point $x^* \in \tilde{B}(x_0; s)$.

We will next show the point x^* is a fixed point. We estimate the distance

$$\begin{aligned} d(x^*, f(x^*)) &\leq d(x^*, x_{n+1}) + d(x_{n+1}, f(x^*)) \\ &\leq d(x^*, x_{n+1}) + \gamma d(x_n, f(x^*)) + \gamma d(x^*, x_{n+1}) \\ &= (1+\gamma) d(x^*, x_{n+1}) + \gamma d(x_n, f(x^*)) \\ &\leq (1+\gamma) d(x^*, x_{n+1}) + \gamma d(x_n, x^*) + \gamma d(x^*, f(x^*)), \end{aligned}$$

implying

$$(1-\gamma) d(x^*, f(x^*)) \leq (1+\gamma) d(x^*, x_{n+1}) + \gamma d(x_n, x^*).$$

We now take $n \longrightarrow \infty$ and we obtain

$$d(x^*, f(x^*)) = 0,$$

which proves that x^* is a fixed point for the Chatterjea contraction f.

In the last part, we need to prove that x^* is the unique fixed point of f, which we will do by contradiction. Assume that there exists another fixed point $y^* \in B(x_0; r)$ such that $y^* \neq x^*$. Then

$$\begin{aligned} d(x^*, y^*) = d(f(x^*), f(y^*)) &\leq \gamma \left[d(x^*, f(y^*)) + d(y^*, f(x^*)) \right] \\ &= 2\gamma d(x^*, y^*), \end{aligned}$$

which implies that $2\gamma \geq 1$, contradicting the hypothesis and proving that x^* is the unique fixed point. □

The last local fixed point theorem proven in this section refers to the Berinde generalized contraction (see [3,5,6]). This extension will not include the uniqueness of the fixed point, which is somehow an expected fact, since Berinde contractions are weakly Picard operators, but (in general) are not Picard operators.

Theorem 2.3 *Let (X, d) be a complete metric space, $x_0 \in X$, a positive number r and let $f : B(x_0; r) \to X$ be a Berinde type contraction. If*

$$d(x_0, f(x_0)) < (1-\alpha) r,$$

then the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ of successive approximations starting from the center of the ball converges to a point x^* which is a fixed point of f.

Proof Let $0 < s < r$ such that

$$d(x_0, f(x_0)) \leq (1-\alpha) s < (1-\alpha) r.$$

The sequence $(f^n(x_0))_{n\in\mathbb{N}}$ of successive approximations has the recurrent form $x_{n+1} = f(x_n)$ for all $n \in \mathbb{N}$. We evaluate the following distance:

$$d(x_1, x_2) = d(f(x_0), f(x_1)) \leq \alpha d(x_0, x_1) + L d(x_1, f(x_0)) = \alpha d(x_0, x_1).$$

Now assume $p(n) : d(x_{n-1}, x_n) \leq \alpha^{n-1} d(x_0, x_1)$ for $n \in \mathbb{N}^*, n \geq 2$ and since

$$\begin{aligned} d(x_n, x_{n+1}) &= d(f(x_{n-1}), f(x_n)) \\ &\leq \alpha d(x_{n-1}, x_n), \end{aligned}$$

it follows that

$$d(x_n, x_{n+1}) s \leq \alpha^n d(x_0, x_1).$$

From the above inequality, we have proven $p(n+1)$, so by mathematical induction $p(n)$ holds for all $n \in \mathbb{N}^*$, $n \geq 2$.

Now we show that the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ is in the closed ball $\tilde{B}(x_0; s)$, for all $n \in \mathbb{N}$. For this purpose, consider an arbitrary $n \in \mathbb{N}$ and we compute the following distance:

$$\begin{aligned} d(x_0, x_n) &\leq d(x_0, x_1) + d(x_1, x_2) + \cdots + d(x_{n-1}, x_n) \\ &\leq d(x_0, x_1) + \alpha d(x_0, x_1) + \cdots + \alpha^{n-1} d(x_0, x_1) \\ &= d(x_0, x_1)(1 + \alpha + \cdots + \alpha^{n-1}) \\ &\leq \frac{1-\alpha^n}{1-\alpha} d(x_0, x_1) \leq \frac{1}{1-\alpha} d(x_0, x_1), \end{aligned}$$

and since $d(x_0, x_1) = d(x_0, f(x_0)) \leq (1-\alpha)s$, we get

$$d(x_0, x_n) \leq s,$$

proving that all the elements of the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ are in the closed ball $\tilde{B}(x_0; s)$.

Next we will show the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ is Cauchy in $\tilde{B}(x_0; s)$. We take $n \in \mathbb{N}$ and $p \in \mathbb{N}^*$ and evaluate

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + \cdots + d(x_{n+p-1}, x_{n+p}) \\ &\leq \alpha^n d(x_0, x_1) + \cdots + \alpha^{n+p-1} d(x_0, x_1) \\ &= \alpha^n d(x_0, x_1) \left(1 + \alpha + \cdots + \alpha^{p-1}\right) \\ &= \alpha^n \frac{1-\alpha^p}{1-\alpha} d(x_0, x_1) \\ &\leq \frac{\alpha^n}{1-q} d(x_0, x_1) \longrightarrow 0 \text{ (as } n \text{ and } p \longrightarrow \infty). \end{aligned}$$

Therefore, we get that the sequence is Cauchy and together with the completeness of the metric space, it is also convergent in $\tilde{B}(x_0; s)$ to a point x^*.

To prove x^* is a fixed point, we will estimate the following distance:

$$\begin{aligned} d(x^*, f(x^*)) &\leq d(x^*, x_{n+1}) + d(f(x_n), f(x^*)) \\ &\leq d(x^*, x_{n+1}) + \alpha d(x_n, x^*) + L d(x^*, x_{n+1}). \end{aligned}$$

This leads us to

$$d(x^*, f(x^*)) \leq (1+L) d(x^*, x_{n+1}) + \alpha d(x_n, x^*)$$

and taking $n \longrightarrow \infty$, we obtain

$$d(x^*, f(x^*)) = 0,$$

which also proves that x^* is a fixed point for f. □

Remark 2.4 For other local or nonself fixed point theorems, see [4–6,9,10,12–15]. Our results improve some corresponding local fixed point theorems, since here we do not impose conditions to get the invariance of the ball, only (weaker) assumptions assuring the convergence of the sequence of Picard iterates starting from the center of the ball to a fixed point.

3 Open mapping results

In this section, we present an application of the local fixed point theorems to open mapping principles. First, we consider the case of Ćirić–Reich–Rus type operators.

Theorem 3.1 *Let $(E, \|\cdot\|)$ be a Banach space, U an open subset of E, and let $f : U \to E$ be a Ćirić–Reich–Rus contraction. Then, the field g is an open operator.*

Proof The operator g is open if and only if for all open subsets $V \subset U$, the set $g(V)$ is open in E. It is enough to prove that for all $y \in g(V)$, there exists an open neighborhood W of y such that $W \subset g(V)$. Let $V \subset U$ open, and we prove that for all $u \in V$ and for all $B(u;r) \subset V$, with $r > 0$, we have that

$$B\left(g(u); \frac{1-\alpha-2\beta}{1-\beta}r\right) \subset g(B(u;r)).$$

For this, let $u \in V$ such that $B(u;r) \subset V$, let $y_0 \in B\left(g(u); \frac{1-\alpha-2\beta}{1-\beta}r\right)$, and consider the operator

$$h\colon B(u;r) \to E, h(x) = y_0 + f(x). \tag{7}$$

Then, h is a Ćirić–Reich–Rus type contraction, since for $x_1, x_2 \in B(u;r)$ arbitrarily chosen, we have

$$\begin{aligned}\|h(x_1) - h(x_2)\| &\le \|f(x_1) - f(x_2)\| \\ &\le \alpha\|x_1 - x_2\| + \beta\left(\|x_1 - f(x_1)\| + \|x_2 - f(x_2)\|\right).\end{aligned}$$

Now, we compute the following distance:

$$\begin{aligned}\|u - h(u)\| &= \|u - y_0 - f(u)\| = \|u - f(u) - y_0\| \\ &= \|g(u) - y_0\| < \frac{1-\alpha-2\beta}{1-\beta}r,\end{aligned}$$

because $y_0 \in B\left(g(u); \frac{1-\alpha-2\beta}{1-\beta}r\right)$. By the local fixed point theorem for Ćirić–Reich–Rus generalized contractions, we obtain that there exists a unique fixed point $x^* \in B(u;r)$, such that for the chosen $y_0 \in B\left(g(u); \frac{1-\alpha-2\beta}{1-\beta}r\right)$, we have

$$y_0 = g(x^*) \in g(B(u;r)) \implies B\left(g(u); \frac{1-\alpha-2\beta}{1-\beta}r\right) \subset g(B(u;r)).$$

We denote the set $W := B\left(g(u); \frac{1-\alpha-2\beta}{1+\beta}r\right)$. Since $B(u;r) \subset V$ and also $W \subset g(B(u;r))$ we obtain that $W \subset g(B(u;r)) \subset g(V)$. Thus, the field operator g is open. □

Now we introduce the open mapping theorem for Chatterjea generalized contractions.

Theorem 3.2 *Let $(E, \|\cdot\|)$ be a Banach space, U an open subset of E, and let $f : U \to E$ be a Chatterjea contraction. Then, the field g is an open operator.*

Proof We say that an operator is open if and only if for any open set $V \subset U$, the set $g(V)$ is open in E. Again, we will prove that for all $y \in g(V)$, there exists an open neighborhood W of y such that $W \subset g(V)$.Let $V \subset U$ be an open set and we show that for all $u \in V$ and for all $B(u;r) \subset V$ arbitrarily chosen we will have

$$B\left(g(u); \frac{1-2\gamma}{1-\gamma}r\right) \subset g(B(u;r)).$$

To show the above inclusion, let $u \in V$ such that $B(u;r) \subset V$, and let $y_0 \in B\left(g(u); \frac{1-2\gamma}{1-\gamma}r\right)$. We consider the same operator h defined at (7), which now is a Chatterjea contraction, because for any arbitrary x_1 and x_2 in $B(u;r)$, we get

$$\begin{aligned}\|h(x_1) - h(x_2)\| &\le \|f(x_1) - f(x_2)\| \\ &\le \gamma\left[\|x_1 - f(x_2)\| + \|x_2 - f(x_1)\|\right].\end{aligned}$$

We now compute

$$\|u - h(u)\| = \|u - y_0 - f(u)\| = \|u - f(u) - y_0\|$$

$$= \|g(u) - y_0\| < \frac{1-2\gamma}{1-\gamma} r.$$

Having this estimation, in view of the local fixed point theorem for the Chatterjea operator, we obtain that there exists a unique fixed point $x^* \in B(u;r)$ such that for the chosen y_0, we have

$$y_0 = g(x^*) \in g(B(u;r)) \implies B\left(g(u); \frac{1-2\gamma}{1-\gamma} r\right) \subset g(B(u;r)).$$

We denote $W := B\left(g(u); \frac{1-2\gamma}{1+\gamma} r\right)$. We already know that $B(u;r) \subset V$ which implies that $g(B(u;r)) \subset g(V)$, and because $W \subset g(B(u;r))$, we can conclude that the field operator g is open in the Banach space $(E, \|\cdot\|)$. □

The final case for this application is the one of Berinde type operators, where we obtain the same conclusion.

Theorem 3.3 *Let $(E, \|\cdot\|)$ be a Banach space, U an open subset of E, and let $f : U \to E$ be a Berinde contraction. Then, the field g is an open operator.*

Proof We say that g is an open operator if and only if for any V open subset of U, the set $g(V)$ is open in E as well. Thus, it is sufficient to show that for all y in $g(V)$, there exists W an open neighborhood of y such that $W \subset g(V)$. We consider the set $V \subset U$ open, and first we will prove that for all u in V, and for all the open balls $B(u;r) \subset V$ we have that the open ball $B(g(u); (1-\alpha)r)$ is a subset of $g(B(u;r))$, where $r > 0$.

Let u in the open set V such that $B(u;r) \subset V$, and y_0 in $B(g(u); (1-\alpha)r)$. Again, we consider the operator h defined at (7), and we state that, in this case, it is a Berinde operator, since for any arbitrary x_1 and x_2 in $B(u;r)$, we get

$$\begin{aligned} \|h(x_1) - h(x_2)\| &\leq \|f(x_1) - f(x_2)\| \\ &\leq \alpha\|x_1 - x_2\| + L\|x_1 - f(x_2)\|. \end{aligned}$$

We still need to compute the following distance

$$\|u - h(u)\| = \|u - y_0 - f(u)\| = \|u - f(u) - y_0\| = \|g(u) - y_0\| < (1-\alpha)r,$$

due to the fact that y_0 is in the open ball $B(g(u); (1-\alpha)r)$. By the local fixed point theorem for the Berinde operator, there exists a fixed point x^* in $B(u;r)$ such that for the chosen point y_0, we have the following:

$$y_0 = g(x^*) \text{ in } g(B(u;r)), \text{ implying } B(g(u); (1-\alpha)r) \subset g(B(u;r)).$$

We consider $W := B(g(u); (1-\alpha)r)$. We already know that $B(u;r) \subset V$, which means that $g(B(u;r)) \subset g(V)$, and since the inclusion $W \subset g(B(u;r))$ happens, then $W \subset g(V)$ concluding that g is an open operator. □

Remark 3.4 Similar results as presented above have been obtained in [10]. Our results complement some theorems from the above mentioned paper, by considering other classes of generalized contractions.

4 Continuation theorems

In the last section of the paper, we will present some continuation results for three classes of generalized contractions: Ćirić–Reich–Rus contractions, Chatterjea contractions and Berinde type contractions. We denote $CR(Y, X)$ the family of all contractions from Y to X and by

$$CR_{\delta Y}(Y, X) := \left\{ f \in CR(Y, X) \text{ such that } f_{|\delta Y} : \delta Y \to X \text{ is fixed point free} \right\}.$$

We begin by defining the concept of (α, β)-contractive family.

Definition 4.1 Let (X, d) be a metric space and (J, ρ) be a connected metric space. We say that the sequence $(H_\lambda)_{\lambda \in J} \subset CR(Y, X)$ is an (α, β)-contractive family if there exist $\alpha \in (0, 1)$, $p \in (0, 1]$ and $M > 0$ such that

$$\begin{aligned} &(i) d(H_\lambda(x_1), H_\lambda(x_2)) \leq \alpha d(x_1, x_2) + \beta\left[d(x_1, H_\lambda(x_1)) + d(x_2, H_\lambda(x_2))\right], \\ &\quad \text{for all } x_1, x_2 \in Y \text{ and } \lambda \in J; \\ &(ii) d(H_\lambda(x), H_\mu(x)) \leq M\left[\rho(\lambda, \mu)\right]^p, \text{ for all } x \in Y \text{ and } \lambda, \mu \in J. \end{aligned}$$

Below we have the continuation principle corresponding to the Ćirić–Reich–Rus generalized contractions.

Theorem 4.2 *Let (X, d) be a complete metric space and Y a closed subset such that $intY \neq \emptyset$. Let (J, ρ) be a connected metric space and $(H_\lambda)_{\lambda \in J}$ be an (α, β)-contractive family from $CR_{\delta Y}(Y, X)$. The following conclusions occur:*

(i) If there exists a point $\lambda_0^* \in J$, such that the equation $H_{\lambda_0^*}(x) = x$ has a solution, then the equation $H_\lambda(x) = x$ has a unique solution for any $\lambda \in J$;

(ii) If $H_\lambda(x_\lambda) = x_\lambda$ for any $\lambda \in J$, then the operator

$$j : J \to intY,\ j(\lambda) = x_\lambda \tag{8}$$

is continuous.

Proof Let x_λ and x_μ be two fixed points of H_λ and H_μ, respectively. Then,

$$\begin{aligned} d(x_\lambda, x_\mu) &= d(H_\lambda(x_\lambda), H_\mu(x_\mu)) \\ &\leq d(H_\lambda(x_\lambda), H_\lambda(x_\mu)) + d(H_\lambda(x_\mu), H_\mu(x_\mu)) \\ &\leq \alpha d(x_\lambda, x_\mu) + \beta d(x_\mu, H_\lambda(x_\mu)) + d(H_\lambda(x_\mu), H_\mu(x_\mu)) \\ &= \alpha d(x_\lambda, x_\mu) + \beta d(H_\mu(x_\mu), H_\lambda(x_\mu)) + d(H_\lambda(x_\mu), H_\mu(x_\mu)) \\ &\leq \alpha d(x_\lambda, x_\mu) + (\beta + 1) M[\rho(\lambda, \mu)]^p. \end{aligned}$$

This inequality implies that

$$d(x_\lambda, x_\mu) \leq \frac{1+\beta}{1-\alpha} M[\rho(\lambda, \mu)]^p, \tag{9}$$

for any λ and μ. Let

$$Q = \{\lambda \in J \mid \exists x_\lambda \in intY \text{ such that } x_\lambda = H_\lambda(x_\lambda)\}. \tag{10}$$

We will show that the set Q is both closed and open. First, for proving that Q is closed, let $(\lambda_n)_{n\in\mathbb{N}} \subset Q$ such that $\lambda_n \longrightarrow \lambda^*$. We now show that $\lambda^* \in Q$.
From $x_{\lambda_n} = H_{\lambda_n}(x_{\lambda_n})$ and $x_{\lambda_m} = H_{\lambda_m}(x_{\lambda_m})$, we know that

$$d(x_{\lambda_n}, x_{\lambda_m}) < \frac{1+\beta}{1-\alpha} M\left[\rho(\lambda_n, \lambda_m)\right]^p.$$

We also know the fact that the sequence $(\lambda_n)_{n\in\mathbb{N}}$ is Cauchy, which implies

$$\rho(\lambda_n, \lambda_m) < \varepsilon := \left[\frac{\varepsilon'(1-\alpha)}{M(1+\beta)}\right]^{\frac{1}{p}}.$$

The last two inequalities tell us that

$$d(x_{\lambda_n}, x_{\lambda_m}) < \varepsilon',$$

proving that the sequence (x_{λ_n}) is Cauchy, and since the space X is complete, we get that it is also convergent, and so, the set Q is closed.

For proving that Q is also open, let $\lambda_0 \in Q$. This implies that there exists a point $x_{\lambda_0} \in intY$ such that $x_{\lambda_0} = H_{\lambda_0}(x_{\lambda_0})$. We will show that there exist $\varepsilon > 0$ and an open ball $B(\lambda_0; \varepsilon) \subset Q$. Since $intY$ is open and $x_{\lambda_0} \in intY$, we obtain that there exists an open ball $B(x_{\lambda_0}; r) \subseteq intY$. Now, let $\varepsilon > 0$ such that $\varepsilon^p < \dfrac{(1-\alpha-2\beta)r}{M(1-\beta)}$, and let an arbitrary $\lambda \in B(\lambda_0; \varepsilon)$. We will now show that $\lambda \in Q$. We begin by estimating the distance between $H_\lambda(x_{\lambda_0})$ and x_{λ_0}:

$$\begin{aligned} d(H_\lambda(x_{\lambda_0}), x_{\lambda_0}) &= d(H_\lambda(x_{\lambda_0}), H_{\lambda_0}(x_{\lambda_0})) \\ &\leq M\,(\rho(\lambda, \lambda_0))^p \leq M\varepsilon^p \\ &< \frac{1-\alpha-2\beta}{1-\beta} r. \end{aligned}$$

From the above relations, we have that

$$H_\lambda : B(x_{\lambda_0}; r) \to X$$

is a Ćirić–Reich–Rus contraction. By the local fixed point theorem for the Ćirić–Reich–Rus operators, we obtain that $\mathrm{Fix}(H_\lambda) \neq \emptyset$, which implies that $\lambda \in Q$.

From what we have proven until now, we get that the operator j is single-valued and

$$d(j(\lambda), j(\mu)) \leq \frac{(1+\beta)}{1-\alpha} M(\rho(\lambda, \mu))^p.$$

Let

$$\rho(\lambda, \mu) < \delta := \left[\frac{\varepsilon(1-\alpha)}{M(1+\beta)}\right]^{\frac{1}{p}},$$

with an arbitrary $\varepsilon > 0$. This immediately implies that

$$d(j(\lambda), j(\mu)) < \varepsilon,$$

which proves the fact that j is continuous. □

Next, we introduce the notion of a γ-contractive family

Definition 4.3 Let (X, d) be a complete metric space and (J, ρ) be a connected metric space. The sequence $(H_\lambda)_{\lambda \in J} \subset CR(Y, X)$ is a γ-contractive family if there exist $\gamma \in \left[0, \frac{1}{2}\right)$, $M > 0$ and $p \in (0, 1]$ such that

$$(i)\ d(H_\lambda(x_1), H_\lambda(x_2)) \leq \gamma\left[d(x_1, H_\lambda(x_2)) + d(x_2, H_\lambda(x_1))\right], \text{ for all } x_1, x_2 \in Y,\ \lambda \in J;$$
$$(ii)\ d(H_\lambda(x), H_\mu(x)) \leq M\left[\rho(\lambda, \mu)\right]^p, \text{ for all } x \in Y \text{ and } \lambda, \mu \in J.$$

Below we have the application for the Chatterjea generalized contractions.

Theorem 4.4 *Let (X, d) be a complete metric space and Y a closed subset such that $\mathrm{int}\, Y \neq \emptyset$. Let (J, ρ) be a connected metric space and $(H_\lambda)_{\lambda \in J}$ be a γ-contractive family from $CR_{\delta Y}(Y, X)$. The following conclusions occur:*

(i) If there exists a point $\lambda_0^* \in J$, such that the equation $H_{\lambda_0^*}(x) = x$ has a solution, then the equation $H_\lambda(x) = x$ has a unique solution for any $\lambda \in J$;

(ii) If $H_\lambda(x_\lambda) = x_\lambda$, for any $\lambda \in J$, then the operator defined at (8) is continuous.

Proof Following the example of Theorem 4.2, we start by estimating the distance between two arbitrary fixed points x_λ and x_μ of the operators H_λ and H_μ, respectively:

$$\begin{aligned} d(x_\lambda, x_\mu) &= d(H_\lambda(x_\lambda), H_\mu(x_\mu)) \\ &\leq d(H_\lambda(x_\lambda), H_\lambda(x_\mu)) + d(H_\lambda(x_\mu), H_\mu(x_\mu)) \\ &\leq \gamma\left[d(x_\lambda, H_\lambda(x_\mu)) + d(x_\mu, H_\lambda(x_\lambda))\right] + d(H_\lambda(x_\mu), H_\mu(x_\mu)) \\ &= \gamma d(x_\lambda, H_\lambda(x_\mu)) + \gamma d(x_\lambda, x_\mu) + d(H_\lambda(x_\mu), H_\mu(x_\mu)) \\ &\leq \gamma d(x_\lambda, x_\mu) + \gamma d(x_\mu, H_\lambda(x_\mu)) + \gamma d(x_\lambda, x_\mu) + d(H_\lambda(x_\mu), H_\mu(x_\mu)), \end{aligned}$$

implying that

$$d(x_\lambda, x_\mu) \leq \frac{M(1+\gamma)}{1-2\gamma}\left[\rho(\lambda, \mu)\right]^p.$$

Now, we consider the set Q defined at (10) and again we try to prove that it is both closed and open, to prove that Q is the whole space J. We begin by proving Q is closed.

Let the sequence $(\lambda_n)_{n \in \mathbb{N}} \subset Q$ such that $\lambda_n \longrightarrow \lambda^*$. We show that $\lambda^* \in Q$. Arbitrarily taking two fixed points x_λ and x_μ of the operators H_λ and H_μ, respectively, we have

$$d(x_\lambda, x_\mu) \leq \frac{M(1+\gamma)}{1-2\gamma}\left[\rho(\lambda, \mu)\right]^p.$$

Since the sequence $(\lambda_n)_{n\in\mathbb{N}}$ is Cauchy, then there exists an $\varepsilon > 0$ such that

$$\rho(\lambda_n, \lambda_m) < \varepsilon := \left[\frac{\varepsilon'(1-2\gamma)}{M(1+\gamma)}\right]^{\frac{1}{p}}, \text{ with } \varepsilon' > 0.$$

Taking into account the two previous inequalities, we get that

$$d(x_\lambda, x_\mu) < \varepsilon',$$

thus the sequence $(x_{\lambda_n})_{\lambda_n\in J}$ is Cauchy in Y, and because the space is complete, it is also convergent. In view of the Hölder continuity, the sequence $(\lambda_n)_{n\in\mathbb{N}}$ is also convergent, showing that Q is closed.

For obtaining that Q is also open, let $\lambda_0 \in Q$ arbitrarily, meaning that there exists a point $x_{\lambda_0} \in intY$ such that $x_{\lambda_0} = H_{\lambda_0}(x_{\lambda_0})$. We will show that there exists an open ball $B(\lambda_0; \varepsilon) \subset Q$, with $\varepsilon > 0$. Since $intY$ is open and $x_{\lambda_0} \in intY$, we get the existence of an open ball $B(x_{\lambda_0}; r) \subseteq intY$, with $r > 0$. We take $\varepsilon > 0$ such that

$$\varepsilon^p < \frac{1-2\gamma}{M(1-\gamma)},$$

an arbitrary $\lambda \in B(\lambda_0; \varepsilon)$, and we prove that $\lambda \in Q$. For this, we compute:

$$\begin{aligned} d(H_\lambda(x_{\lambda_0}), x_{\lambda_0}) &= d(H_\lambda(x_{\lambda_0}), H_{\lambda_0}(x_{\lambda_0})) \\ &\le M\,[\rho(\lambda_0, \lambda)]^p \\ &\le M\varepsilon^p \\ &< \frac{1-2\gamma}{1-\gamma}. \end{aligned}$$

We know the operator

$$H_\lambda : B(x_{\lambda_0}; r) \to X$$

is a Chatterjea contraction. Considering the estimation for the distance between $H_\lambda(x_{\lambda_0})$ and x_{λ_0}, we are in the terms of the local fixed point theorem for the Chatterjea operator, thus getting that $\mathrm{Fix}(H_\lambda) \neq \emptyset$, which proves that λ is indeed in Q.

We already know that the operator j is single-valued, and also

$$d(j(\lambda), j(\mu)) \le \frac{M(1+\gamma)}{1-2\gamma}\,(\rho(\lambda,\mu))^p\,.$$

To prove the continuity of this operator, we consider

$$\rho(\lambda, \mu) < \delta := \left[\frac{1-2\gamma}{M(1+\gamma)}\varepsilon\right]^{\frac{1}{p}},$$

with an arbitrary $\varepsilon > 0$. Considering the two previous inequalities, we get

$$d(j(\lambda), j(\mu)) < \varepsilon,$$

which proves the desired conclusions. □

We also define the notion of an (α, L)-contractive family and introduce the continuation principle in regards of the Berinde type of operators.

Definition 4.5 Let (X, d) be a metric space and (J, ρ) be a connected metric space. We define the (α, L)-contractive family as a sequence $(H_\lambda)_{\lambda\in J}$ included in $CR(Y, X)$ such that there exist $\alpha \in (0, 1)$, $M > 0$ and $p \in (0, 1]$ with

$$(i) d(H_\lambda(x_1), H_\lambda(x_2)) \le \alpha d(x_1, x_2) + L d(x_1, H_\lambda(x_2)), \text{ for all } x_1, x_2 \in Y, \lambda \in J;$$
$$(ii) d(H_\lambda(x), H_\mu(x)) \le M\,[\rho(\lambda,\mu)]^p, \text{ for all } x \in Y \text{ and } \lambda, \mu \in J.$$

The next theorem has similar conclusions as above, but also requires an additional condition for the considered (α, L) - contractive family.

Theorem 4.6 *Let (X, d) be a complete metric space and Y a closed subset such that $intY \neq \emptyset$. Let (J, ρ) be a connected metric space and $(H_\lambda)_{\lambda \in J}$ be an (α, L)-contractive family from $CR_{\delta Y}(Y, X)$. Then, the following conclusions occur:*

(i) If there exists a point $\lambda_0^* \in J$, such that the equation $H_{\lambda_0^*}(x) = x$ has a solution, then the equation $H_\lambda(x) = x$ has a unique solution for any $\lambda \in J$;

(ii) If $H_\lambda(x_\lambda) = x_\lambda$, for any $\lambda \in J$ and also $\alpha + L < 1$, where $\alpha \in (0, 1)$ and $L > 0$, then the operator

$$\begin{aligned} j \colon J &\to intY \\ j(\lambda) &= x_\lambda \end{aligned}$$

is single valued and continuous.

Proof As before, let x_λ be a fixed point of H_λ and x_μ a fixed point of the operator H_μ. Then,

$$\begin{aligned} d(x_\lambda, x_\mu) &= d(H_\lambda(x_\lambda), H_\mu(x_\mu)) \\ &\leq d(H_\lambda(x_\lambda)), H_\lambda(x_\mu)) + d(H_\lambda(x_\mu), H_\mu(x_\mu)) \\ &\leq \alpha d(x_\lambda, x_\mu) + Ld(x_\lambda, H_\lambda(x_\mu)) + d(H_\lambda(x_\mu), H_\mu(x_\mu)) \\ &\leq \alpha d(x_\lambda, x_\mu) + Ld(x_\lambda, x_\mu) + (1 + L)d(H_\mu(x_\mu), H_\lambda(x_\mu)), \end{aligned}$$

which implies that

$$d(x_\lambda, x_\mu) \leq \frac{M(1 + L)}{1 - \alpha - L} (\rho(\lambda, \mu))^p . \tag{11}$$

By the same rationale as before, we consider the set

$$Q = \{\lambda \in J \mid \exists x_\lambda \in intY \text{ such that } x_\lambda = H_\lambda(x_\lambda)\},$$

and we will prove that it is both open and closed. For this, let $(\lambda_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in Q. Then,

$$d(x_{\lambda_m}, x_{\lambda_n}) \leq \frac{M(1 + L)}{1 - \alpha - L} (\rho(\lambda_m, \lambda_n))^p .$$

We know that the sequence is Cauchy, so

$$\rho(\lambda_m, \lambda_n) < \varepsilon := \left[\frac{\varepsilon'(1 - \alpha - L)}{M(1 + L)}\right]^{\frac{1}{p}},$$

which also implies that $d(x_{\lambda_m}, x_{\lambda_n}) < \varepsilon'$, meaning the sequence $(x_{\lambda_n})_{n\in\mathbb{N}}$ is Cauchy in a complete metric space, thus convergent. By the Hölder continuity, we also get that the sequence $(\lambda_n)_{n\in\mathbb{N}}$ is convergent to a point λ^* in Q. This proves that the set Q is closed.

To prove that the set Q is open, we consider λ_0 in Q, thus there exists a point x_{λ_0} in $intY$ such that $x_{\lambda_0} = H_{\lambda_0}(x_{\lambda_0})$. Again, we prove that there exist $\varepsilon > 0$ and an open ball $B(\lambda_0; \varepsilon) \subset Q$. Since $intY$ is open and $x_{\lambda_0} \in intY$, we obtain the existence of an open ball $B(x_{\lambda_0}; r) \subseteq intY$. We consider $\varepsilon > 0$ such that $\varepsilon^p < \frac{(1 - \alpha)r}{M}$, and an arbitrary λ in $B(\lambda_0; \varepsilon)$. We will prove that λ is in Q.

$$\begin{aligned} d(H_\lambda(x_{\lambda_0}), x_{\lambda_0}) &= dH_\lambda(x_{\lambda_0}), H_{\lambda_0}(x_{\lambda_0})) \\ &\leq M\,(\rho(\lambda, \lambda_0))^p \\ &\leq M\varepsilon^p \\ &< (1 - \alpha)r. \end{aligned}$$

Next, we define the operator

$$H_\lambda \colon B(x_{\lambda_0}; r) \to X,$$

as a Berinde contraction. Considering the previous estimation for the distance between $H_\lambda(x_{\lambda_0})$ and x_{λ_0}, we are in the terms of the local fixed point theorem for the Berinde operator, and so, we get that $\mathrm{Fix}(H_\lambda) \neq \emptyset$, thus proving that λ is indeed in Q.

Due to the fact that $\alpha + L < 1$, we already know that the operator j is single-valued. We also have that

$$d(j(\lambda), j(\mu)) \leq \frac{M(1+L)}{1-\alpha-L}\,(\rho(\lambda,\mu))^p\,.$$

Now, to prove the continuity, we consider

$$\rho(\lambda,\mu) < \delta := \left[\frac{1-\alpha-L}{M(1+L)}\varepsilon\right]^{\frac{1}{p}},$$

with an arbitrary $\varepsilon > 0$. From the previous inequality, we obtain

$$d(j(\lambda), j(\mu)) < \varepsilon,$$

concluding that j is a continuous operator. □

Remark 4.1 Further research for generalized contractions in various metric type spaces can be considered following [1,2,13,15].

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