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Successive approximations for random coupled Hilfer fractional differential systems

Received: 10 November 2020 / Accepted: 22 May 2021 / Published online: 14 June 2021
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Abstract In this paper, we study the global convergence of successive approximations as well as the uniqueness of the random solution of a coupled random Hilfer fractional differential system. We prove a theorem on the global convergence of successive approximations to the unique solution of our problem. In the last section, we present an illustrative example.

Mathematics Subject Classification 26A33

1 Introduction

Fractional calculus began in 1695, it is a theory of integrals and derivatives with a real order, several mathematicians interested in it as Leibniz, Liouville, Riemann, Abel, ... (see [17]). Several areas are interested in fractional calculus examples: biology, mechanics, electricity, control theory, biophysics, and other applied sciences [7, 9, 13, 17, 18, 20]. Recently, considerable attention has been given to the existence of solutions of fractional differential equations with Hilfer fractional derivative [10, 11, 13, 18, 19], and the references therein.

Functional differential equations with random effects play a fundamental role in the theory of random dynamical systems [2, 4, 12]. Random operator theory is usually the case that the mathematical models used to describe phenomena in the biological, physical, engineering and systems sciences contain certain parameters or coefficients that have specific interpretations, but whose values are unknown.

Recently, several articles on global convergence of successive approximations as well as the uniqueness of solutions for fractional differential equations were made, we refer [3, 5], and the references therein.

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In [6], the authors studied the existence of random solutions for a random coupled Hilfer and Hadamard fractional differential systems in generalized Banach spaces. In this paper we study the uniformly convergence of successive approximations for the coupled random Hilfer fractional differential system:

$$\begin{cases} \left(D_0^{\alpha_1, \beta_1} u \right) (t, w) = f_1(t, u(t, w), v(t, w), w) \\ \left(D_0^{\alpha_2, \beta_2} v \right) (t, w) = f_2(t, u(t, w), v(t, w), w) \end{cases} ; t \in I := [0, T], w \in \Omega, \quad (1)$$

with the initial conditions:

$$\begin{cases} \left(I_0^{1-\gamma_1} u \right) (0, w) = \phi_1(w) \\ \left(I_0^{1-\gamma_2} v \right) (0, w) = \phi_2(w) \end{cases} ; w \in \Omega, \quad (2)$$

where $T > 0$, $\alpha_i \in (0, 1)$, $\beta_i \in [0, 1]$, $(\Omega, \mathcal{A})$ is a measurable space, $\gamma_i = \alpha_i + \beta_i - \alpha_i \beta_i$, $\phi_i : \Omega \rightarrow \mathbb{R}^m$, $f_i : I \times \mathbb{R}^m \times \mathbb{R}^m \times \Omega \rightarrow \mathbb{R}^m$; $i = 1, 2$, are given functions, $I_0^{1-\gamma_i}$ is the left-sided mixed Riemann–Liouville integral of order $1 - \gamma_i$, and $D_0^{\alpha_i, \beta_i}$ is the generalized Riemann–Liouville derivative (Hilfer) operator of order α_i and type β_i ; $i = 1, 2$.

This paper initiates the study of system (1)–(2) using successive approximations.

2 Preliminaries

We denote by C the Banach space of all continuous functions from I into $\mathbb{R}^m$ with the supremum (uniform) norm $\|\cdot\|_\infty$. As usual, $AC(I)$ denotes the space of absolutely continuous functions from I into $\mathbb{R}^m$. By $L^1(I)$, we denote the space of Lebesgue-integrable functions $v : I \rightarrow \mathbb{R}^m$ with the norm:

$$\|w\|_1 = \int_0^T \|w(t)\| dt.$$

By $C_\gamma(I)$ and $C_\gamma^1(I)$, we denote the weighted spaces of continuous functions defined by

$$C_\gamma(I) = \{v : (0, T] \rightarrow \mathbb{R}^m : t^{1-\gamma} v(t) \in C\}$$

with the norm:

$$\|v\|_{C_\gamma} := \sup_{t \in I} \|t^{1-\gamma} v(t)\|$$

and

$$C_\gamma^1(I) = \left\{ v \in C : \frac{dv}{dt} \in C_\gamma \right\}$$

with the norm:

$$\|v\|_{C_\gamma^1} := \|v\|_\infty + \|v'\|_{C_\gamma}.$$

In addition, by $\mathcal{C} := C_{\gamma_1} \times C_{\gamma_2}$, we denote the product weighted space with the norm:

$$\|(u, v)\|_{\mathcal{C}} = \|u\|_{C_{\gamma_1}} + \|v\|_{C_{\gamma_2}}.$$

Now, we will give some necessary definitions of fractional calculus.

Definition 2.1 [1, 15] The left-sided mixed Riemann–Liouville integral of order $l > 0$ of a function $v \in L^1(I)$ is defined by

$$\left(I_0^l v \right) (t) = \frac{1}{\Gamma(l)} \int_0^t (t-s)^{l-1} v(s) ds; \text{ for a.e. } t \in I,$$

where $\Gamma(\cdot)$ is the Gamma function. For all $l, l_1, l_2 > 0$ and each $v \in C$, we have $I_0^l v \in C$, and

$$\left(I_0^{l_1} I_0^{l_2} v \right) (t) = \left(I_0^{l_1+l_2} v \right) (t); \text{ for a.e. } t \in I.$$

Definition 2.2 [1, 15] The Riemann–Liouville fractional derivative of order $l \in (0, 1]$ of a function $v \in L^1(I)$ is defined by:

$$\begin{aligned} (D_0^l v)(t) &= \left(\frac{d}{dt} I_0^{1-l} v \right)(t) \\ &= \frac{1}{\Gamma(1-l)} \frac{d}{dt} \int_0^t (t-s)^{-l} v(s) ds; \text{ for a.e. } t \in I \end{aligned}$$

Lemma 2.3 [16] Let $l \in (0, 1]$, $\gamma \in [0, 1)$ and $v \in C_{1-\gamma}(I)$. Then

$$(D_0^l I_0^l v)(t) = v(t); \text{ for all } t \in (0, T].$$

In addition, if $I_0^{1-l} v \in C_{1-\gamma}^1(I)$, then

$$(I_0^l D_0^l v)(t) = v(t) - \frac{(I_0^{1-l} v)(0^+)}{\Gamma(r)} t^{l-1}; \text{ for all } t \in (0, T].$$

Definition 2.4 [1, 15] The Caputo fractional derivative of order $l \in (0, 1]$ of a function $v \in L^1(I)$ is defined by

$$\begin{aligned} ({}^c D_0^l v)(t) &= \left(I_0^{1-l} \frac{d}{dt} v \right)(t) \\ &= \frac{1}{\Gamma(1-l)} \int_0^t (t-s)^{-l} \frac{d}{ds} v(s) ds; \text{ for a.e. } t \in I. \end{aligned}$$

Definition 2.5 [13] (Hilfer derivative). Let $\alpha \in (0, 1)$, $\beta \in [0, 1]$, $v \in L^1(I)$, and $I_0^{(1-\alpha)(1-\beta)} v \in AC(I)$. The Hilfer fractional derivative of order α and type β of w is defined as

$$(D_0^{\alpha, \beta} v)(t) = \left(I_0^{\beta(1-\alpha)} \frac{d}{dt} I_0^{(1-\alpha)(1-\beta)} v \right)(t); \text{ for a.e. } t \in I.$$

Property 2.6 Let $\alpha \in (0, 1)$, $\beta \in [0, 1]$, $\gamma = \alpha + \beta - \alpha\beta$, and $v \in L^1(I)$.

1. The operator $(D_0^{\alpha, \beta} v)(t)$ can be written as

$$(D_0^{\alpha, \beta} v)(t) = \left(I_0^{\beta(1-\alpha)} \frac{d}{dt} I_0^{1-\gamma} v \right)(t) = (I_0^{\beta(1-\alpha)} D_0^\gamma v)(t); \text{ for a.e. } t \in I.$$

2. If $D_0^\gamma v$ exists and is in $L^1(I)$, then

$$(I_0^\alpha D_0^{\alpha, \beta} v)(t) = (I_0^\gamma D_0^\gamma v)(t) = v(t) - \frac{I_0^{1-\gamma}(0^+)}{\Gamma(\gamma)} t^{\gamma-1}; \text{ for a.e. } t \in I.$$

Corollary 2.7 Let $\chi \in C_\gamma(I)$. Then, the Cauchy problem

$$\begin{cases} (D_0^{\alpha, \beta} u)(t) = \chi(t); t \in I \\ (I_0^{1-\gamma} u)(t) \Big|_{t=0} = \varphi, \end{cases}$$

has the unique solution

$$u(t) = \frac{\varphi}{\Gamma(\gamma)} t^{\gamma-1} + (I_0^\alpha \chi)(t).$$

Let $\beta_{\mathbb{R}^m}$ be the Borel σ -algebra. A mapping $\xi : \Omega \rightarrow \mathbb{R}^m$ is said to be measurable if for any $B \in \beta_{\mathbb{R}^m}$; one has

$$\xi^{-1}(B) = \{w \in \Omega : \xi(w) \in B\} \subset \mathcal{A}.$$

Definition 2.8 Let $\mathcal{A} \times \beta_{\mathbb{R}^m}$ be the direct product of the σ -algebras $\mathcal{A}$ and $\beta_{\mathbb{R}^m}$ those defined in Ω and $\mathbb{R}^m$, respectively. A mapping $T : \Omega \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is called jointly measurable if for any $D \in \beta_{\mathbb{R}^m}$, one has

$$T^{-1}(D) = \{(w, v) \in \Omega \times \mathbb{R}^m : T(w, v) \in D\} \in \mathcal{A} \times \beta_{\mathbb{R}^m}.$$

Definition 2.9 A function $T : \Omega \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is called jointly measurable if $T(\cdot, v)$ is measurable for all $v \in \mathbb{R}^m$ and $T(w, \cdot)$ is continuous for all: $w \in \Omega$.

A random operator is a mapping $T : \Omega \times \mathbb{R}^m \rightarrow \mathbb{R}^m$, such that $T(w, v)$ is measurable in w for all $v \in \mathbb{R}^m$, and it is expressed as $T(w)v = T(w, v)$; we also say that $T(w)$ is a random operator on $\mathbb{R}^m$. The random operator $T(w)$ on E is called continuous (resp. compact, totally bounded, and completely continuous) if $T(w, v)$ is continuous (resp. compact, totally bounded, and completely continuous) in v for all $w \in \Omega$ (for more details, see [14]).

Definition 2.10 [8] Let $\mathcal{P}(X)$ be the family of all nonempty subsets of X and D be a mapping from Ω into $\mathcal{P}(X)$. A mapping $T : \{(w, x) : w \in \Omega, x \in D(w)\} \rightarrow X$ is called a random operator with stochastic domain D if D is measurable (i.e., for all closed $N \subset X$, $\{w \in \Omega, D(w) \cap N \neq \emptyset\}$ is measurable), and for all open $G \subset X$ and all $x \in X$, $\{w \in \Omega : x \in D(w), T(w, x) \in G\}$ is measurable. T will be called continuous if every $T(w)$ is continuous.

Definition 2.11 . A function $h : I \times \mathbb{R}^m \times \mathbb{R}^m \times \Omega \rightarrow \mathbb{R}^m$ is called random Carathéodory if the following conditions are satisfied:

- (i) The map $(t, w) \rightarrow h(t, y, v, w)$ is jointly measurable for all $y, v \in \mathbb{R}^m$; and
- (ii) The map $(y, v) \rightarrow h(t, y, v, w)$ is continuous for a.e. $t \in I$ and $w \in \Omega$.

3 Successive approximations and uniqueness results

In this section, we will give the main result of the global convergence of approximations of the problem (1) and (2).

Definition 3.1 By a generalized solution of the problem (1) and (2) we mean coupled measurable functions $(u, v) \in C_{\gamma_1} \times C_{\gamma_2}$ that satisfies the system (1) on I and the system (2).

Set $I_\eta := [0, \eta T]$; for any $\eta \in [0, 1]$. Let us introduce the following hypotheses.

- (H₁) The functions $f_i : I \times \mathbb{R}^m \times \mathbb{R}^m \times \Omega \rightarrow \mathbb{R}^m$; $i = 1, 2$, are random Carathéodory,
- (H₂) There exist a constant $\rho > 0$ and continuous functions $g_i : I \times [0, \rho]^m \times [0, \rho]^m \times \Omega \rightarrow \mathbb{R}_+$; $i = 1, 2$, such that $g_i(t, \cdot, \cdot, w)$ is nondecreasing for any $w \in \Omega$ and each $t \in I$, and

$$\left\| f_i(t, u, v, w) - f_i(t, \bar{u}, \bar{v}, w) \right\| \leq g_i(t, \|u - \bar{u}\|_{C_{\gamma_1}}, \|v - \bar{v}\|_{C_{\gamma_2}}, w); \quad i = 1, 2. \quad (3)$$

for any $w \in \Omega$ and each $t \in I$, $u, \bar{u} \in C_{\gamma_1}$, and $v, \bar{v} \in C_{\gamma_2}$, such that $\|u - \bar{u}\|_{C_{\gamma_1}} \leq \rho$, and $\|v - \bar{v}\|_{C_{\gamma_2}} \leq \rho$,

- (H₃) $(V, W) \equiv (0, 0)$ is the only coupled functions in $\Omega \times C_{\gamma_1}(I_\lambda, [0, \rho]) \times C_{\gamma_2}(I_\lambda, [0, \rho])$, respectively, satisfying the integral inequalities:

$$V(t, w) \leq \frac{1}{\Gamma(\alpha_1)} \int_0^{\lambda t} g_1(s, V(s, w), W(s, w), w)(t-s)^{\alpha_1-1} ds, \quad (4)$$

and

$$W(t, w) \leq \frac{1}{\Gamma(\alpha_2)} \int_0^{\lambda t} g_2(s, V(s, w), W(s, w), w)(t-s)^{\alpha_2-1} ds, \quad (5)$$

with $\eta \leq \lambda \leq 1$.

Remark 3.2 From (3), for any $w \in \Omega$ and each $t \in I$, $u \in C_{\gamma_1}$, $v \in C_{\gamma_2}$, and $i = 1, 2$, we get

$$\begin{aligned} \|f_i(t, u, v, w)\| &\leq \|f_i(t, 0, 0, w)\| + g_i(t, \|u\|_{C_{\gamma_1}}, \|v\|_{C_{\gamma_2}}, w) \\ &\leq f_i^*(w) + g_i^*(w), \end{aligned}$$

where

$$f_i^*(w) := \sup_{t \in I} \|f_i(t, 0, 0, w)\|,$$

and

$$g_i^*(w) := \sup_{(t,x,y) \in I \times [0,\rho] \times [0,\rho]} g_i(t, x, y, w); \quad i = 1, 2.$$

Define the operators $L_1 : \mathcal{C} \times \Omega \rightarrow C_{\gamma_1}$, and $L_2 : \mathcal{C} \times \Omega \rightarrow C_{\gamma_2}$ by

$$(L_1(u, v))(t, w) = \frac{\phi_1(w)}{\Gamma(\gamma_1)} t^{\gamma_1-1} + \int_0^t (t-s)^{\alpha_1-1} \frac{f_1(s, u(s, w), v(s, w), w)}{\Gamma(\alpha_1)} ds,$$

and

$$(L_2(u, v))(t, w) = \frac{\phi_2(w)}{\Gamma(\gamma_2)} t^{\gamma_2-1} + \int_0^t (t-s)^{\alpha_2-1} \frac{f_2(s, u(s, w), v(s, w), w)}{\Gamma(\alpha_2)} ds.$$

Consider the operator $L : \mathcal{C} \times \Omega \rightarrow \mathcal{C}$:

$$(L(u, v))(t, w) = ((L_1(u, v))(t, w), (L_2(u, v))(t, w)).$$

For any $w \in \Omega$, we define the successive approximations of the problem (1) and (2) as follows:

$$\begin{aligned} (u_0(t, w), v_0(t, w)) &= (\phi_1(w), \phi_2(w)); \quad t \in I \\ (u_{n+1}(t, w), v_{n+1}(t, w)) &= ((L_1(u_n, v_n))(t, w), (L_2(u_n, v_n))(t, w)); \quad t \in I. \end{aligned}$$

Theorem 3.3 Assume that the hypotheses $(H_1) - (H_3)$ hold. Then the successive approximations $((u_n)_{n \in \mathbb{N}}, (v_n)_{n \in \mathbb{N}})$ are well defined and converge uniformly on I to the unique random solution of problem (1) and (2).

Proof From (H_1) the successive approximations are well defined. Thus, there exist $\theta_1, \theta_2 > 0$, such that $\|u\|_{C_{\gamma_1}} \leq \theta_1$, $\|v\|_{C_{\gamma_1}} \leq \theta_2$. Next, for any $w \in \Omega$, and each $t_1, t_2 \in I$ with $t_1 < t_2$, we have

$$\begin{aligned} & \|t_2^{1-\gamma_1} u_n(t_2, w) - t_1^{1-\gamma_1} u_n(t_1, w)\| \\ & \leq \left\| t_2^{1-\gamma_1} \left(\frac{\phi_1(w)}{\Gamma(\gamma_1)} t_2^{\gamma_1-1} + \int_0^{t_2} (t_2-s)^{\alpha_1-1} \frac{f_1(s, u_{n-1}(s, w), v_{n-1}(s, w), w)}{\Gamma(\alpha_1)} ds \right) \right. \\ & \quad \left. - t_1^{1-\gamma_1} \left(\frac{\phi_1(w)}{\Gamma(\gamma_1)} t_1^{\gamma_1-1} + \int_0^{t_1} (t_1-s)^{\alpha_1-1} \frac{f_1(s, u_{n-1}(s, w), v_{n-1}(s, w), w)}{\Gamma(\alpha_1)} ds \right) \right\| \\ & \leq \left\| t_2^{1-\gamma_1} \int_0^{t_2} (t_2-s)^{\alpha_1-1} \frac{f_1(s, u_{n-1}(s, w), v_{n-1}(s, w), w)}{\Gamma(\alpha_1)} ds \right. \\ & \quad \left. - t_1^{1-\gamma_1} \int_0^{t_1} (t_1-s)^{\alpha_1-1} \frac{f_1(s, u_{n-1}(s, w), v_{n-1}(s, w), w)}{\Gamma(\alpha_1)} ds \right\| \\ & = \left\| t_2^{1-\gamma_1} \int_0^{t_1} (t_2-s)^{\alpha_1-1} \frac{f_1(s, u_{n-1}(s, w), v_{n-1}(s, w), w)}{\Gamma(\alpha_1)} ds \right. \\ & \quad \left. + t_2^{1-\gamma_1} \int_{t_1}^{t_2} (t_2-s)^{\alpha_1-1} \frac{f_1(s, u_{n-1}(s, w), v_{n-1}(s, w), w)}{\Gamma(\alpha_1)} ds \right. \\ & \quad \left. - t_1^{1-\gamma_1} \int_0^{t_1} (t_1-s)^{\alpha_1-1} \frac{f_1(s, u_{n-1}(s, w), v_{n-1}(s, w), w)}{\Gamma(\alpha_1)} ds \right\| \\ & \leq T^{1-\gamma_1} \left\| \int_0^{t_1} (t_2-s)^{\alpha_1-1} \frac{f_1(s, u_{n-1}(s, w), v_{n-1}(s, w), w)}{\Gamma(\alpha_1)} ds \right\| \end{aligned}$$

$$\begin{aligned}
& + \int_{t_1}^{t_2} (t_2 - s)^{\alpha_1 - 1} \frac{f_1(s, u_{n-1}(s, w), v_{n-1}(s, w), w)}{\Gamma(\alpha_1)} ds \\
& - \int_0^{t_1} (t_1 - s)^{\alpha_1 - 1} \frac{f_1(s, u_{n-1}(s, w), v_{n-1}(s, w), w)}{\Gamma(\alpha_1)} ds \Big\| \\
& \leq \frac{T^{1-\gamma_1}}{\Gamma(\alpha_1)} \Big\| \int_0^{t_1} \left((t_2 - s)^{\alpha_1 - 1} - (t_1 - s)^{\alpha_1 - 1} \right) f_1(s, u_{n-1}(s, w), v_{n-1}(s, w), w) ds \\
& + \int_{t_1}^{t_2} (t_2 - s)^{\alpha_1 - 1} f_1(s, u_{n-1}(s, w), v_{n-1}(s, w), w) ds \Big\|.
\end{aligned}$$

Then, from Remark 3.2, we get

$$\begin{aligned}
& \|t_2^{1-\gamma_1} u_n(t_2, w) - t_1^{1-\gamma_1} u_n(t_1, w)\| \leq \frac{T^{1-\gamma_1}}{\Gamma(\alpha_1)} (f_1^*(w) + g_1^*(w)) \\
& \times \left(\int_0^{t_1} |(t_2 - s)^{\alpha_1 - 1} - (t_1 - s)^{\alpha_1 - 1}| ds + \int_{t_1}^{t_2} |(t_2 - s)^{\alpha_1 - 1}| ds \right) \\
& \longrightarrow 0, \text{ as } t_1 \rightarrow t_2.
\end{aligned}$$

Thus,

$$\|t_2^{1-\gamma_1} u_n(t_2, w) - t_1^{1-\gamma_1} u_n(t_1, w)\| \longrightarrow 0, \text{ as } t_1 \rightarrow t_2.$$

In addition, we obtain that

$$\|t_2^{1-\gamma_2} v_n(t_2, w) - t_1^{1-\gamma_2} v_n(t_1, w)\| \longrightarrow 0, \text{ as } t_1 \rightarrow t_2.$$

Hence

$$\|t_2^{1-\gamma_1} u_n(t_2, w) - t_1^{1-\gamma_1} u_n(t_1, w)\| + \|t_2^{1-\gamma_2} v_n(t_2, w) - t_1^{1-\gamma_2} v_n(t_1, w)\| \longrightarrow 0, \text{ as } t_1 \rightarrow t_2.$$

So, the sequence $\{(u_n, v_n); n \in \mathbb{N}\}$ is equi-continuous on I , for any $w \in \Omega$.

Let

$$\tau := \sup \{ \eta \in [0, 1] : \{(u_n, v_n)\} \text{ converges uniformly on } I_\eta, \text{ for any } w \in \Omega \}.$$

If $\tau = 1$, then we have the global convergence of successive approximations. Suppose that $\tau < 1$, then the sequence $\{(u_n, v_n)\}$ converges uniformly on I_τ . Since this sequence is equi-continuous, then it converges uniformly to a continuous function $(\tilde{u}(t), \tilde{v}(t))$. If we prove that there exists $\lambda \in (\tau, 1]$, such that $\{(u_n, v_n)\}$ converges uniformly on I_λ , for any $w \in \Omega$. This will yield a contradiction.

Put $u(t, w) = \tilde{u}(t, w)$ and $v(t, w) = \tilde{v}(t, w)$; for each $t \in I_\tau$ and any $w \in \Omega$.

From (H_3) , there exists a constant $\rho > 0$ and a function $g_i : I \times [0, \rho]^m \times [0, \rho]^m \times \Omega \rightarrow \mathbb{R}_+$ satisfying inequality (3). In addition, there exist $\lambda \in [\tau, 1]$ and $n_0 \in \mathbb{N}$, such that for all $t \in I_\lambda$ and any $w \in \Omega$, and $n, m > n_0$, we have

$$\begin{aligned}
& \|u_n(\cdot, w) - u_m(\cdot, w)\|_{C_{\gamma_1}} \leq \rho, \\
& \|v_n(\cdot, w) - v_m(\cdot, w)\|_{C_{\gamma_2}} \leq \rho.
\end{aligned}$$

For each $t \in I_\lambda$, and any $w \in \Omega$, we put

$$\begin{aligned}
V^{(n,m)}(\cdot, w) &= \|u_n(\cdot, w) - u_m(\cdot, w)\|_{C_{\gamma_1}}, \\
V_k(t, w) &= \sup_{n,m \geq k} V^{(n,m)}(t, w), \\
W^{(n,m)}(\cdot, w) &= \|v_n(\cdot, w) - v_m(\cdot, w)\|_{C_{\gamma_2}}, \\
W_k(t, w) &= \sup_{n,m \geq k} W^{(n,m)}(t, w).
\end{aligned}$$

Since the sequence $(V_k(t, w), W_k(t, w))$ is non-increasing, it is convergent to a function $(V(t, w), W(t, w))$ for each $t \in I_\lambda$, and any $w \in \Omega$. From the equi-continuity of $\{(V_k(t, w), W_k(t, w))\}$ it follows that $\lim_{k \rightarrow \infty} V_k(t, w) = V(t, w)$ and $\lim_{k \rightarrow \infty} W_k(t, w) = W(t, w)$ uniformly on I_λ . Furthermore, for each $t \in I_\lambda$, and any $w \in \Omega$, and for $n, m \geq k$, we have

$$\begin{aligned} V^{(n,m)}(\cdot, w) &= \|u_n(\cdot, w) - u_m(\cdot, w)\|_{C_{\gamma_1}} = \|t^{1-\gamma_1}(u_n(t, w) - u_m(t, w))\| \\ &\leq \sup_{s \in [0, t]} \|t^{1-\gamma_1}(u_n(s, w) - u_m(s, w))\| \\ &\leq \left\| t^{1-\gamma_1} \left[\left(\frac{\phi_1(w)}{\Gamma(\gamma_1)} t^{\gamma_1-1} + \int_0^t (t-s)^{\alpha_1-1} \frac{f_1(s, u_{n-1}(s, w), v_{n-1}(s, w), w)}{\Gamma(\alpha_1)} ds \right) \right. \right. \\ &\quad \left. \left. - \left(\frac{\phi_1(w)}{\Gamma(\gamma_1)} t^{\gamma_1-1} + \int_0^t (t-s)^{\alpha_1-1} \frac{f_1(s, u_{m-1}(s, w), v_{m-1}(s, w), w)}{\Gamma(\alpha_1)} ds \right) \right] \right\| \\ &\leq \frac{t^{1-\gamma_1}}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} \left\| f_1(s, u_{n-1}(s, w), v_{n-1}(s, w), w) \right. \\ &\quad \left. - f_1(s, u_{m-1}(s, w), v_{m-1}(s, w), w) \right\| ds \\ &\leq \frac{1}{\Gamma(\alpha_1)} \int_0^{\lambda T} (t-s)^{\alpha_1-1} s^{1-\gamma_1} \left\| f_1(s, u_{n-1}(s, w), v_{n-1}(s, w), w) \right. \\ &\quad \left. - f_1(s, u_{m-1}(s, w), v_{m-1}(s, w), w) \right\| ds \\ &\leq \frac{1}{\Gamma(\alpha_1)} \int_0^{\lambda T} \left\| f_1(s, u_{n-1}(s, w), v_{n-1}(s, w), w) - f_1(s, u_{m-1}(s, w), v_{m-1}(s, w), w) \right\|_{C_{\gamma_1}} \\ &\quad \times (t-s)^{\alpha_1-1} ds. \end{aligned}$$

Thus, from (3), we get

$$\begin{aligned} V^{(n,m)}(t, w) &\leq \frac{1}{\Gamma(\alpha_1)} \int_0^{\lambda T} g_1(s, \|u_{n-1}(s, w) - u_{m-1}(s, w)\|_{C_{\gamma_1}}, \|v_{n-1}(s, w) - v_{m-1}(s, w)\|_{C_{\gamma_2}}, w) \\ &\quad (t-s)^{\alpha_1-1} ds \\ &= \frac{1}{\Gamma(\alpha_1)} \int_0^{\lambda T} g_1(s, V^{(n-1,m-1)}(s, w), W^{(n-1,m-1)}(s, w), w) (t-s)^{\alpha_1-1} ds. \end{aligned}$$

Hence

$$V_k(t, w) \leq \frac{1}{\Gamma(\alpha_1)} \int_0^{\lambda T} g_1(s, V_{k-1}(s, w), W_{k-1}(s, w), w) (t-s)^{\alpha_1-1} ds.$$

By the Lebesgue dominated convergence theorem, we get

$$V(t, w) \leq \frac{1}{\Gamma(\alpha_1)} \int_0^{\lambda T} g_1(s, V(s, w), W(s, w), w) (t-s)^{\alpha_1-1} ds.$$

In addition, we find that

$$W(t, w) \leq \frac{1}{\Gamma(\alpha_2)} \int_0^{\lambda T} g_2(s, V(s, w), W(s, w), w) (t-s)^{\alpha_2-1} ds.$$

Then, from (H_1) and (H_3) , we get $V \equiv 0$ and $W \equiv 0$ on $I_\lambda \times \Omega$, which yields that $\lim_{k \rightarrow \infty} (V_k(t, w), W_k(t, w)) = (0, 0)$ uniformly on $I_\lambda \times \Omega$. Thus $\{(u_k(t, w), v_k(t, w))\}_{k=1}^\infty$ is a Cauchy sequence on $I_\lambda \times \Omega$. Consequently $\{(u_k(t, w), v_k(t, w))\}_{k=1}^\infty$ is uniformly convergent on I_λ which yields the contradiction.

Thus, $\{(u_k(t, w), v_k(t, w))\}_{k=1}^{\infty}$ converges uniformly on I for any $w \in \Omega$ to a continuous function $(u_*(t, w), v_*(t, w))$. By the Lebesgue dominated convergence theorem, we get

$$\begin{aligned} & \lim_{k \rightarrow \infty} \frac{\phi_1(w)}{\Gamma(\gamma_1)} t^{\gamma_1-1} + \int_0^t (t-s)^{\alpha_1-1} \frac{f(s, u_k(s, w), v_k(s, w), w)}{\Gamma(\alpha_1)} ds \\ &= \frac{\phi_1(w)}{\Gamma(\gamma_1)} t^{\gamma_1-1} + \int_0^t (t-s)^{\alpha_1-1} \frac{f(s, u_*(s, w), v_*(s, w), w)}{\Gamma(\alpha_1)} ds, \end{aligned}$$

and

$$\begin{aligned} & \lim_{k \rightarrow \infty} \frac{\phi_2(w)}{\Gamma(\gamma_2)} t^{\gamma_2-1} + \int_0^t (t-s)^{\alpha_2-1} \frac{f(s, u_k(s, w), v_k(s, w), w)}{\Gamma(\alpha_2)} ds \\ &= \frac{\phi_2(w)}{\Gamma(\gamma_2)} t^{\gamma_2-1} + \int_0^t (t-s)^{\alpha_2-1} \frac{f(s, u_*(s, w), v_*(s, w), w)}{\Gamma(\alpha_2)} ds, \end{aligned}$$

for each $t \in I$. This yields that (u_*, v_*) is a solution of the problem (1) and (2). $\square$

Finally, we show the uniqueness of solutions of the problem (1) and (2). Let (u_1, v_1) and (u_2, v_2) be two solutions. As above, put

$$\tau := \sup\{\eta \in [0, 1] : u_1(t, w) = u_2(t, w), v_1(t, w) = v_2(t, w) \text{ for } t \in I_\eta, \text{ and } w \in \Omega\},$$

and suppose that $\tau < 1$. There exist a constant $\rho > 0$ and a comparison function $g_i : I \times [0, \rho]^m \times [0, \rho]^m \times \Omega \rightarrow \mathbb{R}_+$; $i = 1, 2$, satisfying inequality (3). We choose $\lambda \in (\sigma, 1)$, such that

$$\begin{aligned} & \|u_1(\cdot, w) - u_2(\cdot, w)\|_{C_{\gamma_1}} \leq \rho \text{ and } \|v_1(\cdot, w) - v_2(\cdot, w)\|_{C_{\gamma_2}} \leq \rho; \\ & \|u_1(\cdot, w) - u_2(\cdot, w)\|_{C_{\gamma_1}} \\ & \leq \frac{1}{\Gamma(\alpha_1)} \int_0^{\lambda T} \left\| f_1(s, u_0(s, w), v_0(s, w), w) - f_1(s, u_1(s, w), v_1(s, w), w) \right\|_{C_{\gamma_1}} (t-s)^{\alpha_1-1} ds \\ & \leq \frac{1}{\Gamma(\alpha_1)} \int_0^{\lambda T} g_1(s, \|u_0(s, w) - u_1(s, w)\|_{C_{\gamma_1}}, \|v_0(s, w) - v_1(s, w)\|_{C_{\gamma_2}}, w) (t-s)^{\alpha_1-1} ds. \end{aligned}$$

and

$$\begin{aligned} & \|v_1(\cdot, w) - v_2(\cdot, w)\|_{C_{\gamma_2}} \\ & \leq \frac{1}{\Gamma(\alpha_2)} \int_0^{\lambda T} \left\| f_2(s, u_0(s, w), v_0(s, w), w) - f_2(s, u_1(s, w), v_1(s, w), w) \right\|_{C_{\gamma_2}} (t-s)^{\alpha_2-1} ds \\ & \leq \frac{1}{\Gamma(\alpha_2)} \int_0^{\lambda T} g_2(s, \|u_0(s, w) - u_1(s, w)\|_{C_{\gamma_1}}, \|v_0(s, w) - v_1(s, w)\|_{C_{\gamma_2}}, w) (t-s)^{\alpha_2-1} ds. \end{aligned}$$

Again, by (H_1) and (H_3) , we get $u_1 - u_2 \equiv 0$ and $v_1 - v_2 \equiv 0$ on $I_\lambda \times \Omega$. This gives $u_1 = u_2$ and $v_1 = v_2$ on $I_\lambda \times \Omega$, which yields a contradiction. Consequently, $\tau = 1$ and the solution of the problem (1) and (2) is unique.

4 An example

We equip the space $\mathbb{R}_-^* := (-\infty, 0)$ with the usual σ -algebra consisting of Lebesgue measurable subsets of $\mathbb{R}_-^*$. Consider the following random coupled Hilfer fractional differential system:

$$\begin{cases} (D_0^{\frac{1}{2}, \frac{1}{2}} u)(t, w) = f_1(t, u(t, w), v(t, w), w) \\ (D_0^{\frac{1}{2}, \frac{1}{2}} v)(t, w) = f_2(t, u(t, w), v(t, w), w) \\ (I_0^{\frac{1}{4}} u)(0, w) = 2 \sin w \\ (I_0^{\frac{1}{4}} v)(0, w) = 2 \cos w, \end{cases} \quad ; t \in [0, 1], w \in \mathbb{R}_-^*, \quad (6)$$

where

$$\begin{aligned} f_1(t, u, v, w) &= \frac{w^2 \sin t}{(2 + w^2)(1 + |u| + |v|)}; t \in [0, 1], w \in \mathbb{R}_+^*, \\ f_2(t, u, v, w) &= \frac{w^2 \cos t}{(2 + w^2)(1 + |u| + |v|)}; t \in [0, 1], w \in \mathbb{R}_+^*, \\ \text{with } \alpha_i &= \beta_i = \frac{1}{2}; i = 1, 2, \text{ and } \gamma_i = \frac{3}{4}; i = 1, 2. \end{aligned}$$

For each $u, v, \bar{u}, \bar{v} \in \mathbb{R}$, $p \in \mathbb{N}^*$ and $t \in [0, 1]$ we have

$$\begin{aligned} & \left\| f_1(s, u, v, w) - f_1(s, \bar{u}, \bar{v}, w) \right\|_{C_{\frac{3}{4}}} \\ &= \left\| t^{\frac{1}{4}} \left(\frac{w^2 \sin t}{(2 + w^2)(1 + |u| + |v|)} - \frac{w^2 \sin t}{(2 + w^2)(1 + |\bar{u}| + |\bar{v}|)} \right) \right\| \\ &\leq \left\| \frac{t^{\frac{1}{4}} w^2 \sin t}{(2 + w^2)} \left(\frac{1}{1 + |u| + |v|} - \frac{1}{1 + |\bar{u}| + |\bar{v}|} \right) \right\| \\ &\leq \frac{w^2 t^{\frac{1}{4}}}{(2 + w^2)} \left\| \frac{(|\bar{u}| - |u|) + (|\bar{v}| - |v|)}{(1 + |u| + |v|)(1 + |\bar{u}| + |\bar{v}|)} \right\| \\ &\leq \frac{w^2}{2 + w^2} \left(\|u - \bar{u}\|_{C_{\frac{1}{4}}} + \|v - \bar{v}\|_{C_{\frac{1}{4}}} \right). \end{aligned}$$

In addition, we obtain

$$\left\| f_2(s, u, v, w) - f_2(s, \bar{u}, \bar{v}, w) \right\|_{C_{\frac{3}{4}}} \leq \frac{w^2}{2 + w^2} \left(\|u - \bar{u}\|_{C_{\frac{1}{4}}} + \|v - \bar{v}\|_{C_{\frac{1}{4}}} \right).$$

This means that condition (3) holds for $t \in [0, 1]$, $\rho > 0$ and the comparison functions $g_i : [0, 1] \times [0, \rho] \rightarrow [0, \infty)$; $i = 1, 2$ given by

$$g_i(t, u, v, w) = \frac{w^2}{2 + w^2}(u + v); i = 1, 2.$$

Consequently, Theorem 3.3 implies that the successive approximations (u_n, v_n) ; $n \in \mathbb{N}$, defined by

$$\begin{aligned} (u_0(t), v_0(t)) &= (2 \sin w, 2 \cos w); t \in I, \\ (u_{n+1}(t), v_{n+1}(t)) &= \left((N_1(u_n, v_n))(t, w), (N_2(u_n, v_n))(t, w) \right); t \in I, \end{aligned}$$

where

$$(L_1(u, v))(t, w) = \frac{2 \sin w}{\Gamma(\frac{3}{4})} t^{\frac{3}{4}-1} + \int_0^t (t-s)^{\frac{1}{2}-1} \frac{f_1(s, u(s, w), v(s, w), w)}{\Gamma(\frac{1}{2})} ds,$$

and

$$(L_2(u, v))(t, w) = \frac{2 \cos w}{\Gamma(\frac{3}{4})} t^{\frac{3}{4}-1} + \int_0^t (t-s)^{\frac{1}{2}-1} \frac{f_2(s, u(s, w), v(s, w), w)}{\Gamma(\frac{1}{2})} ds,$$

converge uniformly on $[0, 1]$ to the unique solution of the problem (6).

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