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Study of the growth properties of meromorphic solutions of higher-order linear difference equations

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Abstract In this paper, we investigate the growth of meromorphic solutions of homogeneous and non-homogeneous linear difference equations

$$\begin{aligned} A_k(z)f(z+c_k) + \cdots + A_1(z)f(z+c_1) + A_0(z)f(z) &= 0, \\ A_k(z)f(z+c_k) + \cdots + A_1(z)f(z+c_1) + A_0(z)f(z) &= F, \end{aligned}$$

where $A_k(z), \dots, A_0(z), F(z)$ are meromorphic functions and $c_j (1, \dots, k)$ are non-zero distinct complex numbers. Under some conditions on the coefficients, we extend early results due to Zhou and Zheng, Belaïdi and Benkarouba.

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1 Introduction and statement of main results

In this paper, a meromorphic function means a function that is meromorphic in the whole complex plane $\mathbb{C}$. Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory of meromorphic functions [4, 7, 19]. Recently, many articles focused on complex difference equations [1, 10, 13–15, 20, 21]. The background for these studies lies in the recent difference counterparts of Nevanlinna theory. The key result here is the difference analog of the lemma on the logarithmic derivative obtained by Halburd–Korhonen [5, 6] and Chiang–Feng [3], independently.

In the following, we recall some fundamental definitions which are used later. For all $r \in \mathbb{R}$, we define $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp(\exp_p r)$, $p \in \mathbb{N}$. We also define for all $r > 0$ sufficiently large $\log_1 r := \log r$ and $\log_{p+1} r := \log(\log_p r)$, $p \in \mathbb{N}$. Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$, see [9].

Definition 1.1 [9] Let $p \geq 1$ be an integer. Then, the iterated p -order $\rho_p(f)$ of a meromorphic function f is defined by

$$\rho_p(f) = \limsup_{r \rightarrow +\infty} \frac{\log_p T(r, f)}{\log r},$$

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where $T(r, f)$ is the characteristic function of Nevanlinna (see, [7, 19]). For $p = 1$, this quantity is called order and hyper-order when $p = 2$. If f is an entire function, then the iterated p -order of f is defined as

$$\rho_p(f) = \limsup_{r \rightarrow +\infty} \frac{\log_p T(r, f)}{\log r} = \limsup_{r \rightarrow +\infty} \frac{\log_{p+1} M(r, f)}{\log r},$$

where $M(r, f) = \max_{|z|=r} |f(z)|$.

Definition 1.2 [8, 16, 17] Let $p \geq 1$ be an integer. Then, the iterated lower p -order $\mu_p(f)$ of a meromorphic function f is defined by

$$\mu_p(f) = \liminf_{r \rightarrow +\infty} \frac{\log_p T(r, f)}{\log r}.$$

For $p = 1$, this quantity is called lower order and hyper lower order when $p = 2$. If f is an entire function, then the iterated lower p -order of f is defined as

$$\mu_p(f) = \liminf_{r \rightarrow +\infty} \frac{\log_p T(r, f)}{\log r} = \liminf_{r \rightarrow +\infty} \frac{\log_{p+1} M(r, f)}{\log r}.$$

Definition 1.3 [2, 8] Let f be a meromorphic function of finite and non-zero iterated p -order $\rho_p(f)$. Then, the iterated p -type $\tau_p(f)$ of f is defined by

$$\tau_p(f) = \limsup_{r \rightarrow +\infty} \frac{\log_{p-1} T(r, f)}{r^{\rho_p(f)}} \quad (p \geq 1 \text{ an integer}).$$

If f is an entire function, then the iterated p -type $\tau_{M,p}(f)$ of f with finite and non-zero iterated p -order $\rho_p(f)$ is defined as

$$\tau_{M,p}(f) = \limsup_{r \rightarrow +\infty} \frac{\log_p M(r, f)}{r^{\rho_p(f)}} \quad (p \geq 1 \text{ an integer}).$$

Definition 1.4 [8] The iterated lower p -type $\underline{\tau}_p(f)$ of a meromorphic function f of finite and non-zero iterated lower p -order $\mu_p(f)$ is defined by

$$\underline{\tau}_p(f) = \liminf_{r \rightarrow +\infty} \frac{\log_{p-1} T(r, f)}{r^{\mu_p(f)}} \quad (p \geq 1 \text{ an integer}).$$

If f is an entire function, then the iterated lower p -type $\underline{\tau}_{M,p}(f)$ of f with finite and non-zero iterated lower p -order $\mu_p(f)$ is defined as

$$\underline{\tau}_{M,p}(f) = \liminf_{r \rightarrow +\infty} \frac{\log_p M(r, f)}{r^{\mu_p(f)}} \quad (p \geq 1 \text{ an integer}).$$

Definition 1.5 [8, 16, 17] Let f be a meromorphic function. Then, the iterated p -exponent of convergence of poles $\lambda_p\left(\frac{1}{f}\right)$ of f is defined by

$$\lambda_p\left(\frac{1}{f}\right) = \limsup_{r \rightarrow +\infty} \frac{\log_p N(r, f)}{\log r} \quad (p \geq 1 \text{ an integer}),$$

where $N(r, f)$ is the integrated counting function of poles of f in $\{z : |z| \leq r\}$.

In [3], Chiang and Feng investigated meromorphic solutions of the linear difference equation

$$A_k(z)f(z+k) + A_{k-1}(z)f(z+k-1) + \cdots + A_1(z)f(z+1) + A_0(z)f(z) = 0, \quad (1.1)$$

where $A_k(z), \dots, A_0(z)$ are entire functions and proved the following result.

Theorem 1.6 [3] Let $A_0(z), \dots, A_k(z)$ be entire functions. If there exists an integer l ($0 \leq l \leq k$) such that

$$\rho(A_l) > \max_{0 \leq l \leq k, j \neq l} \{\rho(A_j)\}$$

holds, then every meromorphic solution f ($\neq 0$) of Eq. (1.1) satisfies $\rho(f) \geq \rho(A_l) + 1$.

For the case when there is more than one of coefficients which have the maximal order, Laine and Yang [10] obtained the following result.

Theorem 1.7 [10] *Let $A_0(z), \dots, A_k(z)$ be entire functions of finite order such that among those having the maximal order*

$$\rho = \max_{0 \leq j \leq k} \{\rho(A_j)\},$$

exactly one has its type strictly greater than the others. Then for every meromorphic solution $f (\not\equiv 0)$ of equation

$$A_k(z)f(z+c_k) + A_{k-1}(z)f(z+c_{k-1}) + \dots + A_1(z)f(z+c_1) + A_0(z)f(z) = 0, \quad (1.2)$$

where $c_k, \dots, c_1$ are non-zero distinct complex numbers, we have $\rho(f) \geq \rho + 1$.

Using the concepts of lower order and lower type, Zheng and Tu [20] investigated the growth of solutions of Eq. (1.1) and proved the following result.

Theorem 1.8 [20] *Let $A_0(z), \dots, A_k(z)$ be entire functions such that there exists an integer l ($0 \leq l \leq n$) satisfying*

$$\max \{\rho(A_j) : j = 0, 1, \dots, k, j \neq l\} \leq \mu(A_l) < \infty$$

and

$$\max \{\tau_M(A_j) : \rho(A_j) = \mu(A_l) > 0 : j = 0, 1, \dots, k, j \neq l\} < \tau_M(A_l) < \infty.$$

Then, every meromorphic solution $f (\not\equiv 0)$ of Eq. (1.1) satisfies $\mu(f) \geq \mu(A_l) + 1$.

When the coefficients $A_0(z), \dots, A_k(z)$ are meromorphic, Latreuch and Belaïdi [13] investigated the growth of solutions of Eq. (1.1) and obtained the following result.

Theorem 1.9 [13] *Let $A_0(z), \dots, A_k(z)$ be meromorphic functions such that for some integer l ($0 \leq l \leq k$), we have*

$$\lambda\left(\frac{1}{A_l}\right) < \rho(A_l) = \rho \quad (0 < \rho < \infty), \quad \tau(A_l) = \tau \quad (0 < \tau < \infty).$$

Suppose that

$$\max \{\rho(A_j) : 0 \leq j \leq k, j \neq l\} \leq \rho$$

and

$$\sum_{\rho(A_j)=\rho} \tau(A_j) < \tau.$$

If $f (\not\equiv 0)$ is a meromorphic solution of Eq. (1.1), then $\rho(f) \geq \rho(A_l) + 1$.

Very recently, Zhou and Zheng [21] considered the case of the non-homogeneous equation and got the following result.

Theorem 1.10 [21] *Let $A_j(z)$ ($j = 0, 1, \dots, k$) and $F(z)$ be meromorphic functions. If there exists an integer l ($0 \leq l \leq k$) such that $A_l(z)$ satisfies*

$$\begin{aligned} \lambda\left(\frac{1}{A_l}\right) &< \rho(A_l) < \infty, \\ \max \{\rho(A_j) : j = 0, 1, \dots, k, j \neq l\} &\leq \rho(A_l), \\ \sum_{\rho(A_j)=\rho(A_l), j \neq l} \tau(A_j) &< \tau(A_l) < \infty. \end{aligned}$$

(i) If $\rho(F) < \rho(A_l)$, or $\rho(F) = \rho(A_l)$ and $\sum_{\rho(A_j)=\mu(A_l), j \neq l} \tau(A_j) + \tau(F) < \tau(A_l)$, or $\rho(F) = \rho(A_l)$ and $\sum_{\rho(A_j)=\mu(A_l) > 0, j \neq l} \tau(A_j) < \tau(F)$, then every meromorphic solution $f (\not\equiv 0)$ of equation

$$A_k(z)f(z+c_k) + A_{k-1}(z)f(z+c_{k-1}) + \dots + A_1(z)f(z+c_1) + A_0(z)f(z) = F(z) \quad (1.3)$$

satisfies $\rho(f) \geq \rho(A_l)$. Furthermore, if $F(z) \equiv 0$, then $\rho(f) \geq \rho(A_l) + 1$.

(ii) If $\rho(F) > \rho(A_l)$, then every meromorphic solution f of Eq. (1.3) satisfies $\rho(f) \geq \rho(F)$.

Thus, there arise many interesting questions such as:

Question 1.11 What can be said if we replace the conditions on " $\rho(A_l)$ and $\tau(A_l)$ " in Theorems 1.9 and 1.10 by the conditions on " $\mu(A_l)$ and $\underline{\tau}(A_l)$ "?

Question 1.12 What about the growth of meromorphic solutions in Theorem 1.10 when we use the concepts of iterated lower p -order and iterated lower p -type?

The aim of this paper is to give an answer for the above two questions, and we obtain the following results.

Theorem 1.13 Let $A_j(z)$ ($j = 0, 1, \dots, k$) be meromorphic functions. If there exists an integer l ($0 \leq l \leq k$) such that $A_l(z)$ satisfies

$$\begin{aligned} \lambda\left(\frac{1}{A_l}\right) &< \mu(A_l) < \infty, \\ \max\{\rho(A_j) : j = 0, 1, \dots, k, j \neq l\} &\leq \mu(A_l), \\ \sum_{\rho(A_j)=\mu(A_l), j \neq l} \tau(A_j) &< \underline{\tau}(A_l) < \infty. \end{aligned}$$

Then, every meromorphic solution $f(z) \not\equiv 0$ of Eq. (1.2) satisfies $\mu(f) \geq \mu(A_l) + 1$.

Theorem 1.14 Let $A_j(z)$ ($j = 0, 1, \dots, k$) and $F(z) \not\equiv 0$ be meromorphic functions. If there exists an integer l ($0 \leq l \leq k$) such that $A_l(z)$ satisfies

$$\begin{aligned} \lambda\left(\frac{1}{A_l}\right) &< \mu(A_l) < \infty, \\ \max\{\rho(A_j) : j = 0, 1, \dots, k, j \neq l\} &\leq \mu(A_l), \\ \sum_{\rho(A_j)=\mu(A_l), j \neq l} \tau(A_j) &< \underline{\tau}(A_l) < \infty. \end{aligned}$$

(i) If $\rho(F) < \mu(A_l)$, or $\rho(F) = \mu(A_l)$ and $\sum_{\rho(A_j)=\mu(A_l), j \neq l} \tau(A_j) + \tau(F) < \underline{\tau}(A_l)$, or $\mu(F) = \mu(A_l)$ and $\sum_{\rho(A_j)=\mu(A_l), j \neq l} \tau(A_j) + \underline{\tau}(A_l) < \underline{\tau}(F)$, then every meromorphic solution f of Eq. (1.3) satisfies $\mu(f) \geq \mu(A_l)$.

(ii) If $\mu(F) > \mu(A_l)$, then every meromorphic solution f of Eq. (1.3) satisfies $\mu(f) \geq \mu(F)$.

Next, we consider the properties of meromorphic solutions of the non-homogeneous linear difference Eq. (1.3) by using the concepts of iterated lower p -order and iterated lower p -type.

Theorem 1.15 Let $p \geq 2$ be an integer and $A_j(z)$ ($j = 0, 1, \dots, k$), $F(z) \not\equiv 0$ be meromorphic functions. If there exists $l \in \{0, 1, \dots, k\}$ such that

$$\begin{aligned} \lambda_p\left(\frac{1}{A_l}\right) &< \mu_p(A_l) < \infty, \\ \max\{\rho_p(A_j) : j = 0, 1, \dots, k, j \neq l\} &\leq \mu_p(A_l), \\ \max\{\tau_p(A_j) : \rho(A_j) = \mu_p(A_l), (j \neq l)\} &< \underline{\tau}_p(A_l) < \infty. \end{aligned}$$

(i) If $\rho_p(F) < \mu_p(A_l)$, or $\rho_p(F) = \mu_p(A_l)$ and $\tau_p(F) < \underline{\tau}_p(A_l) = \tau$, or $\mu_p(F) = \mu_p(A_l)$ and $\underline{\tau}_p(F) > \underline{\tau}_p(A_l)$, then every meromorphic solution f of Eq. (1.3) satisfies $\mu_p(f) \geq \mu_p(A_l)$.

(ii) If $\mu_p(F) > \mu_p(A_l)$, then every meromorphic solution f of Eq. (1.3) satisfies $\mu_p(f) \geq \mu_p(F)$.

Remark 1.16 Theorem 1.13 is the improvement of Theorems 1.8–1.9 and Theorem 1.14 is the improvement of Theorem 1.10. Furthermore, Theorem 1.15 is the improvement of Theorem 1.5 [21] and Theorem 1.4 [1].

2 Some auxiliary lemmas

The proofs of our results depend mainly on the following lemmas.

Lemma 2.1 [5] *Let f be a non-constant meromorphic function, $c \in \mathbb{C}$, $\delta < 1$ and $\varepsilon > 0$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r+|c|, f)^{1+\varepsilon}}{r^\delta}\right),$$

for all r outside of a possible exceptional set $E_1 \subset (1, +\infty)$ with finite logarithmic measure $lm(E_1) = \int_{E_1} \frac{dr}{r} < \infty$.

Lemma 2.2 [4] *Let f be a meromorphic function, c be a non-zero complex constant. Then, we have that for $r \rightarrow +\infty$*

$$(1+o(1))T(r-|c|, f(z)) \leq T(r, f(z+c)) \leq (1+o(1))T(r+|c|, f(z)).$$

Consequently for

$$p \in \mathbb{N}_+ = \{1, 2, \dots\}, \rho_p(f(z+c)) = \rho_p(f), \quad \mu_p(f(z+c)) = \mu_p(f).$$

Lemmas 2.1 and 2.2 lead to the following lemma.

Lemma 2.3 [5] *Let f be a non-constant meromorphic function, $c, h \in \mathbb{C}$, $c \neq h$, $\delta < 1$, $\varepsilon > 0$. Then*

$$m\left(r, \frac{f(z+c)}{f(z+h)}\right) = o\left(\frac{T(r+|c-h|+|h|, f)^{1+\varepsilon}}{r^\delta}\right),$$

holds for all r outside of a possible exceptional set $E_2 \subset (1, +\infty)$ with finite logarithmic measure $lm(E_2) = \int_{E_2} \frac{dr}{r} < \infty$.

The following two Lemmas 2.4 and 2.5 are well known for $p = 1$ [15, 20]. For convenience of readers, we give their proofs for $p \geq 2$.

Lemma 2.4 *Let f be a meromorphic function of finite and non-zero iterated lower p -order $\mu_p(f)$. Then, there exists a subset $E_3 \subset (1, +\infty)$ of infinite logarithmic measure such that*

$$\tau_p(f) = \lim_{\substack{r \rightarrow +\infty \\ r \in E_3}} \frac{\log_{p-1} T(r, f)}{r^{\mu_p(f)}}.$$

Consequently for any given $\varepsilon > 0$, for all $r \in E_3$

$$T(r, f) < \exp_{p-1} \left\{ \left(\tau_p(f) + \varepsilon \right) r^{\mu_p(f)} \right\}.$$

Proof By the definition of the iterated lower p -type, there exists a sequence $\{r_n\}_{n=1}^\infty$ tending to ∞ satisfying $(1 + \frac{1}{n})r_n < r_{n+1}$ and

$$\tau_p(f) = \lim_{r_n \rightarrow +\infty} \frac{\log_{p-1} T(r_n, f)}{r_n^{\mu_p(f)}}.$$

Then for any given $\varepsilon > 0$, there exists an n_1 such that for $n \geq n_1$ and any $r \in \left[\frac{n}{n+1}r_n, r_n \right]$, we have

$$\frac{\log_{p-1} T(\frac{n}{n+1}r_n, f)}{r_n^{\mu_p(f)}} \leq \frac{\log_{p-1} T(r, f)}{r^{\mu_p(f)}} \leq \frac{\log_{p-1} T(r_n, f)}{(\frac{n}{n+1}r_n)^{\mu_p(f)}}.$$

It follows that

$$\begin{aligned} \left(\frac{n}{n+1} \right)^{\mu_p(f)} \frac{\log_{p-1} T(\frac{n}{n+1}r_n, f)}{\left(\frac{n}{n+1}r_n \right)^{\mu_p(f)}} &\leq \frac{\log_{p-1} T(r, f)}{r^{\mu_p(f)}} \\ &\leq \frac{\log_{p-1} T(r_n, f)}{r_n^{\mu_p(f)}} \left(\frac{n+1}{n} \right)^{\mu_p(f)}. \end{aligned}$$

Set

$$E_3 = \bigcup_{n=n_1}^{+\infty} \left[\frac{n}{n+1} r_n, r_n \right].$$

Then, we have

$$\lim_{\substack{r \rightarrow +\infty \\ r \in E_3}} \frac{\log_{p-1} T(r, f)}{r^{\mu_p(f)}} = \lim_{r_n \rightarrow +\infty} \frac{\log_{p-1} T(r_n, f)}{r_n^{\mu_p(f)}} = \underline{\tau}_p(f)$$

$$\text{and } lm(E_3) = \int_{E_3} \frac{dr}{r} = \sum_{n=n_1}^{+\infty} \int_{\frac{n}{n+1} r_n}^{r_n} \frac{dt}{t} = \sum_{n=n_1}^{+\infty} \log \left(1 + \frac{1}{n} \right) = +\infty. \quad \square$$

Lemma 2.5 *Let f be a meromorphic function with finite iterated lower p -order $\mu_p(f)$. Then, for any given $\varepsilon > 0$, there exists a subset $E_4 \subset (1, +\infty)$ of infinite logarithmic measure such that*

$$T(r, f) < \exp_{p-1} \left\{ r^{\mu_p(f) + \varepsilon} \right\}.$$

Proof By definition of iterated lower p -order, there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to ∞ satisfying $(1 + \frac{1}{n}) r_n < r_{n+1}$ and

$$\lim_{r_n \rightarrow +\infty} \frac{\log_p T(r_n, f)}{\log r_n} = \mu_p(f).$$

Then for any given $\varepsilon > 0$, there exists an integer n_2 such that for all $n \geq n_2$,

$$T(r_n, f) < \exp_{p-1} \left\{ r_n^{\mu_p(f) + \frac{\varepsilon}{2}} \right\}.$$

Set $E_4 = \bigcup_{n=n_2}^{+\infty} \left[\frac{n}{n+1} r_n, r_n \right]$. Then for $r \in E_4 \subset (1, +\infty)$, we obtain

$$\begin{aligned} T(r, f) &\leq T(r_n, f) < \exp_{p-1} \left\{ r_n^{\mu_p(f) + \frac{\varepsilon}{2}} \right\} \\ &\leq \exp_{p-1} \left\{ \left(\frac{n+1}{n} r \right)^{\mu_p(f) + \frac{\varepsilon}{2}} \right\} < \exp_{p-1} \left\{ r^{\mu_p(f) + \varepsilon} \right\} \end{aligned}$$

and $lm(E_4) = \sum_{n=n_2}^{+\infty} \int_{\frac{n}{n+1} r_n}^{r_n} \frac{dt}{t} = \sum_{n=n_2}^{+\infty} \log \left(1 + \frac{1}{n} \right) = \infty$. Thus, Lemma 2.5 is proved. $\square$

Lemma 2.6 [3] *Let γ, R, R' be real numbers such that $0 < \gamma < 1, R > 0$, and let η be a non-zero complex number. Then, there is a positive constant C_γ depending only on γ such that for a given meromorphic function f we have, when $|z| = r, \max\{1, r + |\eta|\} < R < R'$, the estimate*

$$\begin{aligned} m \left(r, \frac{f(z + \eta)}{f(z)} \right) + m \left(r, \frac{f(z)}{f(z + \eta)} \right) &\leq \frac{2|\eta|R}{(R - r - |\eta|)^2} \left(m(R, f) + m \left(R, \frac{1}{f} \right) \right) \\ &\quad + \frac{2R'}{(R' - R)} \left(\frac{|\eta|}{R - r - |\eta|} + \frac{C_\gamma |\eta|^\gamma}{(1 - \gamma)r^\gamma} \right) \left(N(R', f) + N \left(R', \frac{1}{f} \right) \right). \end{aligned}$$

Lemma 2.7 *Let η_1, η_2 be two complex numbers such that $\eta_1 \neq \eta_2$ and let f be a finite iterated lower p -order meromorphic function. Let $\mu_p(f) = \mu$ be the iterated lower p -order of f . Then, for any given $\varepsilon > 0$, there exists a subset $E_5 \subset (1, +\infty)$ of infinite logarithmic measure such that for $r \in E_5$, we have:*

(i) *If $p = 1$, then*

$$m \left(r, \frac{f(z + \eta_1)}{f(z + \eta_2)} \right) = O \left(r^{\mu - 1 + \varepsilon} \right).$$

(ii) *If $p \geq 2$, then*

$$m \left(r, \frac{f(z + \eta_1)}{f(z + \eta_2)} \right) = O \left(\exp_{p-1} \{ r^{\mu + \varepsilon} \} \right).$$

Proof We have

$$\begin{aligned} m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) &\leq m\left(r, \frac{f(z+\eta_1)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+\eta_2)}\right) \\ &\leq m\left(r, \frac{f(z+\eta_1)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+\eta_1)}\right) \\ &\quad + m\left(r, \frac{f(z)}{f(z+\eta_2)}\right) + m\left(r, \frac{f(z+\eta_2)}{f(z)}\right). \end{aligned} \quad (2.1)$$

Since f has finite iterated lower p -order $\mu_p(f) = \mu < +\infty$, then by Lemma 2.5 for any given ε , $0 < \varepsilon < 2$, there exists a subset $E_5 \subset (1, +\infty)$ of infinite logarithmic measure such that for $r \in E_5$

$$T(r, f) \leq \exp_{p-1}\{r^{\mu+\frac{\varepsilon}{2}}\}. \quad (2.2)$$

Using Lemma 2.6, we obtain from inequality Eq. (2.1)

$$\begin{aligned} m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) &\leq \frac{2|\eta_1|R}{(R-r-|\eta_1|)^2} \left(m(R, f) + m\left(R, \frac{1}{f}\right)\right) \\ &\quad + \frac{2R'}{(R'-R)} \left(\frac{|\eta_1|}{R-r-|\eta_1|} + \frac{C_\gamma |\eta_1|^\gamma}{(1-\gamma)r^\gamma}\right) \left(N(R', f) + N\left(R', \frac{1}{f}\right)\right) \\ &\quad + \frac{2|\eta_2|R}{(R-r-|\eta_2|)^2} \left(m(R, f) + m\left(R, \frac{1}{f}\right)\right) \\ &\quad + \frac{2R'}{(R'-R)} \left(\frac{|\eta_2|}{R-r-|\eta_2|} + \frac{C_\gamma |\eta_2|^\gamma}{(1-\gamma)r^\gamma}\right) \left(N(R', f) + N\left(R', \frac{1}{f}\right)\right) \\ &= \left(\frac{2|\eta_1|R}{(R-r-|\eta_1|)^2} + \frac{2|\eta_2|R}{(R-r-|\eta_2|)^2}\right) \left(m(R, f) + m\left(R, \frac{1}{f}\right)\right) \\ &\quad + \frac{2R'}{(R'-R)} \left(\frac{|\eta_1|}{R-r-|\eta_1|} + \frac{C_\gamma |\eta_1|^\gamma}{(1-\gamma)r^\gamma}\right. \\ &\quad \left.+ \frac{|\eta_2|}{R-r-|\eta_2|} + \frac{C_\gamma |\eta_2|^\gamma}{(1-\gamma)r^\gamma}\right) \left(N(R', f) + N\left(R', \frac{1}{f}\right)\right). \end{aligned} \quad (2.3)$$

By choosing $\gamma = 1 - \frac{\varepsilon}{2}$, $R = 2r$, $R' = 3r$ and $r > \max\{|\eta_1|, |\eta_2|, 1/2\}$ in Eq. (2.3), we get

$$\begin{aligned} m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) &\leq \left(\frac{4|\eta_1|r}{(r-|\eta_1|)^2} + \frac{4|\eta_2|r}{(r-|\eta_2|)^2}\right) \left(m(2r, f) + m\left(2r, \frac{1}{f}\right)\right) \\ &\quad + 6\left(\frac{|\eta_1|}{r-|\eta_1|} + \frac{2C_\gamma |\eta_1|^{1-\frac{\varepsilon}{2}}}{\varepsilon r^{1-\frac{\varepsilon}{2}}} + \frac{|\eta_2|}{r-|\eta_2|} + \frac{2C_\gamma |\eta_2|^{1-\frac{\varepsilon}{2}}}{\varepsilon r^{1-\frac{\varepsilon}{2}}}\right) \left(N(3r, f) + N\left(3r, \frac{1}{f}\right)\right) \\ &\leq 4\left[\frac{4|\eta_1|r}{(r-|\eta_1|)^2} + \frac{4|\eta_2|r}{(r-|\eta_2|)^2}\right. \\ &\quad \left.+ 6\left(\frac{|\eta_1|}{r-|\eta_1|} + \frac{|\eta_2|}{r-|\eta_2|} + \frac{2C_\gamma (|\eta_1|^{1-\frac{\varepsilon}{2}} + |\eta_2|^{1-\frac{\varepsilon}{2}})}{\varepsilon r^{1-\frac{\varepsilon}{2}}}\right)\right] T(3r, f). \end{aligned}$$

From this, using the estimate Eq. (2.2), we have for $p = 1$

$$\begin{aligned} m\left(r, \frac{f(z + \eta_1)}{f(z + \eta_2)}\right) &\leq 4K \left[\frac{4|\eta_1|r}{(r - |\eta_1|)^2} + \frac{4|\eta_2|r}{(r - |\eta_2|)^2} \right. \\ &\quad \left. + 6 \left(\frac{|\eta_1|}{r - |\eta_1|} + \frac{|\eta_2|}{r - |\eta_2|} + \frac{2C_\gamma (|\eta_1|^{1-\frac{\varepsilon}{2}} + |\eta_2|^{1-\frac{\varepsilon}{2}})}{\varepsilon r^{1-\frac{\varepsilon}{2}}} \right) \right] (3r)^{\mu+\frac{\varepsilon}{2}} \\ &\leq Mr^{\mu-1+\varepsilon}, \end{aligned}$$

where $K > 0$, $M > 0$ are some constants. When $p \geq 2$, we obtain

$$\begin{aligned} m\left(r, \frac{f(z + \eta_1)}{f(z + \eta_2)}\right) &\leq 4K \left[\frac{4|\eta_1|r}{(r - |\eta_1|)^2} + \frac{4|\eta_2|r}{(r - |\eta_2|)^2} \right. \\ &\quad \left. + 6 \left(\frac{|\eta_1|}{r - |\eta_1|} + \frac{|\eta_2|}{r - |\eta_2|} + \frac{2C_\gamma (|\eta_1|^{1-\frac{\varepsilon}{2}} + |\eta_2|^{1-\frac{\varepsilon}{2}})}{\varepsilon r^{1-\frac{\varepsilon}{2}}} \right) \right] \exp_{p-1} \left\{ (3r)^{\mu+\frac{\varepsilon}{2}} \right\} \\ &\leq M_1 \exp_{p-1} \left\{ r^{\mu+\varepsilon} \right\}, \end{aligned}$$

where $M_1 > 0$ is some constant. This completes the proof of Lemma 2.7. $\square$

Remark 2.8 We note that Lemmas 2.1, 2.2, 2.3 and 2.6 hold without any finite-order conditions on meromorphic functions.

3 Proof of the theorems

Proof of Theorem 1.13. Let $f \not\equiv 0$ be a meromorphic solution of Eq. (1.2). If f has infinite lower order, then the result holds. Now, we suppose that $\mu(f) < \infty$. We divide Eq. (1.2) by $f(z + c_l)$ to get

$$-A_l(z) = \sum_{j=1, j \neq l}^k A_j(z) \frac{f(z + c_j)}{f(z + c_l)} + A_0(z) \frac{f(z)}{f(z + c_l)}. \quad (3.1)$$

From Lemma 2.7, it follows that for any given $\varepsilon > 0$, there exists a subset $E_5 \subset (1, +\infty)$ of infinite logarithmic measure such that for $r \in E_5$, we have

$$\begin{aligned} T(r, A_l(z)) &= m(r, A_l(z)) + N(r, A_l(z)) \leq \sum_{j=0, j \neq l}^k m(r, A_j(z)) \\ &\quad + \sum_{j=1, j \neq l}^k m\left(r, \frac{f(z + c_j)}{f(z + c_l)}\right) + m\left(r, \frac{f(z)}{f(z + c_l)}\right) + N(r, A_l(z)) + O(1) \\ &\leq \sum_{j=0, j \neq l}^k T(r, A_j(z)) + \sum_{j=1, j \neq l}^k O\left(r^{\mu(f)-1+\varepsilon}\right) + O\left(r^{\mu(f)-1+\varepsilon}\right) + N(r, A_l(z)) + O(1) \\ &\leq \sum_{j=0, j \neq l}^k T(r, A_j(z)) + N(r, A_l(z)) + O\left(r^{\mu(f)-1+\varepsilon}\right). \end{aligned} \quad (3.2)$$

First, we suppose that $b = \max \{\rho(A_j) : j = 0, 1, \dots, k, j \neq l\} < \mu(A_l) = \mu$. Then, for any given ε ($0 < 2\varepsilon < \mu - b$) and sufficiently large r , we have

$$T(r, A_l) \geq r^{\mu(A_l)-\varepsilon} \quad (3.3)$$

and

$$T(r, A_j) \leq r^{b+\varepsilon} \quad (j = 0, 1, \dots, k, j \neq l). \quad (3.4)$$

By the definition of $\lambda\left(\frac{1}{A_l}\right)$, for any given $\varepsilon\left(0 < 2\varepsilon < \mu - \lambda\left(\frac{1}{A_l}\right)\right)$ and sufficiently large r , we have

$$N(r, A_l) \leq r^{\lambda\left(\frac{1}{A_l}\right)+\varepsilon}. \quad (3.5)$$

By substituting the assumptions Eqs. (3.3), (3.4) and (3.5) into Eq. (3.2), for any given $\varepsilon\left(0 < 2\varepsilon < \min\left\{\mu - b, \mu - \lambda\left(\frac{1}{A_l}\right)\right\}\right)$ and sufficiently large $r \in E_5$, we obtain

$$r^{\mu-\varepsilon} \leq kr^{b+\varepsilon} + r^{\lambda\left(\frac{1}{A_l}\right)+\varepsilon} + O\left(r^{\mu(f)-1+\varepsilon}\right).$$

So

$$(1 - o(1))r^{\mu-\varepsilon} \leq O\left(r^{\mu(f)-1+\varepsilon}\right),$$

that is, $\mu(f) \geq \mu + 1 - 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\mu(f) \geq \mu(A_l) + 1$.

Assume

$$\max\{\rho(A_j) : j = 0, 1, \dots, k, j \neq l\} = \mu(A_l) = \mu$$

and $\tau_1 = \sum_{\rho(A_j)=\mu(A_l), j \neq l} \tau(A_j) < \tau(A_l) = \tau$. Then, there exists a set $J \subseteq \{j = 0, 1, \dots, k\} \setminus \{l\}$ such that for $j \in J$, we have $\rho(A_j) = \mu(A_l) = \mu$ with $\tau_1 = \sum_{j \in J} \tau(A_j) < \tau(A_l) = \tau$ and for $j \in \{0, 1, \dots, l-1, l+1, \dots, k\} \setminus J$, we have $\rho(A_j) < \mu(A_l) = \mu$. Hence, for any given $\varepsilon\left(0 < \varepsilon < \frac{\tau-\tau_1}{k+1}\right)$ and sufficiently large r , we have

$$T(r, A_j) \leq (\tau(A_j) + \varepsilon)r^{\mu(A_l)}, \quad j \in J \quad (3.6)$$

and

$$T(r, A_j) \leq r^{\mu_0}, \quad j \in \{0, 1, \dots, l-1, l+1, \dots, k\} \setminus J, \quad (3.7)$$

where $0 < \mu_0 < \mu$. By the definition of lower type, for the above ε and sufficiently large r , we have

$$T(r, A_l) \geq (\tau - \varepsilon)r^{\mu(A_l)}. \quad (3.8)$$

Now, we may choose sufficiently small ε satisfying $0 < \varepsilon < \min\left\{\frac{\mu-\lambda\left(\frac{1}{A_l}\right)}{2}, \frac{\tau-\tau_1}{k+1}\right\}$, by substituting the assumptions Eqs. (3.5), (3.6), (3.7) and (3.8) into Eq. (3.2), for sufficiently large $r \in E_5$, we obtain

$$\begin{aligned} (\tau - \varepsilon)r^\mu &\leq \sum_{j \in J} (\tau(A_j) + \varepsilon)r^\mu + \sum_{j \in \{0, 1, \dots, l-1, l+1, \dots, k\} \setminus J} r^{\mu_0} + r^{\lambda\left(\frac{1}{A_l}\right)+\varepsilon} \\ &\quad + O\left(r^{\mu(f)-1+\varepsilon}\right) \leq (\tau_1 + k\varepsilon)r^\mu + kr^{\mu_0} + r^{\lambda\left(\frac{1}{A_l}\right)+\varepsilon} + O\left(r^{\mu(f)-1+\varepsilon}\right). \end{aligned}$$

It follows that

$$(1 - o(1))(\tau - \tau_1 - (k+1)\varepsilon)r^\mu \leq O\left(r^{\mu(f)-1+\varepsilon}\right),$$

that is, $\mu(f) \geq \mu + 1 - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\mu(f) \geq \mu(A_l) + 1$. $\square$

Proof of Theorem 1.14. Let f be a meromorphic solution of Eq. (1.3). We divide Eq. (1.3) by $f(z + c_l)$ to get

$$-A_l(z) = \sum_{j=1, j \neq l}^k A_j(z) \frac{f(z + c_j)}{f(z + c_l)} + A_0(z) \frac{f(z)}{f(z + c_l)} - \frac{F(z)}{f(z + c_l)}. \quad (3.9)$$

From Lemma 2.2 and Lemma 2.3, it follows that for any given $\varepsilon > 0$, we have

$$\begin{aligned} T(r, A_l(z)) &= m(r, A_l(z)) + N(r, A_l(z)) \\ &\leq \sum_{j=0, j \neq l}^k m(r, A_j(z)) + \sum_{j=1, j \neq l}^k m\left(r, \frac{f(z + c_j)}{f(z + c_l)}\right) + m\left(r, \frac{f(z)}{f(z + c_l)}\right) \\ &\quad + m\left(r, \frac{1}{f(z + c_l)}\right) + m(r, F(z)) + N(r, A_l(z)) + O(1) \\ &\leq \sum_{j=0, j \neq l}^k T(r, A_j(z)) + \sum_{j=1, j \neq l}^k o\left(\frac{(T(r + |c_j - c_l| + |c_l|, f))^{1+\varepsilon}}{r^\delta}\right) \\ &\quad + o\left(\frac{(T(r + |c_l|, f))^{1+\varepsilon}}{r^\delta}\right) + T(r, f(z + c_l)) + T(r, F(z)) + N(r, A_l(z)) + O(1) \\ &\leq \sum_{j=0, j \neq l}^k T(r, A_j(z)) + o\left(\frac{(T(r + 2|c_l|, f))^{1+\varepsilon}}{r^\delta}\right) + (1 + o(1))T(r + |c_l|, f(z)) \\ &\quad + T(r, F(z)) + N(r, A_l(z)) \leq \sum_{j=0, j \neq l}^k T(r, A_j(z)) + T(r, F(z)) \\ &\quad + N(r, A_l(z)) + 2T(r + |c_l|, f(z)) + o\left(\frac{(T(r + 2|c_l|, f))^{1+\varepsilon}}{r^\delta}\right) \\ &\leq \sum_{j=0, j \neq l}^k T(r, A_j(z)) + T(r, F(z)) + N(r, A_l(z)) + 3(T(2r, f(z)))^2 \end{aligned} \quad (3.10)$$

for all r outside of a possible exceptional set E_2 with finite logarithmic measure $\int_{E_2} \frac{dr}{r} < \infty$.

(i) If $\rho(F) < \mu(A_l) = \mu$, then for any given ε ($0 < 2\varepsilon < \mu - \rho(F)$) and sufficiently large r , we have

$$T(r, F) \leq r^{\rho(F)+\varepsilon}. \quad (3.11)$$

First, we suppose that $b = \max\{\rho(A_j) : j = 0, 1, \dots, k, j \neq l\} < \mu(A_l) = \mu$. By substituting the assumptions Eqs. (3.3), (3.4), (3.5) and (3.11) into Eq. (3.10), for any given ε $\left(0 < \varepsilon < \min\left\{\frac{\mu-b}{2}, \frac{\mu-\lambda\left(\frac{1}{A_l}\right)}{2}, \frac{\mu-\rho(F)}{2}\right\}\right)$ and sufficiently large $r \notin E_2$, we obtain

$$r^{\mu-\varepsilon} \leq kr^{b+\varepsilon} + r^{\rho(F)+\varepsilon} + r^{\lambda\left(\frac{1}{A_l}\right)+\varepsilon} + 3(T(2r, f(z)))^2.$$

So

$$(1 - o(1))r^{\mu-\varepsilon} \leq 3(T(2r, f(z)))^2,$$

that is, $\mu(f) \geq \mu - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\mu(f) \geq \mu(A_l)$.

Assume

$$\max\{\rho(A_j) : j = 0, 1, \dots, k, j \neq l\} = \mu(A_l) = \mu$$

and $\tau_1 = \sum_{\rho(A_j)=\mu(A_l), j \neq l} \tau(A_j) < \underline{\tau}(A_l) = \tau$. By substituting the assumptions Eqs. (3.5), (3.6), (3.7), (3.8) and (3.11) into Eq. (3.10), for any given $\varepsilon \left(0 < \varepsilon < \min \left\{ \frac{\tau - \tau_1}{k+1}, \frac{\mu - \lambda \left(\frac{1}{A_l} \right)}{2}, \frac{\mu - \rho(F)}{2} \right\} \right)$ and sufficiently large $r \notin E_2$, we obtain

$$\begin{aligned} (\tau - \varepsilon) r^\mu &\leq \sum_{j \in J} (\tau(A_j) + \varepsilon) r^\mu + \sum_{j \in \{0, 1, \dots, l-1, l+1, \dots, k\} \setminus J} r^{\mu_0} \\ &\quad + r^{\rho(F) + \varepsilon} + r^{\lambda \left(\frac{1}{A_l} \right) + \varepsilon} + 3(T(2r, f(z)))^2 \\ &\leq (\tau_1 + k\varepsilon) r^\mu + kr^{\mu_0} + r^{\rho(F) + \varepsilon} + r^{\lambda \left(\frac{1}{A_l} \right) + \varepsilon} + 3(T(2r, f(z)))^2. \end{aligned}$$

So

$$(1 - o(1))(\tau - \tau_1 - (k+1)\varepsilon) r^\mu \leq 3(T(2r, f(z)))^2,$$

which implies $\mu(f) \geq \mu(A_l)$.

If $\rho(F) = \mu(A_l) = \mu$ and $\sum_{\rho(A_j)=\mu(A_l), j \neq l} \tau(A_j) + \tau(F) = \tau_1 + \tau(F) < \underline{\tau}(A_l) = \tau$, then for any given $\varepsilon > 0$ and sufficiently large r , we have

$$T(r, F) \leq (\tau(F) + \varepsilon) r^{\mu(A_l)}. \quad (3.12)$$

Now, we may choose sufficiently small ε satisfying

$$0 < \varepsilon < \min \left\{ \frac{\mu - \lambda \left(\frac{1}{A_l} \right)}{2}, \frac{\tau - \tau_1 - \tau(F)}{k+2} \right\},$$

by substituting the assumptions Eqs. (3.5), (3.6), (3.7), (3.8) and (3.12) into Eq. (3.10), for sufficiently large $r \notin E_2$, we obtain

$$\begin{aligned} (\tau - \varepsilon) r^\mu &\leq \sum_{j \in J} (\tau(A_j) + \varepsilon) r^\mu + \sum_{j \in \{0, 1, \dots, l-1, l+1, \dots, k\} \setminus J} r^{\mu_0} \\ &\quad + (\tau(F) + \varepsilon) r^\mu + r^{\lambda \left(\frac{1}{A_l} \right) + \varepsilon} + 3(T(2r, f(z)))^2 \\ &\leq (\tau_1 + k\varepsilon) r^\mu + kr^{\mu_0} + (\tau(F) + \varepsilon) r^\mu + r^{\lambda \left(\frac{1}{A_l} \right) + \varepsilon} + 3(T(2r, f(z)))^2. \end{aligned}$$

It follows that

$$(1 - o(1))(\tau - \tau_1 - \tau(F) - (k+2)\varepsilon) r^\mu \leq 3(T(2r, f(z)))^2,$$

which implies $\mu(f) \geq \mu(A_l)$.

If $\mu(F) = \mu(A_l) = \mu$ and $\tau_1 + \underline{\tau}(A_l) < \underline{\tau}(F)$, then for any given $\varepsilon > 0$ and sufficiently large r , we have

$$T(r, F) > (\underline{\tau}(F) - \varepsilon) r^{\mu(A_l)}. \quad (3.13)$$

By Lemma 2.4, for the above ε , there exists a subset E_3 with infinite logarithmic measure such that for all $r \in E_3$, we have

$$T(r, A_l) \leq (\underline{\tau}(A_l) + \varepsilon) r^{\mu(A_l)}. \quad (3.14)$$

By Eq. (1.3) and Lemma 2.2, we obtain

$$\begin{aligned} T(r, F(z)) &\leq \sum_{j=0}^k T(r, A_j(z)) + \sum_{j=1}^k T(r, f(z + c_j)) + T(r, f(z)) + O(1) \\ &\leq \sum_{j=0}^k T(r, A_j(z)) + (1 + o(1))kT(r + |c_s|, f(z)) + T(r, f(z)) + O(1) \\ &\leq \sum_{j=0, j \neq l}^k T(r, A_j(z)) + T(r, A_l(z)) + (2k+1)T(2r, f(z)), \quad |c_s| = \max_{1 \leq j \leq k} \{|c_j|\}. \end{aligned} \quad (3.15)$$

Now, we may choose sufficiently small ε satisfying $0 < \varepsilon < \frac{\tau(F) - \tau_1 - \tau}{k+2}$, by substituting Eqs. (3.6), (3.7), (3.13) and (3.14) into Eq. (3.15) that for $r \in E_3$ sufficiently large, we get

$$\begin{aligned} (\tau(F) - \varepsilon)r^\mu &\leq \sum_{j \in J} (\tau(A_j) + \varepsilon)r^\mu + \sum_{j \in \{0, 1, \dots, l-1, l+1, \dots, k\} \setminus J} r^{\mu_0} \\ &\quad + (\tau + \varepsilon)r^\mu + (2k+1)T(2r, f(z)) \\ &\leq (\tau_1 + k\varepsilon)r^\mu + kr^{\mu_0} + (\tau + \varepsilon)r^\mu + (2k+1)T(2r, f(z)). \end{aligned}$$

It follows that

$$(1 - o(1))(\tau(F) - \tau_1 - \tau - (k+2)\varepsilon)r^\mu \leq (2k+1)T(2r, f(z)),$$

which implies $\mu(f) \geq \mu(A_l)$.

(ii) Next, we consider the case $\mu(F) > \mu(A_l) = \mu$. Let f be a meromorphic solution of Eq. (1.3). Then, for any given ε ($0 < 2\varepsilon < \mu(F) - \mu$) and sufficiently large r , we have

$$T(r, F) \geq r^{\mu(F) - \varepsilon}. \quad (3.16)$$

By Lemma 2.5, for the above ε , there exists a subset E_4 with infinite logarithmic measure such that for all $r \in E_4$, we have

$$T(r, A_l) \leq r^{\mu(A_l) + \varepsilon}. \quad (3.17)$$

If $b = \max \{\rho(A_j) : j = 0, 1, \dots, k, j \neq l\} < \mu(A_l) = \mu$, then by substituting the assumptions Eqs. (3.4), (3.16) and (3.17) into Eq. (3.15), for any given ε ($0 < 2\varepsilon < \mu(F) - \mu$) and sufficiently large $r \in E_4$, we obtain

$$r^{\mu(F) - \varepsilon} \leq kr^{b+\varepsilon} + r^{\mu+\varepsilon} + (2k+1)T(2r, f(z)).$$

Thus,

$$(1 - o(1))r^{\mu(F) - \varepsilon} \leq (2k+1)T(2r, f(z)),$$

that is, $\mu(f) \geq \mu(F) - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\mu(f) \geq \mu(F)$.

If

$$\max \{\rho(A_j) : j = 0, 1, \dots, k, j \neq l\} = \mu(A_l) = \mu$$

and $\tau_1 = \sum_{\rho(A_j)=\mu(A_l), j \neq l} \tau(A_j) < \tau(A_l) = \tau$, then by substituting the assumptions Eqs. (3.6), (3.7), (3.14) and (3.16) into Eq. (3.15), for any given ε ($0 < 2\varepsilon < \mu(F) - \mu$) and sufficiently large $r \in E_3$, we obtain

$$\begin{aligned} r^{\mu(F) - \varepsilon} &\leq \sum_{j \in J} (\tau(A_j) + \varepsilon)r^\mu + \sum_{j \in \{0, 1, \dots, l-1, l+1, \dots, k\} \setminus J} r^{\mu_0} + T(r, A_l(z)) \\ &\quad + (2k+1)T(2r, f(z)) \leq (\tau_1 + k\varepsilon)r^\mu + kr^{\mu_0} + (\tau + \varepsilon)r^\mu + (2k+1)T(2r, f(z)). \end{aligned}$$

Thus,

$$(1 - o(1))r^{\mu(F) - \varepsilon} \leq (2k+1)T(2r, f(z)),$$

that is, $\mu(f) \geq \mu(F) - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\mu(f) \geq \mu(F)$. $\square$

Proof of Theorem 1.15 Let f be a meromorphic solution of Eq. (1.3).

(i) If $\rho_p(F) < \mu_p(A_l) = \mu$, then for any given ε ($0 < 2\varepsilon < \mu - \rho_p(F)$) and sufficiently large r , we have

$$T(r, F) \leq \exp_{p-1} \left\{ r^{\rho_p(F) + \varepsilon} \right\}. \quad (3.18)$$

First, we suppose that $b = \max \{\rho_p(A_j) : j = 0, 1, \dots, k, j \neq l\} < \mu_p(A_l) = \mu$. Then, for any given ε ($0 < 2\varepsilon < \mu - b$) and sufficiently large r , we have

$$T(r, A_l) \geq \exp_{p-1} \left\{ r^{\mu_p(A_l) - \varepsilon} \right\} \quad (3.19)$$

and

$$T(r, A_j) \leq \exp_{p-1} \left\{ r^{b+\varepsilon} \right\} \quad (j = 0, 1, \dots, k, j \neq l). \quad (3.20)$$

By the definition of $\lambda_p\left(\frac{1}{A_l}\right)$, for any given $\varepsilon\left(0 < 2\varepsilon < \mu - \lambda_p\left(\frac{1}{A_l}\right)\right)$ and sufficiently large r , we obtain

$$N(r, A_l) \leq \exp_{p-1} \left\{ r^{\lambda_p\left(\frac{1}{A_l}\right) + \varepsilon} \right\}. \quad (3.21)$$

By substituting the assumptions Eqs. (3.18), (3.19), (3.20) and (3.21) into Eq. (3.10), for any given $\varepsilon\left(0 < \varepsilon < \min\left\{\frac{\mu-b}{2}, \frac{\mu-\lambda_p\left(\frac{1}{A_l}\right)}{2}, \frac{\mu-\rho_p(F)}{2}\right\}\right)$ and sufficiently large $r \notin E_2$, we have

$$\begin{aligned} \exp_{p-1} \{r^{\mu-\varepsilon}\} &\leq k \exp_{p-1} \{r^{b+\varepsilon}\} + \exp_{p-1} \{r^{\rho_p(F)+\varepsilon}\} \\ &\quad + \exp_{p-1} \left\{ r^{\lambda_p\left(\frac{1}{A_l}\right) + \varepsilon} \right\} + 3(T(2r, f(z)))^2. \end{aligned}$$

So

$$(1 - o(1)) \exp_{p-1} \{r^{\mu-\varepsilon}\} \leq 3(T(2r, f(z)))^2,$$

that is, $\mu_p(f) \geq \mu - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\mu_p(f) \geq \mu_p(A_l)$. Assume

$$\begin{aligned} \max \{\rho_p(A_j) : j = 0, 1, \dots, k, j \neq l\} &= \mu_p(A_l) = \mu, \\ \tau_1 = \max \{\tau_p(A_j) : \rho_p(A_j) &= \mu_p(A_l), (j \neq l)\} < \tau_p(A_l) = \tau. \end{aligned}$$

Then, there exists a set $J \subseteq \{j = 0, 1, \dots, k\} \setminus \{l\}$ such that for $j \in J$, we have $\rho_p(A_j) = \mu_p(A_l) = \mu$ with $\tau_1 = \max \{\tau_p(A_j) : \rho_p(A_j) = \mu_p(A_l), (j \neq l)\} < \tau_p(A_l) = \tau$ and for $j \in \{0, 1, \dots, l-1, l+1, \dots, k\} \setminus J$, we have $\rho_p(A_j) < \mu_p(A_l) = \mu$. Hence, for any given $\varepsilon\left(0 < \varepsilon < \frac{\tau-\tau_1}{2}\right)$ and sufficiently large r , we have

$$T(r, A_j) \leq \exp_{p-1} \{(\tau_1 + \varepsilon) r^{\mu_p(A_l)}\}, \quad j \in J \quad (3.22)$$

and

$$T(r, A_j) \leq \exp_{p-1} \{r^{\mu_0}\}, \quad j \in \{0, 1, \dots, l-1, l+1, \dots, k\} \setminus J, \quad (3.23)$$

where $0 < \mu_0 < \mu$. By the definition of lower p -type, for the above ε and sufficiently large r , we have

$$T(r, A_l) \geq \exp_{p-1} \{(\tau - \varepsilon) r^{\mu_p(A_l)}\}. \quad (3.24)$$

By substituting the assumptions Eqs. (3.21), (3.22), (3.23), (3.24) into Eq. (3.10), for any given $\varepsilon\left(0 < 2\varepsilon < \min\left\{\mu - \lambda_p\left(\frac{1}{A_l}\right), \mu - \rho_p(F), \tau - \tau_1\right\}\right)$ and sufficiently large $r \notin E_2$, we obtain

$$\begin{aligned} \exp_{p-1} \{(\tau - \varepsilon) r^\mu\} &\leq O(\exp_{p-1} \{(\tau_1 + \varepsilon) r^\mu\}) + O(\exp_{p-1} \{r^{\mu_0}\}) \\ &\quad + \exp_{p-1} \{r^{\rho_p(F)+\varepsilon}\} + \exp_{p-1} \left\{ r^{\lambda_p\left(\frac{1}{A_l}\right) + \varepsilon} \right\} + 3(T(2r, f(z)))^2. \end{aligned}$$

So

$$(1 - o(1)) \exp_{p-1} \{(\tau - \varepsilon) r^\mu\} \leq 3(T(2r, f(z)))^2,$$

which implies $\mu_p(f) \geq \mu_p(A_l)$.

If $\rho_p(F) = \mu_p(A_l) = \mu$ and $\tau_p(F) < \tau_p(A_l) = \tau$, then for any given $\varepsilon\left(0 < \varepsilon < \frac{\tau-\tau_p(F)}{2}\right)$ and sufficiently large r , we have

$$T(r, F) \leq \exp_{p-1} \{(\tau_p(F) + \varepsilon) r^{\mu_p(A_l)}\}. \quad (3.25)$$

First, we suppose that $b = \max \{\rho_p(A_j) : j = 0, 1, \dots, k, j \neq l\} < \mu_p(A_l) = \mu$. We may choose sufficiently small ε satisfying

$$0 < \varepsilon < \min \left\{ \frac{\mu - \lambda_p\left(\frac{1}{A_l}\right)}{2}, \frac{\mu - b}{2}, \frac{\tau - \tau_p(F)}{2} \right\},$$

by substituting the assumptions Eqs. (3.20), (3.21), (3.24) and (3.25) into Eq. (3.10) for sufficiently large $r \notin E_2$, we obtain

$$\begin{aligned} \exp_{p-1} \{(\tau - \varepsilon) r^\mu\} &\leq k \exp_{p-1} \left\{ r^{b+\varepsilon} \right\} + \exp_{p-1} \{(\tau_p(F) + \varepsilon) r^\mu\} \\ &\quad + \exp_{p-1} \left\{ r^{\lambda_p\left(\frac{1}{A_l}\right)+\varepsilon} \right\} + 3 (T(2r, f(z)))^2. \end{aligned}$$

So

$$(1 - o(1)) \exp_{p-1} \{(\tau - \varepsilon) r^\mu\} \leq 3 (T(2r, f(z)))^2,$$

which implies $\mu_p(f) \geq \mu_p(A_l)$.

Now, we suppose that

$$\begin{aligned} \max \{ \rho_p(A_j) : j = 0, 1, \dots, k, j \neq l \} &= \mu_p(A_l) = \mu, \\ \tau_1 = \max \{ \tau_p(A_j) : \rho_p(A_j) &= \mu_p(A_l), (j \neq l) \} < \tau_p(A_l) = \tau. \end{aligned}$$

We may choose sufficiently small ε satisfying

$$0 < \varepsilon < \min \left\{ \frac{\mu - \lambda_p\left(\frac{1}{A_l}\right)}{2}, \frac{\tau - \tau_1}{2}, \frac{\tau - \tau_p(F)}{2} \right\},$$

by substituting the assumptions Eqs. (3.21), (3.22), (3.23), (3.24) and (3.25) into Eq. (3.10) for sufficiently large $r \notin E_2$, we obtain

$$\begin{aligned} \exp_{p-1} \{(\tau - \varepsilon) r^\mu\} &\leq O(\exp_{p-1} \{(\tau_1 + \varepsilon) r^\mu\}) + O(\exp_{p-1} \{r^{\mu_0}\}) \\ &\quad + \exp_{p-1} \{(\tau_p(F) + \varepsilon) r^\mu\} + \exp_{p-1} \left\{ r^{\lambda_p\left(\frac{1}{A_l}\right)+\varepsilon} \right\} + 3 (T(2r, f(z)))^2. \end{aligned}$$

So

$$(1 - o(1)) \exp_{p-1} \{(\tau - \varepsilon) r^\mu\} \leq 3 (T(2r, f(z)))^2,$$

which implies $\mu_p(f) \geq \mu_p(A_l)$.

If $\mu_p(F) = \mu_p(A_l) = \mu$ and $\tau_p(F) > \tau_p(A_l) = \tau$, then for any given $\varepsilon \left(0 < \varepsilon < \frac{\tau_p(F) - \tau}{2} \right)$ and sufficiently large r , we have

$$T(r, F) > \exp_{p-1} \left\{ \left(\tau_p(F) - \varepsilon \right) r^{\mu_p(A_l)} \right\}. \quad (3.26)$$

By Lemma 2.4, for the above ε there exists a subset E_3 with infinite logarithmic measure such that for all $r \in E_3$, we have

$$T(r, A_l) \leq \exp_{p-1} \left\{ \left(\tau_p(A_l) + \varepsilon \right) r^{\mu_p(A_l)} \right\}. \quad (3.27)$$

First, we suppose that $b = \max \{ \rho_p(A_j) : j = 0, 1, \dots, k, j \neq l \} < \mu_p(A_l) = \mu$. We may choose sufficiently small ε satisfying $0 < \varepsilon < \min \left\{ \frac{\mu - b}{2}, \frac{\tau_p(F) - \tau}{2} \right\}$, by substituting Eqs. (3.20), (3.26) and (3.27) into Eq. (3.15) for sufficiently large $r \in E_3$, we obtain

$$\begin{aligned} \exp_{p-1} \left\{ \left(\tau_p(F) - \varepsilon \right) r^\mu \right\} &\leq k \exp_{p-1} \left\{ r^{b+\varepsilon} \right\} \\ &\quad + \exp_{p-1} \{(\tau + \varepsilon) r^\mu\} + (2k + 1) T(2r, f(z)). \end{aligned}$$

So

$$(1 - o(1)) \exp_{p-1} \left\{ \left(\tau_p(F) - \varepsilon \right) r^\mu \right\} \leq (2k + 1) T(2r, f(z)),$$

which implies $\mu_p(f) \geq \mu_p(A_l)$.

Now, we suppose that

$$\begin{aligned} \max \{ \rho_p(A_j) : j = 0, 1, \dots, k, j \neq l \} &= \mu_p(A_l) = \mu, \\ \tau_1 = \max \{ \tau_p(A_j) : \rho_p(A_j) &= \mu_p(A_l), (j \neq l) \} < \tau_p(A_l) = \tau. \end{aligned}$$

We may choose sufficiently small ε satisfying $0 < \varepsilon < \frac{\tau_p(F) - \tau}{2}$, by substituting Eqs. (3.22), (3.23), (3.26) and (3.27) into Eq. (3.15), for sufficiently large $r \in E_3$, we obtain

$$\exp_{p-1} \left\{ \left(\tau_p(F) - \varepsilon \right) r^\mu \right\} \leq O \left(\exp_{p-1} \left\{ (\tau_1 + \varepsilon) r^\mu \right\} \right) + O \left(\exp_{p-1} \left\{ r^{\mu_0} \right\} \right) \\ + \exp_{p-1} \left\{ (\tau + \varepsilon) r^\mu \right\} + (2k + 1) T(2r, f(z)).$$

So

$$(1 - o(1)) \exp_{p-1} \left\{ \left(\tau_p(F) - \varepsilon \right) r^\mu \right\} \leq (2k + 1) T(2r, f(z)),$$

which implies $\mu_p(f) \geq \mu_p(A_l)$.

(ii) Next, we consider the case $\mu_p(F) > \mu_p(A_l) = \mu$. Let f be a meromorphic solution of Eq. (1.3). Then, for any given ε ($0 < 2\varepsilon < \mu_p(F) - \mu$) and sufficiently large r , we have

$$T(r, F) \geq \exp_{p-1} \left\{ r^{\mu_p(F) - \varepsilon} \right\}. \quad (3.28)$$

By Lemma 2.5, for the above ε , there exists a subset E_4 with infinite logarithmic measure such that for all $r \in E_4$, we have

$$T(r, A_l) \leq \exp_{p-1} \left\{ r^{\mu_p(A_l) + \varepsilon} \right\}. \quad (3.29)$$

First, we suppose that $b = \max \left\{ \rho_p(A_j) : j = 0, 1, \dots, k, j \neq l \right\} < \mu_p(A_l) = \mu$. We may choose sufficiently small ε satisfying $0 < 2\varepsilon < \mu_p(F) - \mu$, by substituting Eqs. (3.20), (3.28) and (3.29) into Eq. (3.15) for sufficiently large $r \in E_4$, we obtain

$$\exp_{p-1} \left\{ r^{\mu_p(F) - \varepsilon} \right\} \leq k \exp_{p-1} \left\{ r^{b + \varepsilon} \right\} + \exp_{p-1} \left\{ r^{\mu + \varepsilon} \right\} + (2k + 1) T(2r, f(z)).$$

So

$$(1 - o(1)) \exp_{p-1} \left\{ r^{\mu_p(F) - \varepsilon} \right\} \leq (2k + 1) T(2r, f(z)),$$

that is, $\mu_p(f) \geq \mu_p(F) - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\mu_p(f) \geq \mu_p(F)$.

Now, we suppose that

$$\max \left\{ \rho_p(A_j) : j = 0, 1, \dots, k, j \neq l \right\} = \mu_p(A_l) = \mu, \\ \tau_1 = \max \left\{ \tau_p(A_j) : \rho_p(A_j) = \mu_p(A_l), (j \neq l) \right\} < \tau_p(A_l) = \tau.$$

Now, we may choose sufficiently small ε satisfying $0 < 2\varepsilon < \mu_p(F) - \mu$, by substituting Eqs. (3.22), (3.23), (3.27) and (3.28) into Eq. (3.15) for sufficiently large $r \in E_3$, we get

$$\exp_{p-1} \left\{ r^{\mu_p(F) - \varepsilon} \right\} \leq O \left(\exp_{p-1} \left\{ (\tau_1 + \varepsilon) r^\mu \right\} \right) + O \left(\exp_{p-1} \left\{ r^{\mu_0} \right\} \right) \\ + \exp_{p-1} \left\{ (\tau + \varepsilon) r^\mu \right\} + (2k + 1) T(2r, f(z)).$$

It follows that

$$(1 - o(1)) \exp_{p-1} \left\{ r^{\mu_p(F) - \varepsilon} \right\} \leq (2k + 1) T(2r, f(z)),$$

that is, $\mu_p(f) \geq \mu_p(F) - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\mu_p(f) \geq \mu_p(F)$. $\square$

4 Examples

Example 4.1 Consider the homogeneous difference equation with meromorphic coefficients

$$A_2(z) f(z+2) + A_1(z) f(z+1) + A_0(z) f(z) = 0, \quad (4.1)$$

where

$$\begin{aligned} A_0(z) &= -e^{-z^2 \log \pi} \sec^2(z), \quad A_1(z) = e^{(4z^2+3z+1) \log \frac{1}{\pi}}, \\ A_2(z) &= e^{(7z^2+12z+8) \log \frac{1}{\pi}} \tan^2(z). \end{aligned}$$

Clearly, $A_j(z)$, $j = 0, 1, 2$ satisfy

$$\begin{aligned} \lambda\left(\frac{1}{A_2}\right) &= 1 < \mu(A_2) = 2, \\ \max\{\rho(A_j) : j = 0, 1\} &= \mu(A_2) = 2, \\ \tau(A_0) + \tau(A_1) &= \frac{\log \pi}{\pi} + \frac{4 \log \pi}{\pi} = \frac{5 \log \pi}{\pi} < \frac{7 \log \pi}{\pi} = \tau(A_2). \end{aligned}$$

As we see, the conditions of Theorem 1.13 are verified, where $l = 2$. The function

$$f(z) = e^{z^3 \log \pi}$$

is a solution of Eq. (4.1) and f satisfies $\mu(f) = 3 \geq \mu(A_2) + 1 = 2 + 1$.

Example 4.2 (i) Consider the non-homogeneous difference equation with meromorphic coefficients

$$A_2(z) f\left(z + \frac{\pi}{2}\right) + A_1(z) f\left(z - \frac{\pi}{2}\right) + A_0(z) f(z) = F(z), \quad (4.2)$$

where

$$\begin{aligned} A_0(z) &= e^{z^2+6\pi z}, \quad A_1(z) = e^{-6z^2-\frac{3}{2}\pi^2} \tan^2(4z), \\ A_2(z) &= -e^{z^2-\frac{3}{2}\pi^2}, \quad F(z) = \frac{e^{-6\pi z}}{\cos^2(4z)}. \end{aligned}$$

Case 1. $\rho(F) < \mu(A_l)$. We have

$$\begin{aligned} \lambda\left(\frac{1}{A_1}\right) &= 1 < \mu(A_1) = 2, \quad \max\{\rho(A_j) : j = 0, 2\} = \mu(A_1) = 2, \\ \sum_{\rho(A_j)=\mu(A_1), j \neq 1} \tau(A_j) &= \tau(A_0) + \tau(A_2) = \frac{2}{\pi} < \frac{6}{\pi} = \tau(A_1), \\ \rho(F) &= 1 < \mu(A_1) = 2. \end{aligned}$$

As we see, the conditions of Theorem 1.14 (i) are verified. The meromorphic function

$$f(z) = \frac{e^{6z^2}}{\sin^2(4z)}$$

is a solution of Eq. (4.2) and f satisfies

$$\mu(f) = 2 \geq \mu(A_1) = 2.$$

Case 2. $\rho(F) = \mu(A_l)$ and $\sum \tau(A_j) + \tau(F) < \tau(A_l)$. In Eq. (4.2), for

$$\begin{aligned} A_0(z) &= e^{z^2+\frac{3}{2}\pi^2}, \quad A_1(z) = e^{-8z^2+6\pi z} \tan^2(4z), \\ A_2(z) &= -e^{z^2-6\pi z}, \quad F(z) = \frac{e^{-2z^2+\frac{3}{2}\pi^2}}{\cos^2(4z)}, \end{aligned}$$

we have

$$\begin{aligned}\lambda\left(\frac{1}{A_1}\right) &= 1 < \mu(A_1) = 2, \\ \max\{\rho(A_j) : j = 0, 2\} &= \mu(A_1) = \rho(F) = 2, \\ \sum_{\rho(A_j)=\mu(A_1), j \neq 1} \tau(A_j) + \tau(F) &= \tau(A_0) + \tau(A_2) + \tau(F) = \frac{1}{\pi} + \frac{1}{\pi} + \frac{2}{\pi} = \frac{4}{\pi} < \frac{8}{\pi} = \tau(A_1).\end{aligned}$$

Hence, the conditions of Theorem 1.14 (i) are verified. We see that the meromorphic function $f(z) = \frac{e^{6z^2}}{\sin^2(4z)}$ is a solution of Eq. (4.2) that satisfies $\mu(f) = 2 \geq \mu(A_1) = 2$.

Case 3. $\mu(F) = \mu(A_l)$ and $\sum \tau(A_j) + \tau(A_l) < \tau(F)$. In Eq. (4.2), for

$$\begin{aligned}A_0(z) &= e^{z^2 + \frac{3}{2}\pi^2}, \quad A_1(z) = e^{3z^2 + 6\pi z} \tan^2(4z), \\ A_2(z) &= -e^{z^2 - 6\pi z}, \quad F(z) = \frac{e^{9z^2 + \frac{3}{2}\pi^2}}{\cos^2(4z)},\end{aligned}$$

we have

$$\begin{aligned}\lambda\left(\frac{1}{A_1}\right) &= 1 < \mu(A_1) = 2, \\ \max\{\rho(A_j) : j = 0, 2\} &= \mu(A_1) = \mu(F) = 2, \\ \sum_{\rho(A_j)=\mu(A_1), j \neq 1} \tau(A_j) + \tau(A_1) &= \tau(A_0) + \tau(A_2) + \tau(A_1) = \frac{1}{\pi} + \frac{1}{\pi} + \frac{3}{\pi} = \frac{5}{\pi} < \frac{9}{\pi} = \tau(F).\end{aligned}$$

Thus, the conditions of Theorem 1.14 (i) are satisfied. We see that the meromorphic function $f(z) = \frac{e^{6z^2}}{\sin^2(4z)}$ is a solution of Eq. (4.2) that satisfies $\mu(f) = 2 \geq \mu(A_1) = 2$. (ii) $\mu(F) > \mu(A_l)$. In Eq. (4.2), for

$$\begin{aligned}A_0(z) &= e^{3\pi z}, \quad A_1(z) = e^{8\pi z - \frac{3}{2}\pi^2}, \\ A_2(z) &= -e^{-3\pi z - \frac{3}{2}\pi^2}, \quad F(z) = \frac{e^{6z^2 + 2\pi z}}{\sin^2(4z)},\end{aligned}$$

we have

$$\begin{aligned}\lambda\left(\frac{1}{A_1}\right) &= 0 < \mu(A_1) = 1, \\ \max\{\rho(A_j) : j = 0, 2\} &= \mu(A_1) = 1, \\ \sum_{\rho(A_j)=\mu(A_1), j \neq 1} \tau(A_j) &= \tau(A_0) + \tau(A_2) = 6 < 8 = \tau(A_1), \\ \mu(F) &= 2 > \mu(A_1) = 1.\end{aligned}$$

Obviously, the conditions of Theorem 1.14 (ii) are verified. The meromorphic function

$$f(z) = \frac{e^{6z^2}}{\sin^2(4z)}$$

is solution of Eq. (4.2) and f satisfies

$$\mu(f) = 2 \geq \mu(F) = 2.$$

Example 4.3 (i) Consider the non-homogeneous difference equation with meromorphic coefficients

$$A_2(z) f(z + 2\pi i) + A_1(z) f(z + i\pi) + A_0(z) f(z) = F(z). \quad (4.3)$$

Case 1. $\rho_2(F) < \mu_2(A_1)$. In Eq. (4.3), for

$$\begin{aligned} A_0(z) &= -ze^z, \quad A_1(z) = \frac{(z + i\pi)e^{-\cosh(2z) + \cosh(z) + z}}{z}, \\ A_2(z) &= (z + 2i\pi)e^z, \quad F(z) = \frac{e^z}{z}, \end{aligned}$$

we have

$$\begin{aligned} \lambda_2\left(\frac{1}{A_1}\right) &= 0 < \mu_2(A_1) = 1, \\ \max\{\rho_2(A_j) \ (j = 0, 2), \rho_2(F)\} &= 0 < \mu_2(A_1) = 1. \end{aligned}$$

As we see, the conditions of Theorem 1.15 (i) are verified. The meromorphic function

$$f(z) = \frac{e^{\cosh(2z) + \cosh(z)}}{z}$$

is a solution of Eq. (4.3) and f satisfies

$$\mu_2(f) = 1 \geq \mu_2(A_1) = 1.$$

Case 2. $\rho_2(F) = \mu_2(A_1)$ and $\tau_2(F) < \tau_2(A_1)$. In Eq. (4.3), for

$$\begin{aligned} A_0(z) &= -ze^{\sin(\frac{z}{\pi}) + z}, \quad A_1(z) = \frac{(z + i\pi)e^{-\cosh(2z)}}{z}, \\ A_2(z) &= (z + 2i\pi)e^{\sin(\frac{z}{\pi}) + z}, \quad F(z) = \frac{e^{-\cosh(z)}}{z}, \end{aligned}$$

we have

$$\begin{aligned} \lambda_2\left(\frac{1}{A_1}\right) &= 0 < \mu_2(A_1) = 1, \quad \max\{\rho_2(A_j) \ (j = 0, 2), \rho_2(F)\} = \mu_2(A_1) = 1, \\ \tau_2(F) &= \frac{1}{\pi} < \frac{2}{\pi} = \tau_2(A_1). \end{aligned}$$

Thus, the conditions of Theorem 1.15 (i) are verified. The meromorphic function

$$f(z) = \frac{e^{\cosh(2z) + \cosh(z)}}{z}$$

is a solution of Eq. (4.3) and f satisfies

$$\mu_2(f) = 1 \geq \mu_2(A_1) = 1.$$

Case 3. $\mu_2(F) = \mu_2(A_1)$ and $\tau_2(F) > \tau_2(A_1)$. In Eq. (4.3), for

$$\begin{aligned} A_0(z) &= -ze^{\sin(\frac{z}{\pi}) + z}, \quad A_1(z) = \frac{(z + i\pi)e^{\cosh(z)}}{z}, \\ A_2(z) &= (z + 2i\pi)e^{\sin(\frac{z}{\pi}) + z}, \quad F(z) = \frac{e^{\cosh(2z)}}{z}, \end{aligned}$$

we have

$$\begin{aligned}\lambda_2\left(\frac{1}{A_1}\right) &= 0 < \mu_2(A_1) = 1, \\ \max\{\rho_2(A_j) \mid (j = 0, 2), \mu_2(F)\} &= \mu_2(A_1) = 1, \\ \tau_2(F) &= \frac{2}{\pi} > \frac{1}{\pi} = \tau_2(A_1).\end{aligned}$$

Hence, the conditions of Theorem 1.15 (i) are verified. The meromorphic function

$$f(z) = \frac{e^{\cosh(2z) + \cosh(z)}}{z}$$

is a solution of Eq. (4.3) and f satisfies

$$\mu_2(f) = 1 \geq \mu_2(A_1) = 1.$$

(ii) $\mu_2(F) > \mu_2(A_l)$. In Eq. (4.3), for

$$\begin{aligned}A_0(z) &= -ze^z, \quad A_1(z) = \frac{(z + i\pi)e^{\cosh(\sqrt{z})}}{z}, \\ A_2(z) &= (z + 2i\pi)e^z, \quad F(z) = \frac{e^{\cosh(2z) - \cosh(z) + \cosh(\sqrt{z})}}{z},\end{aligned}$$

we have

$$\begin{aligned}\lambda_2\left(\frac{1}{A_1}\right) &= 0 < \mu_2(A_1) = \frac{1}{2}, \quad \max\{\rho_2(A_j) : j = 0, 2\} = 0 < \mu_2(A_1) = \frac{1}{2}, \\ \mu_2(F) &= 1 > \mu_2(A_1) = \frac{1}{2}.\end{aligned}$$

Obviously, the conditions of Theorem 1.15 (ii) are verified. The meromorphic function

$$f(z) = \frac{e^{\cosh(2z) + \cosh(z)}}{z}$$

is a solution of Eq. (4.3) and f satisfies

$$\mu_2(f) = 1 \geq \mu_2(F) = 1.$$

5 Conclusion and perspectives

In this paper, we do not consider the case when there is no dominating coefficient. In [10], Laine and Yang raised the following question : Whether all meromorphic solutions $f (\not\equiv 0)$ of Eq. (1.2) satisfy

$$\rho(f) \geq \max_{0 \leq l \leq k} \{\rho(A_l)\} + 1,$$

even if there is no dominating coefficient. In the paper [12], Lan and Chen considered the case when there are no dominating coefficients for the difference equation

$$A_1(z)f(z+1) + A_0(z)f(z) = 0,$$

where $A_1(z)$ and $A_0(z)$ are entire functions of finite order. Under some restrictions on the coefficients, they gave an answer to the posed question, see also the papers [11, 18]. So, it is interesting further research in this direction.

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