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Representations of the necklace braid group $\mathcal{NB}_n$ of dimension 4 ($n = 2, 3, 4$)

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Abstract We consider the irreducible representations each of dimension 2 of the necklace braid group $\mathcal{NB}_n$ ($n = 2, 3, 4$). We then consider the tensor product of the representations of $\mathcal{NB}_n$ ($n = 2, 3, 4$) and determine necessary and sufficient condition under which the constructed representations are irreducible. Finally, we determine conditions under which the irreducible representations of $\mathcal{NB}_n$ ($n = 2, 3, 4$) of degree 2 are unitary relative to a hermitian positive definite matrix.

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1 Introduction

Bullivant et al. [4] studied representations of the necklace braid group $\mathcal{NB}_n$, especially those obtained as extensions of representations of the braid group B_n and the loop braid group LB_n (see [3, 5]). They showed that any irreducible B_n representation extends to $\mathcal{NB}_n$ in a standard way. Moreover, they proved that any local representation of B_n , coming from a braided vector space, can be extended to $\mathcal{NB}_n$.

A link $\mathcal{L}_n = K_0 \cup K_1 \cup \dots \cup K_n$ is called a necklace if:

- K_0 is an Euclidean circle of center O and of radius 1
- each K_i is an Euclidean circle whose center O_i belongs to K_0 and radius r_i such that $0 < r_i < \frac{1}{2}$ for $0 < i \leq n$
- the plane of each K_i is the one containing the line (OO_i) and perpendicular to the plane of K_0 .
- If $O_i = O_j$ then $r_i \neq r_j$

The motion group $\mathcal{NB}_n$, the necklace braid group, as described in [1] is identified with the fundamental group of the configuration space $\mathcal{L}_n$. The group $\mathcal{NB}_n$ is generated by the elements $\sigma_1, \dots, \sigma_n$ and τ where σ_i is the motion, up to homotopy, of passing the i th circle through the $(i + 1)$ th circle, while τ corresponds to shifting each circle one position in the counterclockwise direction.

In Sect. 2, we consider the irreducible representations defined on the necklace braid group $\mathcal{NB}_n$ ($n = 2, 3, 4$) each of dimension 2. In Sect. 3, we construct the tensor product of representations on $\mathcal{NB}_n$. Then, we discuss the irreducibility of the tensor product of the representations of $\mathcal{NB}_n$ ($n = 2, 3, 4$). Theorem 3.5 gives necessary and sufficient condition for the irreducibility of tensor product of the representations of $\mathcal{NB}_4$. Theorem 3.17 provides necessary and sufficient conditions under which the tensor product of the representations of $\mathcal{NB}_3$ is

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Fig. 1 Necklace $\mathcal{L}_n$

irreducible. Theorem 3.18 gives necessary and sufficient condition for the irreducibility of the tensor product of the representations of $\mathcal{NB}_2$. In Sect. 4, we prove that the irreducible representations of dimension 2 of $\mathcal{NB}_n$ are unitary relative to hermitian positive definite matrix in the case $n = 3, 4$ (see Propositions 4.4, 4.5 and Proposition 4.6).

Definition 1.1 [1] The necklace braid group $\mathcal{NB}_n$ is identified with the fundamental group of the configuration space of $\mathcal{L}_n$.

The following theorem gives a presentation of the necklace braid group $\mathcal{NB}_n$ by generators and relations.

Theorem 1.2 [1] The necklace braid group $\mathcal{NB}_n$ has a presentation by generators $\sigma_1, \dots, \sigma_n, \tau$ and relations as follows:

- (B1) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for $1 \leq i \leq n$;
- (B2) $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| \neq 1$;
- (N1) $\tau \sigma_i \tau^{-1} = \sigma_{i+1}$ for $1 \leq i \leq n \pmod{n}$;
- (N2) $\tau^{2n} = 1$

Here, indices are taken modulo n , with $\sigma_{n+1} = \sigma_1$ and $\sigma_0 = \sigma_n$. The relations (B1) and (B2) are those for the braid group B_n (see [2]).

2 Irreducible representations of dimension 2 of $\mathcal{NB}_n$ ($n = 2, 3, 4$)

Consider the necklace braid group $\mathcal{NB}_n$ for $n = 2, 3, 4$. We have the following proposition (see [4]).

Proposition 2.1 Any irreducible representation of dimension 2 of $\mathcal{NB}_n$ ($n = 2, 3, 4$) is isomorphic to the representation $\rho = \rho(T, t, a, c, d)$ that is defined by:

$$\rho(\tau) = \begin{pmatrix} Tt & 0 \\ 0 & t \end{pmatrix} \text{ and } \rho(\sigma_i) = \begin{pmatrix} a & T^{i-1} \\ cT^{1-i} & d \end{pmatrix},$$

with the following conditions:

n	T	Conditions
2	$T = -1$	$c = a^2 - ad + d^2, c \neq 0$ and $a \neq d$
3	$T = e^{\pm i 2\pi/3}$	$a = \omega d, c \neq 0, c \neq \omega d^2$ and $\omega = e^{\pm i \pi/3}$
4	$T = -1$	$c = a^2 - ad + d^2, c \neq 0$ and $a \neq d$
	$T = \pm i$	$c = -d^2, a = d \neq 0$

Here, t is any $(2n)^{\text{th}}$ root of unity.

Proof Let (ρ, V) be an irreducible representation of dimension 2 of $\mathcal{NB}_n$ (for $n = 2, 3, 4$). From the fact that τ has order $2n$, we may choose a basis for V such that $\rho(\tau) = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$, where t_1 and t_2 are $(2n)^{\text{th}}$ roots of unity. Since ρ is irreducible representation, it follows that $t_1 \neq t_2$. Similarly we have that $\rho(\sigma_1)$ is neither

upper nor lower triangular, because $(1, 0)$ or $(0, 1)$ would generate an invariant subspace. Due to rescaling, we may assume that $\rho(\sigma_1) = \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}$. Since $\rho(\sigma_1)$ is neither diagonal nor triangular, we have that $c \neq 0$. To require invertibility, we assume that $ad \neq c$.

Using $\rho(\sigma_1) = \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}$ and the condition (N1): $\tau\sigma_i\tau^{-1} = \sigma_{i+1}$, we get

$$\rho(\sigma_i) = \begin{pmatrix} a & T^{i-1} \\ cT^{1-i} & d \end{pmatrix},$$

where $T = t_1t_2^{-1}$. Note that $T \neq 1$ because $t_1 \neq t_2$. Set $t_2 = t$ then $t_1 = Tt$ and $\rho(\tau)$ has the form $\begin{pmatrix} Tt & 0 \\ 0 & t \end{pmatrix}$.

Since $\sigma_{n+1} = \sigma_1$, we have $T^n = T^{-n} = 1$. Thus, T is a primitive n th root of unity.

Now, we check the conditions (B1): $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$. By Lemma 1.2 in [4], it is sufficient to check (B1) for $i = 1$ (i.e. $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$).

By direct calculations, we get

$$\rho(\sigma_1)\rho(\sigma_2)\rho(\sigma_1) = \rho(\sigma_2)\rho(\sigma_1)\rho(\sigma_2) \Leftrightarrow (a^2 - ad + d^2)T + c(1 + T + T^2) = 0.$$

For $n = 2$, we have that $T = -1$ and $c = a^2 - ad + d^2$. Under the assumption that $ad \neq c$, we have $a \neq d$.

For $n = 3$, we have that $T = e^{\pm 2\pi i/3}$. Then, $a = \omega d$ where $\omega = e^{\pm \pi i/3}$. Under the assumption that $ad \neq c$, we have $c \neq \omega d^2$.

For $n = 4$, we have that $T = -1, \pm i$.

- If $T = -1$, then $c = a^2 - ad + d^2$ with $a \neq d$.
- If $T = \pm i$, then $c = -a^2 + ad - d^2$.

It remains to verify the condition (B2): $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i - j| > 1$ when $n = 4$. By Lemma 1.2 in [4] we just check (B2) for $i = 1$ and $j = 3$.

If $T = -1$, then it is clear that $\rho(\sigma_1)\rho(\sigma_3) = \rho(\sigma_3)\rho(\sigma_1)$.

If $T = \pm i$, then

$$\begin{aligned} \rho(\sigma_1)\rho(\sigma_3) = \rho(\sigma_3)\rho(\sigma_1) &\Leftrightarrow \begin{pmatrix} a & 1 \\ c & d \end{pmatrix} \begin{pmatrix} a & -1 \\ -c & d \end{pmatrix} = \begin{pmatrix} a & -1 \\ -c & d \end{pmatrix} \begin{pmatrix} a & 1 \\ c & d \end{pmatrix} \\ &\Leftrightarrow \begin{pmatrix} a^2 - c & -a + d \\ ca - dc & -c + d^2 \end{pmatrix} = \begin{pmatrix} a^2 - c & a - d \\ -ca + dc & -c + d^2 \end{pmatrix} \\ &\Leftrightarrow a = d \end{aligned}$$

It follows that $c = a^2 + ad - d^2 = -d^2$. □

3 Representations of dimension 4 of $\mathcal{NB}_n$ ($n = 2, 3, 4$)

Consider two irreducible representations $\rho_1 = \rho(T_1, t_1, a_1, c_1, d_1)$ and $\rho_2 = \rho(T_2, t_2, a_2, c_2, d_2)$ of $\mathcal{NB}_n$ ($n = 2, 3, 4$) each of dimension 2.

$$\begin{aligned} \rho_1(\tau) &= \begin{pmatrix} T_1t_1 & 0 \\ 0 & t_1 \end{pmatrix} \text{ and } \rho_1(\sigma_i) = \begin{pmatrix} a_1 & T_1^{i-1} \\ c_1T_1^{1-i} & d_1 \end{pmatrix}; \\ \rho_2(\tau) &= \begin{pmatrix} T_2t_2 & 0 \\ 0 & t_2 \end{pmatrix} \text{ and } \rho_2(\sigma_i) = \begin{pmatrix} a_2 & T_2^{i-1} \\ c_2T_2^{1-i} & d_2 \end{pmatrix}. \end{aligned}$$

Definition 3.1 Consider the tensor product $\rho_1 \otimes \rho_2$ given by $\rho_1 \otimes \rho_2(\alpha) = \rho_1(\alpha) \otimes \rho_2(\alpha)$, where α is a generator of $\mathcal{NB}_n$ ($n = 2, 3, 4$). We get the following matrices for the generators τ and σ_1 .

$$\rho(\tau) = t_1 t_2 \begin{pmatrix} T_1 T_2 & 0 & 0 & 0 \\ 0 & T_1 & 0 & 0 \\ 0 & 0 & T_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \rho(\sigma_1) = \begin{pmatrix} a_1 a_2 & a_1 & a_2 & 1 \\ a_1 c_2 & a_1 d_2 & c_2 & d_2 \\ c_1 a_2 & c_1 & d_1 a_2 & d_1 \\ c_1 c_2 & c_1 d_2 & d_1 c_2 & d_1 d_2 \end{pmatrix}.$$

3.1 Representations of $\mathcal{NB}_4$

In this section, we study the irreducibility of the representation ρ of the necklace braid group $\mathcal{NB}_4$. Actually, we have four cases: (3.1.1) $T_1 = T_2 = -1$, (3.1.2) $T_1 = T_2 = \pm i$, (3.1.3) $T_1 = -T_2 = \pm i$ and (3.1.4) $T_1 = -1, T_2 = \pm i$.

In what follows, suppose that $a_1 a_2 d_1 d_2 \neq 0$.

3.1.1 Case $T_1 = T_2 = -1$

Direct computations show that the representation ρ is given by

$$\rho(\tau) = t_1 t_2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \rho(\sigma_1) = \rho(\sigma_3) = \begin{pmatrix} a_1 a_2 & a_1 & a_2 & 1 \\ a_1 c_2 & a_1 d_2 & c_2 & d_2 \\ c_1 a_2 & c_1 & d_1 a_2 & d_1 \\ c_1 c_2 & c_1 d_2 & d_1 c_2 & d_1 d_2 \end{pmatrix}$$

and

$$\rho(\sigma_2) = \rho(\sigma_4) = \begin{pmatrix} a_1 a_2 & -a_1 & -a_2 & 1 \\ -a_1 c_2 & a_1 d_2 & c_2 & -d_2 \\ -c_1 a_2 & c_1 & d_1 a_2 & -d_1 \\ c_1 c_2 & -c_1 d_2 & -d_1 c_2 & d_1 d_2 \end{pmatrix}.$$

The eigenvalues of $\rho(\tau)$ are $\lambda_1 = t_1 t_2$ and $\lambda_2 = -t_1 t_2$. Both are of multiplicities 2. The corresponding eigenvectors are $\alpha_1 e_1 + \alpha_4 e_4 = (\alpha_1, 0, 0, \alpha_4)$ for $\lambda_1 = t_1 t_2$ and $\alpha_2 e_2 + \alpha_3 e_3 = (0, \alpha_2, \alpha_3, 0)$ for $\lambda_2 = -t_1 t_2$, where $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \subset \mathbb{C}$.

We now determine conditions under which the representation ρ is irreducible.

Proposition 3.2 *The representation $\rho : \mathcal{NB}_4 \rightarrow GL(4, \mathbb{C})$ has no non-trivial proper invariant subspaces of dimension 1 if and only if*

$$a_1 a_2 \neq d_1 d_2 \quad \text{and} \quad a_1 d_2 \neq a_2 d_1.$$

Proof The subspaces of dimension 1 that are invariant under $\rho(\tau)$ are those spanned by one of the following vectors: $e_1, e_2, e_3, e_4, e_1 + y e_4$ and $e_2 + x e_3$ for $x \neq 0$ and $y \neq 0$.

$$\rho(\sigma_1)(e_1) = \begin{pmatrix} a_1 a_2 \\ a_1 c_2 \\ c_1 a_2 \\ c_1 c_2 \end{pmatrix} \notin \langle e_1 \rangle \text{ because, } c_1 c_2 \neq 0. \text{ So, } \langle e_1 \rangle \text{ is not invariant.}$$

$$\rho(\sigma_1)(e_2) = \begin{pmatrix} a_1 \\ a_1 d_2 \\ c_1 \\ c_1 d_2 \end{pmatrix} \notin \langle e_2 \rangle \text{ because, } c_1 \neq 0. \text{ So, } \langle e_2 \rangle \text{ is not invariant.}$$

$$\rho(\sigma_1)(e_3) = \begin{pmatrix} a_2 \\ c_2 \\ d_1 a_2 \\ d_1 c_2 \end{pmatrix} \notin \langle e_3 \rangle \text{ because, } c_2 \neq 0. \text{ Thus, } \langle e_3 \rangle \text{ is not invariant.}$$

$$\rho(\sigma_1)(e_4) = \begin{pmatrix} 1 \\ d_2 \\ d_1 \\ d_1 d_2 \end{pmatrix} \notin \langle e_4 \rangle. \text{ So, } \langle e_4 \rangle \text{ is not invariant.}$$

Now consider the subspaces of the form $\langle e_1 + ye_4 \rangle$ with $y \neq 0$

$$\text{Since } \rho(\sigma_i)(e_1 + ye_4) = \begin{pmatrix} a_1 a_2 + y \\ \pm a_1 c_2 \pm d_2 y \\ \pm c_1 a_2 \pm d_1 y \\ c_1 c_2 + d_1 d_2 y \end{pmatrix}, \text{ it follows that } \langle e_1 + ye_4 \rangle \text{ is invariant under } \rho(\sigma_i) \text{ for } 1 \leq i \leq n.$$

$$\text{It follows that } \begin{pmatrix} a_1 a_2 + y \\ \pm a_1 c_2 \pm d_2 y \\ \pm c_1 a_2 \pm d_1 y \\ c_1 c_2 + d_1 d_2 y \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \\ 0 \\ \alpha y \end{pmatrix} \text{ for some } \alpha \in \mathbb{C} \setminus \{0\}.$$

This is equivalent to the system:

$$\begin{aligned} & \begin{cases} (a_1 a_2 + y)y = c_1 c_2 + d_1 d_2 y \\ a_1 c_2 + d_2 y = 0 \\ c_1 a_2 + d_1 y = 0 \end{cases} \\ \Leftrightarrow & \begin{cases} y^2 + (a_1 a_2 - d_1 d_2)y - c_1 c_2 = 0 \\ y = -\frac{a_1 c_2}{d_2} \\ y = -\frac{c_1 a_2}{d_1} \end{cases} \\ \Leftrightarrow & \begin{cases} \frac{a_1^2 c_2^2}{d_2^2} - \frac{a_1^2 a_2 c_2}{d_2} + d_1 a_1 c_2 - c_1 c_2 = 0 \\ -\frac{a_1 c_2}{d_2} = -\frac{c_1 a_2}{d_1} \end{cases} \\ \Leftrightarrow & \begin{cases} a_1^2 c_2 - a_1^2 a_2 d_2 + d_1 a_1 d_2^2 - c_1 d_2^2 = 0 \\ a_1 d_1 c_2 = a_2 d_2 c_1 \end{cases} \\ \Leftrightarrow & \begin{cases} a_1^2 (c_2 - a_2 d_2) - d_2^2 (c_1 - a_1 d_1) = 0 \\ a_1 d_1 (a_2^2 - a_2 d_2 + d_2^2) = a_2 d_2 (a_1^2 - a_1 d_1 + d_1^2) \end{cases} \\ \Leftrightarrow & \begin{cases} a_1^2 (a_2 - d_2)^2 - d_2^2 (a_1 - d_1)^2 = 0 \\ a_1 d_1 a_2^2 - a_1 d_1 a_2 d_2 + a_1 d_1 d_2^2 = a_2 d_2 a_1^2 - a_2 d_2 a_1 d_1 + a_2 d_2 d_1^2 \end{cases} \\ \Leftrightarrow & \begin{cases} a_1^2 (a_2 - d_2)^2 - d_2^2 (a_1 - d_1)^2 = 0 \\ a_1 d_1 a_2^2 - 2a_1 d_1 a_2 d_2 + a_1 d_1 d_2^2 = a_2 d_2 a_1^2 - 2a_2 d_2 a_1 d_1 + a_2 d_2 d_1^2 \end{cases} \\ \Leftrightarrow & \begin{cases} a_1^2 (a_2 - d_2)^2 - d_2^2 (a_1 - d_1)^2 = 0 \\ a_1 d_1 (a_2^2 - 2a_2 d_2 + d_2^2) = a_2 d_2 (a_1^2 - 2a_1 d_1 + d_1^2) \end{cases} \\ \Leftrightarrow & \begin{cases} a_1^2 (a_2 - d_2)^2 = d_2^2 (a_1 - d_1)^2 \\ a_1 d_1 (a_2 - d_2)^2 = a_2 d_2 (a_1 - d_1)^2 \end{cases} \\ \Leftrightarrow & \begin{cases} \frac{a_1^2}{d_2^2} = \frac{(a_1 - d_1)^2}{(a_2 - d_2)^2} \\ \frac{a_1 d_1}{a_2 d_2} = \frac{(a_1 - d_1)^2}{(a_2 - d_2)^2} \end{cases} \\ \Leftrightarrow & \frac{a_1}{d_2} = \frac{d_1}{a_2} \\ \Leftrightarrow & a_1 a_2 = d_1 d_2. \end{aligned}$$

We consider the subspaces of the form $\langle e_2 + xe_3 \rangle$ with $x \neq 0$.

$$\rho(\sigma_j)(e_2 + xe_3) = \begin{pmatrix} \pm a_1 \pm a_2x \\ a_1d_2 + c_2x \\ c_1 + d_1a_2x \\ \pm c_1d_2 \pm d_1c_2x \end{pmatrix} \text{ for } 1 \leq j \leq 4.$$

If we assume that the subspace $\langle e_2 + xe_3 \rangle$ is invariant, then $\begin{pmatrix} a_1 + a_2x \\ a_1d_2 + c_2x \\ c_1 + d_1a_2x \\ c_1d_2 + d_1c_2x \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha \\ \alpha x \\ 0 \end{pmatrix}$ for some $\alpha \in \mathbb{C}^*$.

This is equivalent to:

$$\begin{aligned} & \begin{cases} a_1 + a_2x = 0, \\ c_1d_2 + d_1c_2x = 0, \\ a_1d_2 + c_2x = \alpha \text{ and} \\ c_1 + d_1a_2x = \alpha x \end{cases} \\ \iff & x = -\frac{a_1}{a_2}, \quad x = -\frac{c_1d_2}{d_1c_2}, \quad \text{and} \quad c_2x^2 + (a_1d_2 - d_1a_2)x - c_1 = 0 \\ \iff & c_2\frac{a_1^2}{a_2^2} - \frac{a_1^2d_2}{a_2} + d_1a_1 - c_1 = 0 \quad \text{and} \quad a_1c_2d_1 = a_2c_1d_2 \\ \iff & c_2a_1^2 - a_1^2d_2a_2 + d_1a_1a_2^2 - c_1a_2^2 = 0 \quad \text{and} \quad a_1d_1(a_2^2 - a_2d_2 + d_2^2) = a_2d_2(a_1^2 - a_1d_1 + d_1^2) \\ \iff & a_1^2(c_2 - d_2a_2) = a_2^2(-d_1a_1 + c_1) \quad \text{and} \quad a_1d_1(a_2^2 - 2a_2d_2 + d_2^2) = a_2d_2(a_1^2 - 2a_1d_1 + d_1^2) \\ \iff & a_1^2(a_2 - d_2)^2 = a_2^2(a_1 - d_1)^2 \quad \text{and} \quad a_1d_1(a_2 - d_2)^2 = a_2d_2(a_1 - d_1)^2 \\ \iff & \frac{a_1^2}{a_2^2} = \frac{(a_1 - d_1)^2}{(a_2 - d_2)^2} = \frac{a_1d_1}{a_2d_2} \\ \iff & \frac{a_1}{a_2} = \frac{d_1}{d_2} \\ \iff & a_1d_2 = a_2d_1. \end{aligned}$$

□

Proposition 3.3 *The representation $\rho : \mathcal{NB}_4 \rightarrow GL(4, \mathbb{C})$ has no non-trivial proper invariant subspaces of dimension 2 if and only if*

$$a_1a_2 \neq d_1d_2 \text{ and } a_1d_2 \neq a_2d_1.$$

Proof The subspaces of dimension 2 that are possibly invariant are those spanned by the following subsets of vectors: $\{e_1, he_2 + je_3\}$, $\{e_4, re_2 + se_3\}$, $\{e_2, ke_1 + me_4\}$, $\{e_3, pe_1 + qe_4\}$, $\{e_1 + xe_4, e_2 + ye_3\}$, where $\{h, j, r, s, k, m, p, q, x, y\} \subset \mathbb{C}$.

$$\rho(\sigma_1)(e_1) = \begin{pmatrix} a_1a_2 \\ a_1c_2 \\ c_1a_2 \\ c_1c_2 \end{pmatrix} \notin \langle e_1, he_2 + je_3 \rangle \text{ because } c_1c_2 \neq 0. \text{ So } \langle e_1, he_2 + je_3 \rangle \text{ are not invariant subspaces}$$

for all $h, j \in \mathbb{C}$.

$$\rho(\sigma_1)(e_4) = \begin{pmatrix} 1 \\ d_2 \\ d_1 \\ d_1d_2 \end{pmatrix} \notin \langle e_4, re_2 + se_3 \rangle. \text{ So, } \langle e_4, re_2 + se_3 \rangle \text{ are not invariant subspaces for all } r, s \in \mathbb{C}.$$

$$\rho(\sigma_1)(e_2) = \begin{pmatrix} a_1 \\ a_1d_2 \\ c_1 \\ c_1d_2 \end{pmatrix} \notin \langle e_2, ke_1 + me_4 \rangle \text{ because } c_1 \neq 0. \text{ So, the subspaces } \langle e_2, ke_1 + me_4 \rangle \text{ are not invariant}$$

for all $k, m \in \mathbb{C}$.

$\rho(\sigma_1)(e_3) = \begin{pmatrix} a_2 \\ c_2 \\ d_1 a_2 \\ d_1 c_2 \end{pmatrix} \notin \langle e_3, pe_1 + qe_4 \rangle$ because $c_2 \neq 0$. So, the subspaces $\langle e_3, pe_1 + qe_4 \rangle$ are not invariant for all $p, q \in \mathbb{C}$.

Now consider the subspaces of the form $\langle e_1 + xe_4, e_2 + ye_3 \rangle$ where $x, y \in \mathbb{C}^*$. Put $u = e_1 + xe_4$ and $v = e_2 + ye_3$ then

$$\rho(\sigma_j)(u) = \begin{pmatrix} a_1 a_2 + x \\ \pm a_1 c_2 \pm d_2 x \\ \pm c_1 a_2 \pm d_1 x \\ c_1 c_2 + d_1 d_2 x \end{pmatrix} \text{ and } \rho(\sigma_j)(v) = \begin{pmatrix} \pm a_1 \pm a_2 y \\ a_1 d_2 + c_2 y \\ c_1 + d_1 a_2 y \\ \pm c_1 d_2 \pm d_1 c_2 y \end{pmatrix}.$$

The subspace $\langle u, v \rangle$ is invariant if and only if $\rho(\sigma_j)(u) = \alpha u + \alpha' v$ and $\rho(\sigma_j)(v) = \beta u + \beta' v$ for some $\alpha, \alpha', \beta, \beta' \in \mathbb{C}$.

$$\text{This is equivalent to } \begin{cases} a_1 a_2 + x = \alpha \\ \pm(a_1 c_2 + d_2 x) = \alpha' \\ \pm(c_1 a_2 + d_1 x) = \alpha' y \\ c_1 c_2 + d_1 d_2 x = \alpha x \end{cases} \text{ and } \begin{cases} \pm(a_1 + a_2 y) = \beta \\ a_1 d_2 + c_2 y = \beta' \\ c_1 + d_1 a_2 y = \beta' y \\ \pm(c_1 d_2 + d_1 c_2 y) = \beta x \end{cases}.$$

By eliminating α, α', β and β' , we get the following system.

$$c_1 c_2 + d_1 d_2 x = a_1 a_2 x + x^2, \quad (3.1a)$$

$$c_1 a_2 + d_1 x = a_1 c_2 y + d_2 x y, \quad (3.1b)$$

$$c_1 d_2 + d_1 c_2 y = a_1 x + a_2 x y, \quad (3.1c)$$

$$c_1 + d_1 a_2 y = a_1 d_2 y + c_2 y^2. \quad (3.1d)$$

The Eqs. (3.1a) and (3.1d) lead to four solutions:

$$x = \frac{-a_1 a_2 + d_1 d_2 \pm \sqrt{4c_1 c_2 + (a_1 a_2 - d_1 d_2)^2}}{2}; \quad y = \frac{a_2 d_1 - a_1 d_2 \pm \sqrt{4c_1 c_2 + (a_2 d_1 - a_1 d_2)^2}}{2c_2}.$$

Substitute each of these solutions into the Eqs. (3.1b) and (3.1c) then, using Mathematica, we get the following relations:

$(a_2 = d_2), (a_1 = d_1), (d_1 = e^{\pm\pi i/3} a_1), (d_1 d_2 = a_1 a_2), (d_1 a_2 = a_1 d_2), (a_1 = a_2 = 0), (a_1 = d_2 = 0), (d_1 = a_2 = 0), (d_1 = d_2 = 0), (d_1 = a_1, a_2 = 0), (d_1 = a_1, d_2 = 0), (d_1 = e^{\pm\pi i/3} a_1, a_2 = 0)$ or $(d_1 = e^{\pm\pi i/3} a_1, d_2 = 0)$

But we have $a_1 \neq d_1, a_2 \neq d_2, a_1 a_2 \neq d_1 d_2, a_1 d_2 \neq a_2 d_1$ and $a_1 d_1 a_2 d_2 \neq 0$. So, the only 2 relations left are $d_1 = e^{\pm\pi i/3} a_1$ which lead to $c_1 = a_1^2 - a_1 d_1 + d_1^2 = 0$. This contradicts the fact that $c_1 \neq 0$.

Therefore $\langle u, v \rangle$ is not invariant under $\rho(\sigma_j)$ for $1 \leq j \leq 4$. $\square$

Proposition 3.4 $\rho : \mathcal{NB}_4 \rightarrow GL(4, \mathbb{C})$ has no non-trivial proper invariant subspaces of dimension 3 if and only if

$$a_1 a_2 \neq d_1 d_2 \text{ and } a_1 d_2 \neq a_2 d_1.$$

Proof The subspaces of dimension 3 that are possibly invariant are those spanned by the following subsets of vectors: $\{e_1, e_4, e_3\}, \{e_1, e_4, e_2\}, \{e_1, e_2, e_3\}, \{e_4, e_2, e_3\}, \{e_1, e_4, e_2 + xe_3\}, \{e_1 + ye_4, e_2, e_3\}$, where $x, y \in \mathbb{C}^*$.

$$\rho(\sigma_1)(e_3) = \begin{pmatrix} a_2 \\ c_2 \\ d_1 a_2 \\ d_1 c_2 \end{pmatrix} \notin \langle e_1, e_4, e_3 \rangle \text{ since } c_2 \neq 0. \text{ Thus, } \langle e_1, e_4, e_3 \rangle \text{ is not invariant.}$$

$$\rho(\sigma_1)(e_2) = \begin{pmatrix} a_1 \\ a_1 d_2 \\ c_1 \\ c_1 d_2 \end{pmatrix} \notin \langle e_1, e_4, e_2 \rangle \text{ since } c_1 \neq 0. \text{ So, } \langle e_1, e_4, e_2 \rangle \text{ is not invariant}$$

$$\rho(\sigma_1)(e_1) = \begin{pmatrix} a_1 a_2 \\ a_1 c_2 \\ c_1 a_2 \\ c_1 c_2 \end{pmatrix} \notin \langle e_1, e_2, e_3 \rangle \text{ since } c_1 c_2 \neq 0. \text{ So, } \langle e_1, e_2, e_3 \rangle \text{ is not invariant.}$$

$\rho(\sigma_1)(e_4) = \begin{pmatrix} 1 \\ d_2 \\ d_1 \\ d_1 d_2 \end{pmatrix} \notin \langle e_4, e_2, e_3 \rangle$. So, $\langle e_4, e_2, e_3 \rangle$ is not invariant.

Now, consider the subspace $\langle e_1, e_4, e_2 + x e_3 \rangle$ with $x \neq 0$.

Note that $\rho(\sigma_j)(e_1) = \begin{pmatrix} a_1 a_2 \\ \pm a_1 c_2 \\ \pm c_1 a_2 \\ c_1 c_2 \end{pmatrix}$, $\rho(\sigma_j)(e_4) = \begin{pmatrix} 1 \\ \pm d_2 \\ \pm d_1 \\ d_1 d_2 \end{pmatrix}$ and $\rho(\sigma_j)(e_2 + x e_3) = \begin{pmatrix} \pm a_1 \pm a_2 x \\ a_1 d_2 + c_2 x \\ c_1 + d_1 a_2 x \\ \pm c_1 d_2 \pm d_1 c_2 x \end{pmatrix}$.

Assume that the subspace $\langle e_1, e_4, e_2 + x e_3 \rangle$ is invariant. It follows that $\{\rho(\sigma_1)(e_1), \rho(\sigma_1)(e_4), \rho(\sigma_1)(e_2 + x e_3)\} \subset \langle e_1, e_4, e_2 + x e_3 \rangle$. This is equivalent to the following system.

$$c_1 a_2 = a_1 c_2 x, \quad (3.2a)$$

$$d_1 = d_2 x, \quad (3.2b)$$

$$c_2 x^2 + (a_1 d_2 - d_1 a_2) x - c_1 = 0. \quad (3.2c)$$

We determine conditions for which the system above is consistent.

From Eqs. (3.2a) and (3.2b), we get $x = \frac{c_1 a_2}{a_1 c_2} = \frac{d_1}{d_2}$ then substitute it into the Eq. (3.2c) to obtain $c_2 \frac{d_1^2}{d_2^2} +$

$$a_1 d_1 - \frac{d_1^2 a_2}{d_2} - c_1 = 0.$$

$$\implies a_2 d_2 c_1 = a_1 d_1 c_2 \text{ and } c_2 d_1^2 - a_2 d_2 d_1^2 = d_2^2 c_1 - d_2^2 a_1 d_1$$

$$\implies a_2 d_2 (a_1^2 - a_1 d_1 + d_1^2) = a_1 d_1 (a_2^2 - a_2 d_2 + d_2^2) \text{ and } d_1^2 (c_2 - a_2 d_2) = d_2^2 (c_1 - a_1 d_1)$$

$$\implies a_2 d_2 (a_1^2 - 2a_1 d_1 + d_1^2) = a_1 d_1 (a_2^2 - 2a_2 d_2 + d_2^2) \text{ and } d_1^2 (a_2 - d_2)^2 = d_2^2 (a_1 - d_1)^2$$

$$\implies \frac{a_2 d_2}{a_1 d_1} = \frac{(a_2 - d_2)^2}{(a_1 - d_1)^2} = \frac{d_2^2}{d_1^2}$$

$$\implies \frac{a_2}{a_1} = \frac{d_2}{d_1}$$

$$\implies a_2 d_1 = a_1 d_2.$$

This gives a contradiction.

Now, consider the subspace $\langle e_1 + y e_4, e_2, e_3 \rangle$ with $y \neq 0$. Note that $\rho(\sigma_j)(e_2) = \begin{pmatrix} \pm a_1 \\ a_1 d_2 \\ c_1 \\ \pm c_1 d_2 \end{pmatrix}$, $\rho(\sigma_j)(e_3) =$

$$\begin{pmatrix} \pm a_2 \\ c_2 \\ d_1 a_2 \\ \pm d_1 c_2 \end{pmatrix} \text{ and } \rho(\sigma_j)(e_1 + y e_4) = \begin{pmatrix} a_1 a_2 + y \\ \pm a_1 c_2 \pm d_2 y \\ \pm c_1 a_2 \pm d_1 y \\ c_1 c_2 + d_1 d_2 y \end{pmatrix}.$$

Assume that the subspace $\langle e_1 + y e_4, e_2, e_3 \rangle$ is invariant. It follows that the vectors $\rho(\sigma_i)(e_2)$, $\rho(\sigma_i)(e_3)$ and $\rho(\sigma_i)(e_1 + y e_4)$ are linear combinations of $e_1 + y e_4$, e_2 , and e_3 .

$$\implies \frac{c_1 d_2}{a_1} = \frac{d_1 c_2}{a_2} = y \text{ and } y^2 + (a_1 a_2 - d_1 d_2) y - c_1 c_2 = 0$$

$$\implies a_2 d_2 c_1 = a_1 d_1 c_2 \text{ and } \frac{c_1^2 d_2^2}{a_1^2} + a_2 d_2 c_1 - \frac{c_1 d_1 d_2^2}{a_1} - c_1 c_2 = 0$$

$$\implies a_2 d_2 (a_1^2 - a_1 d_1 + d_1^2) = a_1 d_1 (a_2^2 - a_2 d_2 + d_2^2) \text{ and } c_1 d_2^2 + a_2 d_2 a_1^2 - d_1 d_2^2 a_1 - c_2 a_1^2 = 0$$

$$\implies a_2 d_2 (a_1^2 - 2a_1 d_1 + d_1^2) = a_1 d_1 (a_2^2 - 2a_2 d_2 + d_2^2) \text{ and } d_2^2 (c_1 - d_1 a_1) = a_1^2 (c_2 - a_2 d_2)$$

$$\implies a_2 d_2 (a_1 - d_1)^2 = a_1 d_1 (a_2 - d_2)^2 \text{ and } d_2^2 (a_1 - d_1)^2 = a_1^2 (a_2 - d_2)^2$$

$$\implies \frac{a_2 d_2}{a_1 d_1} = \frac{d_2^2}{a_1^2} = \frac{(a_2 - d_2)^2}{(a_1 - d_1)^2}$$

$$\implies \frac{a_2}{d_1} = \frac{d_2}{a_1}.$$

This gives a contradiction. $\square$

We, thus, get our main theorem.

Theorem 3.5 *The representation $\rho : \mathcal{NB}_4 \rightarrow GL(4, \mathbb{C})$ is irreducible if and only if $a_1a_2 \neq d_1d_2$ and $a_1d_2 \neq a_2d_1$.*

3.1.2 Case $T_1 = T_2 = \pm i$

In the case $T_1 = T_2 = i$, the representation ρ of $\mathcal{NB}_4$ is given by

$$\begin{aligned} \rho(\tau) &= t_1 t_2 \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \rho(\sigma_1) = \begin{pmatrix} d_1 d_2 & d_1 & d_2 & 1 \\ -d_1 d_2^2 & d_1 d_2 & -d_2^2 & d_2 \\ -d_1^2 d_2 & -d_1^2 & d_1 d_2 & d_1 \\ d_1^2 d_2^2 & -d_1^2 d_2 & -d_1 d_2^2 & d_1 d_2 \end{pmatrix}, \\ \rho(\sigma_2) &= \begin{pmatrix} d_1 d_2 & i d_1 & i d_2 & -1 \\ i d_1 d_2^2 & d_1 d_2 & -d_2^2 & i d_2 \\ i d_1^2 d_2 & -d_1^2 & d_1 d_2 & i d_1 \\ -d_1^2 d_2^2 & i d_1^2 d_2 & i d_1 d_2^2 & d_1 d_2 \end{pmatrix}, \quad \rho(\sigma_3) = \begin{pmatrix} d_1 d_2 & -d_1 & -d_2 & 1 \\ d_1 d_2^2 & d_1 d_2 & -d_2^2 & -d_2 \\ d_1^2 d_2 & -d_1^2 & d_1 d_2 & -d_1 \\ d_1^2 d_2^2 & d_1^2 d_2 & d_1 d_2^2 & d_1 d_2 \end{pmatrix}, \\ \rho(\sigma_4) &= \begin{pmatrix} d_1 d_2 & -i d_1 & -i d_2 & -1 \\ i d_1 d_2^2 & d_1 d_2 & -d_2^2 & -i d_2 \\ i d_1^2 d_2 & -d_1^2 & d_1 d_2 & -i d_1 \\ -d_1^2 d_2^2 & -i d_1^2 d_2 & -i d_1 d_2^2 & d_1 d_2 \end{pmatrix}. \end{aligned}$$

Likewise, we write the matrices for $T_1 = T_2 = -i$.

Proposition 3.6 *The representation ρ is reducible.*

Proof Let S be the subspace spanned by the vector $v = d_2 e_2 - d_1 e_3$. Note that e_2 and e_3 are eigen-vectors of $\rho(\tau)$ corresponding the same eigenvalue. So, S is invariant under $\rho(\tau)$. It is easy to show that $\rho(\sigma_j)(v) = 2d_1 d_2 v$ for $j = 1, 2, 3, 4$. This completes the proof. $\square$

3.1.3 Case $T_1 = -T_2 = \pm i$

In the case $T_1 = -T_2 = i$, the representation ρ of $\mathcal{NB}_4$ is given by

$$\begin{aligned} \rho(\tau) &= t_1 t_2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \rho(\sigma_1) = \begin{pmatrix} d_1 d_2 & d_1 & d_2 & 1 \\ -d_1 d_2^2 & d_1 d_2 & -d_2^2 & d_2 \\ -d_1^2 d_2 & -d_1^2 & d_1 d_2 & d_1 \\ d_1^2 d_2^2 & -d_1^2 d_2 & -d_1 d_2^2 & d_1 d_2 \end{pmatrix}, \\ \rho(\sigma_2) &= \begin{pmatrix} d_1 d_2 & -i d_1 & i d_2 & 1 \\ -i d_1 d_2^2 & d_1 d_2 & d_2^2 & i d_2 \\ i d_1^2 d_2 & d_1^2 & d_1 d_2 & -i d_1 \\ d_1^2 d_2^2 & i d_1^2 d_2 & -i d_1 d_2^2 & d_1 d_2 \end{pmatrix}, \quad \rho(\sigma_3) = \begin{pmatrix} d_1 d_2 & -d_1 & -d_2 & 1 \\ d_1 d_2^2 & d_1 d_2 & -d_2^2 & -d_2 \\ d_1^2 d_2 & -d_1^2 & d_1 d_2 & -d_1 \\ d_1^2 d_2^2 & d_1^2 d_2 & d_1 d_2^2 & d_1 d_2 \end{pmatrix}, \\ \rho(\sigma_4) &= \begin{pmatrix} d_1 d_2 & i d_1 & -i d_2 & 1 \\ i d_1 d_2^2 & d_1 d_2 & d_2^2 & -i d_2 \\ -i d_1^2 d_2 & d_1^2 & d_1 d_2 & i d_1 \\ d_1^2 d_2^2 & -i d_1^2 d_2 & i d_1 d_2^2 & d_1 d_2 \end{pmatrix}. \end{aligned}$$

Likewise, we write the matrices for $T_1 = -T_2 = -i$.

Proposition 3.7 *The representation ρ is reducible.*

Proof Let S be the subspace spanned by the vector $v = e_1 + d_1 d_2 e_4$. The subspace S is invariant under $\rho(\tau)$ because, e_1 and e_4 are eigenvectors of $\rho(\tau)$ corresponding to the same eigenvalue. By direct calculations we get $\rho(\sigma_j)(v) = 2d_1 d_2 v \in S$ for $j = 1, 2, 3, 4$. Hence, S is invariant under $\rho(\sigma_j)$. $\square$

3.1.4 Case $T_1 = -1, T_2 = \pm i$

Without loss of generality, assume that $T_1 = -1$ and $T_2 = i$. Then, we have $T_1 T_2 = -i$ and

$$\rho(\tau) = t_1 t_2 \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \rho(\sigma_1) = \begin{pmatrix} a_1 d_2 & a_1 & d_2 & 1 \\ -a_1 d_2^2 & a_1 d_2 & -d_2^2 & d_2 \\ c_1 d_2 & c_1 & d_1 d_2 & d_1 \\ -c_1 d_2^2 & c_1 d_2 & -d_1 d_2^2 & d_1 d_2 \end{pmatrix}$$

Likewise, we write the matrices for $T_1 = -1, T_2 = -i$.

Note that the vectors e_j ($j = 1, 2, 3, 4$) are eigenvectors of $\rho(\tau)$ corresponding to four distinct eigenvalues (each of multiplicity 1).

Proposition 3.8 *The representation ρ is irreducible.*

Proof We have $\rho(\sigma_1)(e_1) = \begin{pmatrix} a_1 d_2 \\ -a_1 d_2^2 \\ c_1 d_2 \\ -c_1 d_2^2 \end{pmatrix}$, $\rho(\sigma_1)(e_2) = \begin{pmatrix} a_1 \\ a_1 d_2 \\ c_1 \\ c_1 d_2 \end{pmatrix}$, $\rho(\sigma_1)(e_3) = \begin{pmatrix} d_2 \\ -d_2^2 \\ d_1 d_2 \\ -d_1 d_2^2 \end{pmatrix}$ and $\rho(\sigma_1)(e_4) = \begin{pmatrix} 1 \\ d_2 \\ d_1 \\ d_1 d_2 \end{pmatrix}$. Note that $a_1 \neq 0, d_1 \neq 0, c_1 \neq 0$ and $d_2 \neq 0$. This implies that none of the proper subspaces

spanned by the vectors e_j are invariant under $\rho(\sigma_1)$. Likewise, we show that there are no invariant subspaces of dimensions 2 and 3. Hence ρ is irreducible. $\square$

3.2 Representation of $\mathcal{NB}_3$

Consider two irreducible representations of dimension 2 of the necklace braid group $\mathcal{NB}_3$ $\rho_1 = \rho(T_1, t_1, \omega_1, c_1, d_1)$ and $\rho_2 = \rho(T_2, t_2, \omega_2, c_2, d_2)$ (see Proposition 2.1).

Recall that $T_1 = e^{\pm i 2\pi/3}, T_2 = e^{\pm i 2\pi/3}, \omega_1 = e^{\pm i \pi/3}$ and $\omega_2 = e^{\pm i \pi/3}$.

Let ρ be the representation of $\mathcal{NB}_3$, which is defined in Definition 3.1. We have

$$\rho(\tau) = t_1 t_2 \begin{pmatrix} T_1 T_2 & 0 & 0 & 0 \\ 0 & T_1 & 0 & 0 \\ 0 & 0 & T_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\rho(\sigma_j) = \begin{pmatrix} \omega_1 \omega_2 d_1 d_2 & \omega_1 d_1 T_2^{j-1} & \omega_2 d_2 T_1^{j-1} & (T_1 T_2)^{j-1} \\ \omega_1 d_1 c_2 T_2^{1-j} & \omega_1 d_1 d_2 & c_2 T_1^{j-1} T_2^{1-j} & d_2 T_1^{j-1} \\ \omega_2 c_1 d_2 T_1^{1-j} & c_1 T_1^{1-j} T_2^{j-1} & \omega_2 d_2 d_1 & d_1 T_2^{j-1} \\ c_1 c_2 (T_1 T_2)^{1-j} & c_1 d_2 T_1^{1-j} & d_1 c_2 T_2^{1-j} & d_1 d_2 \end{pmatrix}.$$

for $j = 1, 2, 3$. We have two cases (1) $T_1 = e^{2\pi i/3}, T_2 = e^{-2\pi i/3}$ and (2) $T_1 = T_2 = e^{\pm 2\pi i/3}$.

We determine a necessary and sufficient condition under which ρ is an irreducible representation of $\mathcal{NB}_3$ of dimension 4.

In what follows, suppose that $d_1 d_2 \neq 0$.

3.2.1 Case $T_1 = e^{2\pi i/3}, T_2 = e^{-2\pi i/3}$

In this case, we have

$$\rho(\tau) = t_1 t_2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 & 0 \\ 0 & 0 & e^{-2\pi i/3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then, the eigenvalues of $\rho(\tau)$ are $\lambda_1 = t_1 t_2$ (of multiplicity 2) and $\lambda_2 = e^{2\pi i/3} t_1 t_2$ and $\lambda_3 = e^{-2\pi i/3} t_1 t_2$. The corresponding eigenvectors are $x_1 e_1 + x_4 e_4$ for λ_1 and e_2 and e_3 for λ_2 and λ_3 , respectively, where $\{x_1, x_4\} \subset \mathbb{C}^*$.

Lemma 3.9 *If $T_1 = e^{2\pi i/3}$, $T_2 = e^{-2\pi i/3}$, $\omega_1 = e^{\pm i\pi/3}$ and $\omega_2 = e^{\pm i\pi/3}$, then the representation ρ of $\mathcal{NB}_3$ has no invariant proper subspaces of dimension 1 if and only if*

$$\omega_1 \omega_2 \neq 1 \text{ or } \omega_1 d_1^2 c_2 \neq \omega_2 d_2^2 c_1.$$

Proof By direct calculations, we check that each of the subspaces spanned by the vectors e_j ($j = 1, 2, 3$) are not invariant under $\rho(\sigma_1)$.

It remains to consider the subspace S spanned by the vector $v = e_1 + x e_4$ with $x \in \mathbb{C}^*$. Since $T_1 T_2 = 1$, it follows that

$$\rho(\sigma_j)(v) = \begin{pmatrix} \omega_1 \omega_2 d_1 d_2 + x \\ \omega_1 d_1 c_2 T_2^{1-j} + d_2 T_1^{j-1} x \\ \omega_2 c_1 d_2 T_1^{1-j} + d_1 T_2^{j-1} x \\ c_1 c_2 + d_1 d_2 x \end{pmatrix} = \begin{pmatrix} \omega_1 \omega_2 d_1 d_2 + x \\ (\omega_1 d_1 c_2 + d_2 x) T_1^{j-1} \\ (\omega_2 c_1 d_2 + d_1 x) T_2^{j-1} \\ c_1 c_2 + d_1 d_2 x \end{pmatrix}.$$

So

$$\begin{aligned} \rho(\sigma_j)(v) \in S &\Leftrightarrow \rho(\sigma_j)(v) = \lambda v \text{ for some } \lambda \in \mathbb{C} \\ &\Leftrightarrow x = \frac{-\omega_1 d_1 c_2}{d_2} = \frac{-\omega_2 c_1 d_2}{d_1} \text{ and } x^2 + (\omega_1 \omega_2 - 1) d_1 d_2 x - c_1 c_2 = 0 \\ &\Leftrightarrow \omega_1 d_1^2 c_2 = \omega_2 d_2^2 c_1 \text{ and } (\omega_1 \omega_2 - 1)(c_2 - \omega_2 d_2^2) = 0 \\ &\Leftrightarrow \omega_1 d_1^2 c_2 = \omega_2 d_2^2 c_1 \text{ and } \omega_1 \omega_2 - 1 = 0. \end{aligned}$$

Therefore the subspace S , that is spanned by $v = e_1 - \frac{\omega_1 d_1 c_2}{d_2} e_4$, is invariant under $\rho(\sigma_j)$ if $\omega_1 \omega_2 = 1$ and $\omega_1 d_1^2 c_2 = \omega_2 d_2^2 c_1$. This gives a contradiction. $\square$

Lemma 3.10 *If $T_1 = e^{2\pi i/3}$, $T_2 = e^{-2\pi i/3}$, $\omega_1 = e^{\pm i\pi/3}$ and $\omega_2 = e^{\pm i\pi/3}$ then the representation ρ of $\mathcal{NB}_3$ has no invariant proper subspaces of dimension 2.*

Proof The possible two-dimensional invariant subspaces candidates to study are the following:

1. $\langle e_i, e_j \rangle$ for $i \neq j$
2. $\langle e_2, e_1 + x e_4 \rangle$ for $x \neq 0$
3. $\langle e_3, e_1 + y e_4 \rangle$ for $y \neq 0$

Since $c_1 \neq 0$, $c_2 \neq 0$, $d_1 \neq 0$ and $d_2 \neq 0$, it follows that none of the subspaces mentioned above is invariant under $\rho(\sigma_1)$. $\square$

Lemma 3.11 *If $T_1 = e^{2\pi i/3}$, $T_2 = e^{-2\pi i/3}$, $\omega_1 = e^{\pm i\pi/3}$ and $\omega_2 = e^{\pm i\pi/3}$ then the representation ρ of $\mathcal{NB}_3$ has no invariant proper subspaces of dimension 3 if and only if*

$$\omega_1 \omega_2 \neq 1 \text{ or } \omega_1 d_1^2 c_2 \neq \omega_2 d_2^2 c_1.$$

Proof The possible invariant three-dimensional invariant subspaces to study are:

1. $\langle e_i, e_j, e_k \rangle$ where i, j, k are pairwise distinct
2. $\langle e_2, e_3, e_1 + x e_4 \rangle$ for $x \neq 0$

By direct computations, it is easy to check that all the subspaces in case 1 are not invariant.

In case 2, we let S be a subspace of the form $\langle e_2, e_3, e_1 + x e_4 \rangle$ with $x \neq 0$.

$$\rho(\sigma_j)(e_2) = \begin{pmatrix} \omega_1 d_1 T_2^{j-1} \\ \omega_1 d_1 d_2 \\ c_1 T_1^{1-j} T_2^{j-1} \\ c_1 d_2 T_1^{1-j} \end{pmatrix}, \quad \rho(\sigma_j)(e_3) = \begin{pmatrix} \omega_2 d_2 T_1^{j-1} \\ c_2 T_1^{j-1} T_2^{1-j} \\ \omega_2 d_1 d_2 \\ d_1 c_2 T_2^{1-j} \end{pmatrix}$$

and

$$\rho(\sigma_j)(e_1 + xe_4) = \begin{pmatrix} \omega_1\omega_2d_1d_2 + x \\ \omega_1d_1c_2T_2^{1-j} + d_2T_1^{j-1}x \\ \omega_2c_1d_2T_1^{1-j} + d_1T_2^{j-1}x \\ c_1c_2 + d_1d_2x \end{pmatrix},$$

Therefore

$$\rho(\sigma_j)(e_2) \in S \Leftrightarrow x = \frac{c_1d_2}{\omega_1d_1}$$

and

$$\begin{aligned} \rho(\sigma_j)(e_3) \in S &\Leftrightarrow x = \frac{d_1c_2}{\omega_2d_2}, \\ \rho(\sigma_j)(e_1 + xe_4) \in S &\Leftrightarrow \begin{pmatrix} \omega_1\omega_2d_1d_2 + x \\ \omega_1d_1c_2T_2^{1-j} + d_2T_1^{j-1}x \\ \omega_2c_1d_2T_1^{1-j} + d_1T_2^{j-1}x \\ c_1c_2 + d_1d_2x \end{pmatrix} = \begin{pmatrix} \lambda \\ \alpha \\ \beta \\ \lambda x \end{pmatrix} \text{ for some } \alpha, \beta, \gamma \in \mathbb{C}. \end{aligned}$$

This is equivalent to

$$x^2 + (\omega_1\omega_2d_1d_2 - d_1d_2)x - c_1c_2 = 0.$$

Therefore, S is invariant under $\rho(\sigma_j)$ ($j = 1, 2, 3$) if and only if

$$x = \frac{c_1d_2}{\omega_1d_1} = \frac{c_2d_1}{\omega_2d_2} \quad (3.3)$$

and

$$c_1^2d_2^2 + \omega_1d_1^2c_1d_2^2(\omega_1\omega_2 - 1) - \omega_1d_1^2c_1c_2 = 0. \quad (3.4)$$

The Eqs. (3.3) and (3.4) imply that

$$\omega_1d_1^2c_2 = \omega_2d_2^2c_1$$

and

$$(\omega_1\omega_2 - 1)(\omega_2d_2^2 - c_2) = 0.$$

Since $\omega_2d_2^2 \neq c_2$, it follows that

$$\omega_1\omega_2 = 1$$

Therefore the subspace S , which is spanned by $v = e_1 - \frac{c_1d_2}{\omega_1d_1}e_4$, is invariant under $\rho(\sigma_j)$ if $\omega_1\omega_2 = 1$ and $\omega_1d_1^2c_2 = \omega_2d_2^2c_1$. This gives a contradiction. $\square$

Now, we have the following proposition.

Proposition 3.12 *If $T_1 = e^{i2\pi/3}$, $T_2 = e^{-i2\pi/3}$, $\omega_1 = e^{\pm i\pi/3}$ and $\omega_2 = e^{\pm i\pi/3}$ then ρ is irreducible representation of $\mathcal{NB}_3$ if and only if $\omega_1\omega_2 \neq 1$ or $\omega_2d_2^2c_1 \neq \omega_1d_1^2c_2$.*

3.2.2 Case $(T_1 = T_2 = e^{2\pi i/3})$ and $(T_1 = T_2 = e^{-2\pi i/3})$

In this case, we have $T_1 = T_2$. Then,

$$\rho(\tau) = t_1 t_2 \begin{pmatrix} e^{\mp i 2\pi/3} & 0 & 0 & 0 \\ 0 & e^{\pm i 2\pi/3} & 0 & 0 \\ 0 & 0 & e^{\pm i 2\pi/3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\rho(\sigma_j) = \begin{pmatrix} \omega_1 \omega_2 d_1 d_2 & \omega_1 d_1 T_1^{j-1} & \omega_2 d_2 T_1^{j-1} & (T_1)^{2j-2} \\ \omega_1 d_1 c_2 T_1^{1-j} & \omega_1 d_1 d_2 & c_2 & d_2 T_1^{j-1} \\ \omega_2 c_1 d_2 T_1^{1-j} & c_1 & \omega_2 d_2 d_1 & d_1 T_1^{j-1} \\ c_1 c_2 (T_1)^{2-2j} & c_1 d_2 T_1^{1-j} & d_1 c_2 T_1^{1-j} & d_1 d_2 \end{pmatrix}$$

for $j = 1, 2, 3$. The eigenvalues of $\rho(\tau)$ are then $\lambda_2 = t_1 t_2 e^{\pm i 2\pi/3}$ (of multiplicity 2) and $\lambda_1 = e^{\mp i 2\pi/3} t_1 t_2$ and $\lambda_3 = t_1 t_2$. The corresponding eigenvectors are $x e_2 + y e_3$ for λ_2 and e_1 and e_4 for λ_1 and λ_3 , respectively.

Lemma 3.13 *If $T_1 = T_2 = e^{\pm 2\pi i/3}$, $\omega_1 = e^{\pm i\pi/3}$ and $\omega_2 = e^{\pm i\pi/3}$ then the representation ρ of $\mathcal{NB}_3$ has no invariant proper subspaces of dimension 1 if and only if $\omega_1 \neq \omega_2$ or $\omega_1 d_1^2 c_2 \neq \omega_2 d_2^2 c_1$.*

Proof By direct calculations, we check that each of the subspaces spanned by the vectors e_j are not invariant under $\rho(\sigma_1)$.

It remains to consider the subspace S spanned by the vector $v = e_2 + x e_3$ with $x \in \mathbb{C}^*$. Since $T_1 = T_2$, it follows that

$$\rho(\sigma_j)(v) = \begin{pmatrix} \omega_1 d_1 T_1^{j-1} + \omega_2 d_2 T_1^{j-1} x \\ \omega_1 d_1 d_2 + c_2 x \\ c_1 + \omega_2 d_2 d_1 x \\ c_1 d_2 T_1^{1-j} + d_1 c_2 T_1^{1-j} x \end{pmatrix} = \begin{pmatrix} (\omega_1 d_1 + \omega_2 d_2 x) T_1^{j-1} \\ \omega_1 d_1 d_2 + c_2 x \\ c_1 + \omega_2 d_2 d_1 x \\ (c_1 d_2 + d_1 c_2 x) T_1^{1-j} \end{pmatrix}.$$

So

$$\begin{aligned} \rho(\sigma_j)(v) \in S &\Leftrightarrow \rho(\sigma_j)(v) = \lambda v \text{ for some } \lambda \in \mathbb{C} \\ &\Leftrightarrow x = \frac{-\omega_1 d_1}{\omega_2 d_2} = \frac{-c_1 d_2}{d_1 c_2} \text{ and } c_2 x^2 + (\omega_1 - \omega_2) d_1 d_2 x - c_1 = 0 \\ &\Leftrightarrow \omega_1 d_1^2 c_2 = \omega_2 d_2^2 c_1 \text{ and } c_2 \omega_1^2 d_1^2 + (\omega_2 - \omega_1) d_1^2 d_2^2 \omega_1 \omega_2 - c_1 \omega_2^2 d_2^2 = 0 \\ &\Leftrightarrow \omega_1 d_1^2 c_2 = \omega_2 d_2^2 c_1 \text{ and } (\omega_1 - \omega_2)(c_2 - \omega_2 d_2^2) = 0 \\ &\Leftrightarrow \omega_1 d_1^2 c_2 = \omega_2 d_2^2 c_1 \text{ and } \omega_1 = \omega_2 \text{ because } c_2 \neq \omega_2 d_2^2 \end{aligned}$$

Therefore the subspace S , which is spanned by $v = e_2 - \frac{\omega_1 d_1}{\omega_2 d_2} e_3$, is invariant under $\rho(\sigma_j)$ if $\omega_1 = \omega_2$ and $\omega_1 d_1^2 c_2 = \omega_2 d_2^2 c_1$. This gives a contradiction. $\square$

Lemma 3.14 *If $T_1 = T_2 = e^{\pm 2\pi i/3}$ and $\omega_1 = e^{\pm i\pi/3}$ and $\omega_2 = e^{\pm i\pi/3}$, then the representation ρ of $\mathcal{NB}_3$ has no invariant proper subspaces of dimension 2.*

Proof The possible two-dimensional invariant subspaces candidates to study are the following:

1. $\langle e_i, e_j \rangle$ for $i \neq j$
2. $\langle e_1, e_2 + x e_3 \rangle$ for $x \neq 0$
3. $\langle e_4, e_2 + y e_3 \rangle$ for $y \neq 0$

Since $c_1 \neq 0$, $c_2 \neq 0$, $d_1 \neq 0$ and $d_2 \neq 0$, it follows that none of the subspaces mentioned above is invariant under $\rho(\sigma_1)$. $\square$

Lemma 3.15 *If $T_1 = T_2 = e^{\pm 2\pi i/3}$, $\omega_1 = e^{\pm i\pi/3}$ and $\omega_2 = e^{\pm i\pi/3}$, then the representation ρ of $\mathcal{NB}_3$ has no invariant proper subspaces of dimension 3 if and only if $\omega_1 \neq \omega_2$ or $\omega_1 d_1^2 c_2 \neq \omega_2 d_2^2 c_1$.*

Proof The possible invariant three-dimensional invariant subspaces to study are:

1. $\langle e_i, e_j, e_k \rangle$ where i, j, k are pairwise distinct
2. $\langle e_1, e_4, e_2 + xe_3 \rangle$ for $x \neq 0$

By direct computations, it is easy to check that all the subspaces in case 1 are not invariant under $\rho(\sigma_1)$.

In case 2, we let S be a subspace of the form $\langle e_1, e_4, e_2 + xe_3 \rangle$ with $x \neq 0$.

$$\rho(\sigma_j)(e_1) = \begin{pmatrix} \omega_1 \omega_2 d_1 d_2 \\ \omega_1 d_1 c_2 T_1^{1-j} \\ \omega_2 c_1 d_2 T_1^{1-j} T_1^{j-1} \\ c_1 c_2 T_1^{2-2j} \end{pmatrix}, \quad \rho(\sigma_j)(e_4) = \begin{pmatrix} T_1^{2j-2} \\ d_2 T_1^{j-1} \\ d_1 T_1^{j-1} \\ d_1 d_2 \end{pmatrix}$$

and

$$\rho(\sigma_j)(e_2 + xe_3) = \begin{pmatrix} \omega_1 d_1 T_1^{j-1} + \omega_2 d_2 T_1^{j-1} x \\ \omega_1 d_1 d_2 + c_2 x \\ c_1 + \omega_2 d_2 d_1 x \\ c_1 d_2 T_1^{1-j} + d_1 c_2 T_1^{1-j} x \end{pmatrix}.$$

Therefore, we get

$$\rho(\sigma_j)(e_1) \in S \Leftrightarrow x = \frac{\omega_2 c_1 d_2}{\omega_1 d_1 c_2}$$

and

$$\rho(\sigma_j)(e_4) \in S \Leftrightarrow x = \frac{d_1}{d_2},$$

$$\rho(\sigma_j)(e_2 + xe_3) \in S \Leftrightarrow \begin{pmatrix} \omega_1 d_1 T_1^{j-1} + \omega_2 d_2 T_1^{j-1} x \\ \omega_1 d_1 d_2 + c_2 x \\ c_1 + \omega_2 d_2 d_1 x \\ c_1 d_2 T_1^{1-j} + d_1 c_2 T_1^{1-j} x \end{pmatrix} = \begin{pmatrix} \alpha \\ \lambda \\ \lambda x \\ \beta \end{pmatrix} \text{ for some } \alpha, \beta, \gamma \in \mathbb{C}.$$

This is equivalent to

$$c_2 x^2 + (\omega_1 - \omega_2) d_1 d_2 x - c_1 = 0.$$

Therefore, S is invariant under $\rho(\sigma_j)$ ($j = 1, 2, 3$) if and only if:

$$x = \frac{d_1}{d_2} = \frac{\omega_2 c_1 d_2}{\omega_1 d_1 c_2} \quad (3.5)$$

and

$$c_2 d_1^2 + (\omega_1 - \omega_2) d_1^2 d_2^2 - c_1 d_2^2 = 0. \quad (3.6)$$

The Eqs. (3.5) and (3.6) imply that

$$\omega_1 d_1^2 c_2 = \omega_2 d_2^2 c_1$$

and

$$(\omega_1 - \omega_2)(\omega_1 d_1^2 - c_1) = 0.$$

Since $\omega_1 d_1^2 \neq c_1$, it follows that $\omega_1 = \omega_2$, which gives a contradiction. Therefore, the subspace S , that is spanned by $v = e_2 - \frac{d_1}{d_2} e_3$, is invariant under $\rho(\sigma_j)$ if $\omega_1 = \omega_2$ and $\omega_1 d_1^2 c_2 = \omega_2 d_2^2 c_1$. $\square$

Proposition 3.16 *If $T_1 = T_2 = e^{\pm 2\pi i/3}$, $\omega_1 = e^{\pm i\pi/3}$ and $\omega_2 = e^{\pm i\pi/3}$ then ρ is an irreducible representation of $\mathcal{NB}_3$ if and only if $\omega_1 \neq \omega_2$ or $\omega_2 d_2^2 c_1 \neq \omega_1 d_1^2 c_2$.*

Proof The proof follows directly from Lemmas 3.13, 3.14 and 3.15. $\square$

Theorem 3.17 *The representation ρ of $\mathcal{NB}_3$ is irreducible if and only if one of the following conditions hold.*

1. $T_1 = e^{2\pi i/3}, T_2 = e^{-2\pi i/3}$ and $\omega_1\omega_2 \neq 1$
2. $T_1 = e^{2\pi i/3}, T_2 = e^{-2\pi i/3}$ and $\omega_1d_1^2c_2 \neq \omega_2d_2^2c_1$
3. $T_1 = T_2 = e^{\pm 2\pi i/3}$ and $\omega_1 \neq \omega_2$
4. $T_1 = T_2 = e^{\pm 2\pi i/3}$ and $\omega_1d_1^2c_2 \neq \omega_2d_2^2c_1$

Proof The proof follows directly from Propositions 3.12 and 3.16 $\square$

3.3 Representation of $\mathcal{NB}_2$

In this section, we consider the representation ρ given by Definition 3.1. In this case, we have $T_1 = T_2 = -1$. A similar work to that done for $\mathcal{NB}_4$ (Theorem 3.5), we get the following theorem.

Theorem 3.18 *The representation ρ is irreducible if and only if $a_1a_2 \neq d_1d_2$ and $a_1d_2 \neq a_2d_1$.*

4 Unitary representations of $\mathcal{NB}_n$ of dimension 2 ($n = 2, 3, 4$)

Definition 4.1 A square matrix A is unitary relative to a matrix M if $AMA^* = M$. Here, A^* is the conjugate transpose of A .

Definition 4.2 A representation $\rho : G \rightarrow GL(n, \mathbb{C})$ of a group G is called unitary if $\rho(g)$ is unitary for every element $g \in G$.

We consider the irreducible representations of $\mathcal{NB}_n$, given by Proposition 2.1. Then, we determine a necessary and sufficient condition under which these representations are unitary relative to some hermitian positive definite matrix.

Recall that

$$\rho(\tau) = \begin{pmatrix} Tt & 0 \\ 0 & t \end{pmatrix}, \quad \rho(\sigma_j) = \begin{pmatrix} a & T^{j-1} \\ cT^{1-j} & d \end{pmatrix},$$

where t is a $(2n)$ th root of unity and $T \in \{-1, i, -i, e^{i2\pi/3}, e^{-i2\pi/3}\}$.

Lemma 4.3 *Let ρ be a complex irreducible representation of $\mathcal{NB}_n$ ($n = 2, 3, 4$), given by Proposition 2.1. If ρ is unitary relative to a matrix $M = \begin{pmatrix} x & y \\ z & u \end{pmatrix}$, then $y = z = 0$ and*

$$\begin{cases} (|a|^2 - 1)x + u = 0 \\ |c|^2x + (|d|^2 - 1)u = 0 \\ a\bar{c}x + \bar{d}u = 0 \end{cases}.$$

Proof Since ρ is unitary relative to a matrix $M = \begin{pmatrix} x & y \\ z & u \end{pmatrix}$, we have $\rho(\tau)M\rho(\tau)^* = M$. Thus, we get

$$\begin{pmatrix} Tt & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} x & y \\ z & u \end{pmatrix} \begin{pmatrix} \bar{T}\bar{t} & 0 \\ 0 & \bar{t} \end{pmatrix} = \begin{pmatrix} x & y \\ z & u \end{pmatrix}.$$

Then, we get

$$\begin{pmatrix} x & Ty \\ \bar{T}z & u \end{pmatrix} = \begin{pmatrix} x & y \\ z & u \end{pmatrix}.$$

It follows that

$$Ty = y \text{ and } \bar{T}z = z.$$

Therefore, $y = z = 0$ because $T \neq 1$.

We also have $\rho(\sigma_j)M\rho(\sigma_j)^* = M$ ($j = 1, 2, 3, 4$).

Then, we get

$$\begin{pmatrix} a & T^{j-1} \\ cT^{1-j} & d \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{c}\bar{T}^{1-j} \\ \bar{T}^{j-1} & \bar{d} \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & u \end{pmatrix}.$$

So we have

$$\begin{pmatrix} |a|^2x + u & a\bar{c}\bar{T}^{1-j}x + \bar{d}T^{j-1}u \\ c\bar{a}T^{1-j}x + d\bar{T}^{j-1}u & |c|^2x + |d|^2u \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & u \end{pmatrix}.$$

Since $\bar{T} = T^{-1}$, it follows that

$$\begin{pmatrix} |a|^2x + u & (a\bar{c}x + \bar{d}u)T^{j-1} \\ (c\bar{a}x + du)T^{1-j} & |c|^2x + |d|^2u \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & u \end{pmatrix}.$$

Then we get

$$\begin{cases} (|a|^2 - 1)x + u = 0 \\ |c|^2x + (|d|^2 - 1)u = 0 \\ a\bar{c}x + \bar{d}u = 0 \end{cases}.$$

□

Proposition 4.4 Let $\rho : \mathcal{NB}_4 \rightarrow GL(2, \mathbb{C})$ be the irreducible representation of $\mathcal{NB}_4$ given in Proposition 2.1 ($T = -1$). Then, ρ is unitary relative to a hermitian positive definite matrix M if and only if

$$|a| < 1$$

and

$$a = d + e^{i\theta}, \begin{cases} d = \delta' - \frac{1}{2}i & \text{if } \theta = \frac{\pi}{2} \bmod 2\pi \\ d = \delta' + \frac{1}{2}i & \text{if } \theta = -\frac{\pi}{2} \bmod 2\pi \\ d = -\frac{1}{2}\sec\theta - (\tan\theta)\delta + i\delta, & \text{if } \theta \neq \pm\frac{\pi}{2} \bmod 2\pi \end{cases},$$

where $\theta, \delta', \delta \in \mathbb{R}$

Proof Recall that

$$\rho(\tau) = \begin{pmatrix} -t & 0 \\ 0 & t \end{pmatrix} \text{ and } \rho(\sigma_j) = \begin{pmatrix} a & (-1)^{j-1} \\ c(-1)^{1-j} & d \end{pmatrix},$$

where t is an eighth root of unity and $c = a^2 - ad + d^2$ such that $a \neq d$ and $c \neq 0$. Suppose that ρ is unitary relative to matrix M . By Lemma 4.3, we have $M = \begin{pmatrix} x & 0 \\ 0 & u \end{pmatrix}$ and

$$\begin{cases} (|a|^2 - 1)x + u = 0 \\ |c|^2x + (|d|^2 - 1)u = 0 \\ a\bar{c}x + \bar{d}u = 0 \end{cases}.$$

To have a non-trivial solution of the first two equations of the system above, the determinant of the system is zero and so

$$(|a|^2 - 1)(|d|^2 - 1) = |c|^2. \quad (4.1)$$

The first equation yields

$$u = (1 - |a|^2)x. \quad (4.2)$$

From the third equation of the last system, we conclude that $ad \neq 0$ and

$$x = -\frac{\bar{d}u}{a\bar{c}}. \quad (4.3)$$

The Eq. (4.2) and the Eq. (4.3) imply that

$$a\bar{c} = (|a|^2 - 1)\bar{d}. \quad (4.4)$$

Using the fact $c = a^2 - ad + d^2$ and the Eqs. (4.1) and (4.4), we get

$$\bar{d} = -a(\bar{a} - \bar{d})^2 \text{ and } \bar{d}(a - d)^2 = -a.$$

Thus, we get

$$|a - d|^2 = 1 \text{ and } \frac{a}{\bar{d}} = -(a - d)^2.$$

Hence, there exists $\theta \in \mathbb{R}$ such that

$$a - d = e^{i\theta} \text{ and } a = -\bar{d}e^{2i\theta}.$$

Therefore we get

$$a = d + e^{i\theta} \text{ and } \bar{d}e^{2i\theta} + e^{i\theta} + d = 0.$$

Set $d = \delta' + i\delta$ where $\delta', \delta \in \mathbb{R}$. Then

$$(1 + e^{2i\theta})\delta' + i\delta(1 - e^{2i\theta}) + e^{i\theta} = 0. \quad (4.5)$$

To solve the Eq. (4.5), consider two cases: (1) $1 + e^{2i\theta} = 0$, (2) $1 + e^{2i\theta} \neq 0$.

If $1 + e^{2i\theta} = 0$ then $d = \delta' \mp \frac{i}{2}$ and $a = \delta' \pm \frac{i}{2}$.

If $1 + e^{2i\theta} \neq 0$ then $d = -\frac{1}{2} \sec \theta - (\tan \theta)\delta + \delta i$.

Therefore, we have

$$a = d + e^{i\theta} \text{ and } \begin{cases} d = \delta' - \frac{1}{2}i & \text{if } \theta = \frac{\pi}{2} \bmod 2\pi \\ d = \delta' + \frac{1}{2}i & \text{if } \theta = -\frac{\pi}{2} \bmod 2\pi \\ d = -\frac{1}{2} \sec \theta - (\tan \theta)\delta + \delta i, & \text{if } \theta \neq \pm \frac{\pi}{2} \bmod 2\pi \end{cases},$$

where $\theta, \delta', \delta \in \mathbb{R}$.

The condition $|a| < 1$ follows directly from the Eq. (4.2) and the fact that M is positive definite. The matrix M becomes $\begin{pmatrix} 1 & 0 \\ 0 & 1 - |a|^2 \end{pmatrix}$ up to a constant. $\square$

Proposition 4.5 *Let $\rho : \mathcal{NB}_4 \rightarrow GL(2, \mathbb{C})$ be the irreducible representation of $\mathcal{NB}_4$ given by Proposition 2.1 ($T = \pm i$). The representation ρ is unitary relative to a hermitian positive definite matrix M if and only if*

$$|d|^2 = \frac{1}{2}.$$

Proof Recall that

$$\rho(\tau) = \begin{pmatrix} \pm it & 0 \\ 0 & t \end{pmatrix} \text{ and } \rho(\sigma_j) = \begin{pmatrix} d & (\pm i)^{j-1} \\ -d^2(\pm i)^{1-j} & d \end{pmatrix},$$

where t is an eighth root of unity and $d \neq 0$.

Suppose that ρ is unitary relative to hermitian positive definite matrix M . Then, by Lemma 4.3, we have $M = \begin{pmatrix} x & 0 \\ 0 & u \end{pmatrix}$ and

$$\begin{cases} (|a|^2 - 1)x + u = 0 \\ |c|^2x + (|d|^2 - 1)u = 0 \\ a\bar{c}x + \bar{d}u = 0 \end{cases}.$$

Since $a = d$ and $c = -d^2$, it follows that

$$\begin{cases} (|d|^2 - 1)x + u = 0 \\ |d|^4x + (|d|^2 - 1)u = 0 \\ -d\bar{d}^2x + \bar{d}u = 0 \end{cases}.$$

To have a non-trivial solution of the first two equations of the last system, the determinant of the system is zero and so

$$(|d|^2 - 1)(|d|^2 - 1) = |d|^4.$$

The first and third equations of the last system imply that

$$x = \frac{u}{d\bar{d}} = \frac{u}{1 - |d|^2}.$$

Then, we get

$$|d|^2 = \frac{1}{2}$$

and the matrix M becomes $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ up to a constant. $\square$

Proposition 4.6 Let $\rho : \mathcal{NB}_3 \rightarrow GL(2, \mathbb{C})$ be the irreducible representation of $\mathcal{NB}_4$ that is defined, in Proposition 2.1 ($T = e^{\pm 2\pi i/3}$). The representation ρ is unitary relative to a hermitian positive definite matrix M if and only if $|d| < 1$ and

$$c = \frac{\omega d(|d|^2 - 1)}{\bar{d}}.$$

Proof Recall that

$$\rho(\tau) = \begin{pmatrix} e^{\pm 2\pi i/3}t & 0 \\ 0 & t \end{pmatrix} \text{ and } \rho(\sigma_j) = \begin{pmatrix} \omega d & e^{\pm(j-1)2\pi i/3} \\ e^{\pm(1-j)2\pi i/3}c & d \end{pmatrix},$$

where t is a sixth root of unity and $\omega = e^{\pm i\pi/3}$ such that $c \neq 0$.

Suppose that ρ is unitary relative to a matrix M . Then, by Lemma 4.3, we have $M = \begin{pmatrix} x & 0 \\ 0 & u \end{pmatrix}$ and

$$\begin{cases} (|a|^2 - 1)x + u = 0 \\ |c|^2x + (|d|^2 - 1)u = 0 \\ a\bar{c}x + \bar{d}u = 0 \end{cases}.$$

Since $a = \omega d$, it follows that

$$\begin{cases} (|d|^2 - 1)x + u = 0 \\ |c|^2x + (|d|^2 - 1)u = 0 \\ \omega d\bar{c}x + \bar{d}u = 0 \end{cases}.$$

So

$$x = -\frac{\bar{d}u}{\omega d\bar{c}} = \frac{u}{1 - |d|^2} \quad (4.6)$$

Since $\omega^{-1} = \bar{\omega}$, it follows that

$$c = \frac{\omega d(|d|^2 - 1)}{\bar{d}}.$$

The condition $|d| < 1$ follows from the Eq. (4.6) and the fact that M is a positive definite. The matrix M becomes $\begin{pmatrix} 1 & 0 \\ 0 & 1 - |d|^2 \end{pmatrix}$ up to a constant. $\square$

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