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Generalized Hsiung–Minkowski formulae on manifolds with density

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Abstract In this work, using the weighted symmetric functions σ_k^∞ and the weighted Newton transformations T_k^∞ introduced by Case (Alias et al. Proc Edinb Math Soc 46(02):465–488, 2003), we derive some generalized integral formulae for close hypersurfaces in weighted manifolds. We also give some examples and applications of these formulae.

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1 Introduction

The classical integral identities of Minkowski type (see [6,17,19,24]) read as follows. Let $x : M^n \longrightarrow \overline{M}^{n+1}$ be a close oriented hypersurface immersed into a space form $\overline{M}^{n+1}$ with a unit normal vector field N. Then we have for $1 \leq r \leq n$:

$$\int\limits_{M^n} H_{r-1} \mathrm{d}v + \int\limits_{M^n} \langle Y, N\rangle\, H_r \mathrm{d}v = 0 \tag{1}$$

where Y is a conformal vector field. i.e. there exists a smooth function ϕ, such as

$$\mathcal{L}_Y \langle , \rangle = 2\phi \langle , \rangle$$

and H_r denotes the rth mean curvatures of M^n.

Minkowski formulae for hypersurfaces were first obtained by Hsiung [17] in the Euclidian space (generalizing Minkowski result for $r = 0$) and later by Bivens [11] in the Euclidean sphere and hyperbolic space. These results were generalized by Alencar and Colares [3] by using the $(r + 1)$-mean curvature linearized operator L_r of the hypersurface.

If the ambient space is not a space form, then the rth Minkowski formula is given by (see [6,22])

$$\int\limits_{M^n} \langle \operatorname{div} T_k, Y\rangle \,\mathrm{d}v + c_r \int\limits_{M^n} (\phi H_k + \langle Y, N\rangle\, H_{k+1})\,\mathrm{d}v = 0 \tag{2}$$

where $c_r = (r+1)\binom{n}{r+1}$.

Surprisingly, many geometric results can be deduced from these simple formulas, such as Alexandrov's theorem or characterizations of certain hypersurfaces (see [2,5,8,9,17,18,20,23,24]).

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It is interesting to know if the formulae (1) and (2) can be extended in other cases, and applied to generalize the aforementioned results.

In this work, using the weighted symmetric functions σ_k^∞ and the weighted Newton transformations T_k^∞ introduced by Case [12], we obtain some integral formulae on weighted manifolds. These formulae generalize (1) and (2). We also give some special cases and applications of these formulae.

Recall that a weighted manifold is a triplet $(M^n, \langle,\rangle, dv_f = e^{-f}dv)$, where M^n is a complete n- dimensional Riemannian manifold, dv is the standard volume element of M^n and $f : M^n \longrightarrow \mathbb{R}$ is a smooth function.

2 Preliminaries

In this section we collect some basic facts and definitions about manifolds with density which are needed in this article. We also give the definitions and some properties of the weighted symmetric functions and the weighted Newton transformations. For more details see [4,12–15,21,25].

Let $x : M^n \longrightarrow \overline{M}^{n+1}$ be an isometric immersion of a closed oriented n-dimensional Riemannian manifold M^n into an $(n+1)$-dimensional weighted Riemannian manifold $(\overline{M}^{n+1}, \langle,\rangle, dv_f)$.

The function $f : \overline{M}^{n+1} \longrightarrow \mathbb{R}$ restricted to M^n induces a weighted measure $e^{-f}dv$ on M^n. Thus, we have an induced weighted manifold $M_f^n = (M^n, \langle,\rangle, e^{-f}dv)$; where dv is the standard volume element of M^n.

We define the second fundamental form (or the shape operator) A of x with respect to the Gauss map N by :

$$AX = -\left(\overline{\nabla}_X N\right)^{\intercal}$$

where $\intercal$ symbolizes the projection above the tangent bundle of M^n and $\overline{\nabla}$ is the Levi-Civita connection of the metric of $\overline{M}^{n+1}$.

It is well known that A is a linear self-adjoint operator and at each point $p \in M^n$, it has real eigenvalues $\mu_1, ..., \mu_n$ (the principal curvatures).

The weighted elementary symmetric functions $\sigma_k^\infty : \mathbb{R}\times\mathbb{R}^n \longrightarrow \mathbb{R}$ are defined recursively [12] by :

$$\begin{cases} \sigma_0^\infty(\mu_0,\mu) = 1, \\ k\sigma_k^\infty(\mu_0,\mu) = \sigma_{k-1}^\infty(\mu_0,\mu)\sum_{j=0}^n \mu_j + \sum_{i=1}^{k-1}\sum_{j=1}^n (-1)^i\,\sigma_{k-1-i}^\infty(\mu_0,\mu)\mu_j^i \qquad \text{for } k\geq 1 \end{cases}$$

where $\mu_0 \in \mathbb{R}$ and $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$. In particular for $\mu_0 = 0$, $\sigma_k^\infty(0,\mu) = \sigma_k(\mu)$ are the classical elementary symmetric functions defined in [23].

Definition 2.1 [12] The weighted Newton transformations $T_k^\infty(\mu_0,\mu)$ are defined inductively from A by :

$$\begin{cases} T_0^\infty(\mu_0,\mu) = I \\ T_k^\infty(\mu_0,\mu) = \sigma_k^\infty(\mu_0,\mu)I - AT_{k-1}^\infty(\mu_0,\mu) \quad \text{for } k\geq 1 \end{cases}$$

or equivalently

$$T_k^\infty(\mu_0,\mu) = \sum_{j=0}^k (-1)^j\,\sigma_{k-j}^\infty(\mu_0,\mu)A^j$$

where I stands for the identity on the Lie algebra of vector fields $\varkappa(M)$, $\sigma_k^\infty(\mu_0, A) = \sigma_k^\infty(\mu_0,\mu_1,\ldots,\mu_n)$ and $\mu_1,\ldots,\mu_n$ are the eigenvalues of A.

It should be noted that $T_k^\infty(0, A) = T_k(A)$ is the classical Newton transformations introduced in [23].

We have the following properties of σ_k^∞ and T_k^∞ (see [12] for the proof).

Proposition 2.2 [12] *For $\mu_0, \mu_1 \in \mathbb{R}$ and $\mu \in \mathbb{R}^n$, we have*

$$\sigma_k^\infty(\mu_0+\mu_1,\mu) = \sum_{j=0}^k \frac{\mu_1^j}{j!}\sigma_{k-j}^\infty(\mu_0,\mu).$$

In particular,

$$\sigma_k^{\infty}(\mu_1, \mu) = \sum_{j=0}^{k} \frac{\mu_1^j}{j!} \sigma_{k-j}(\mu) \tag{2.1}$$

$$trace(AT_k^{\infty}(\mu_0, \mu)) = (k+1)\sigma_{k+1}^{\infty}(\mu_0, \mu) - \mu_0 \sigma_k^{\infty}(\mu_0, \mu). \tag{2.2}$$

For $i \in \{1, \ldots, n\}$ *we have*

$$\sigma_{k,i}^{\infty}(\mu_0, \mu) = \sigma_k^{\infty}(\mu_0, \mu) - \mu_i \sigma_{k-1,i}^{\infty}(\mu_0, \mu)$$

and the i*th eigenvalue of* $T_k^{\infty}(\mu_0, \mu)$ *is equal to* $\sigma_{k,i}^{\infty}(\mu_0, \mu)$, *where* $\sigma_{k,i}^{\infty}(\mu_0, \mu) = \sigma_k^{\infty}(\mu_0, \mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \ldots, \mu_n)$.

For $k \geq 1$ *we have :*

$$\sigma_{k-1}^{\infty}(\mu_0, \mu).\sigma_{k+1}^{\infty}(\mu_0, \mu) \leq \frac{k}{k+1}\left(\sigma_k^{\infty}(\mu_0, \mu)\right)^2 \tag{2.3}$$

with equality if and only if :

(1) $\mu = 0$ *or*
(2) $\mu_0 = 0$ *and, up to reindexing, it holds that* $\mu_1 = \cdots = \mu_{n+1-k} = 0$.

Puting $\mu_0 = \langle \nabla f, N \rangle$, we can see that

$$\sigma_1^{\infty}(\langle \nabla f, N \rangle, A) = \sigma_1(A) + \langle \nabla f, N \rangle$$

is nothing but the definition of the (normalized) weighted mean curvature of the hypersurface M^n studied by Gromov [16].

The variations of a functional whose integrant is the rth weighted curvature on the hypersurface of a closed Riemannian manifold was given in [10].

The rest of this section will be devoted to computing the divergence of the weighted Newton transformation T_k^{∞}. For this purpose recall that the divergence of the weighted Newton transformations is defined by :

$$\operatorname{div}_f T_k^{\infty} = e^f \operatorname{div}\left(e^{-f} T_k^{\infty}\right)$$

where

$$\operatorname{div}\left(T_k^{\infty}\right) = \operatorname{trace}\left(\nabla T_k^{\infty}\right) = \sum_{i=0}^{n} \nabla_{e_i}\left(T_k^{\infty}\right)(e_i)$$

and $\{e_1, ..., e_n\}$ is a local orthonormal frame of the tangent space of M^n.

Lemma 2.3 *The weighted divergence of the weighted Newton transformations* T_k^{∞} *are inductively given by the following formula*

$$\operatorname{div}_f T_0^{\infty} = -\nabla f$$

and

$$\operatorname{div}_f T_k^{\infty} = -\sigma_k^{\infty} \nabla f + \sigma_{k-1}^{\infty} \nabla \mu_0 - A \operatorname{div}_f T_{k-1}^{\infty} - \sum_{i=1}^{n} \left(\overline{R}(N, T_{k-1}^{\infty}(e_i))e_i\right)^{\top} \quad \text{for } k \geq 1.$$

For the proof see [1].

Corollary 2.4 *If* $\overline{M}$ *has constant sectional curvature, then*

$$\operatorname{div}_f T_k^{\infty} = -T_k^{\infty}(\nabla f) + T_{k-1}^{\infty}(\nabla \mu_0).$$

3 Main results

In this section we will derive some general integral formulae for close oriented hypersurface M^n in a weighted manifold $\overline{M}^{n+1}$. Our idea here is to compute the weighted divergence $\operatorname{div}_f\left(T_k^\infty Y^\top\right)$ and $\left\langle \operatorname{div}_f T_k^\infty, Y\right\rangle$, where Y is a conformal vector field. Let $x : M^n \longrightarrow \overline{M}_f^{n+1}$ be an n-dimensional close oriented hypersurface in an $(n+1)$-dimensional weighted Riemannian manifold $\overline{M_f}^{n+1}$.

Let $p \in M^n$ and $\{e_1, ..., e_n\}$ be an orthonormal basis of T_pM^n. We can choose a global vector field N such that $\{e_1, ..., e_{n-1}, N\}$ is an orthonormal basis of $T_p\overline{M}^{n+1}$.

Suppose now the existence of a closed conformal vector field Y on $\overline{M}^{n+1}$; that is to say there exists a $\phi \in C^\infty(\overline{M}^{n+1})$ such that

$$\overline{\nabla}_V Y = \phi V$$

or equivalently

$$\left\langle \overline{\nabla}_V Y, W\right\rangle + \left\langle \overline{\nabla}_W Y, V\right\rangle = 2\phi \langle V, W\rangle$$

for every vector fields V, W over $\overline{M}^{n+1}$.

If $\{e_1, ..., e_n\}$ is an orthonormal basis of T_pM^n that diagonalizes A, then

$$\begin{aligned}
\left\langle \operatorname{div}_f T_k^\infty, Y\right\rangle &= \left\langle \operatorname{div}_f T_k^\infty, Y^\top + \langle Y, N\rangle N\right\rangle \\
&= \left\langle e^f \operatorname{div}\left(e^{-f} T_k^\infty\right), Y^\top\right\rangle \\
&= e^f \operatorname{div}\left(e^{-f} T_k^\infty Y^\top\right) - \sum_{i=0}^{n} \left\langle T_k^\infty e_i, \nabla_{e_i} Y^\top\right\rangle \\
&= e^f \operatorname{div}\left(e^{-f} T_k^\infty Y^\top\right) - \sum_{i=0}^{n} \left\langle \sigma_{k,i}^\infty e_i, \nabla_{e_i} Y^\top\right\rangle \\
&= \operatorname{div}_f\left(T_k^\infty Y^\top\right) - \sum_{i=0}^{n} \left\langle e_i, \nabla_{\sigma_{k,i}^\infty e_i} Y^\top\right\rangle \\
&= \operatorname{div}_f\left(T_k^\infty Y^\top\right) - \sum_{i=0}^{n} \left\langle e_i, \nabla_{T_k^\infty e_i} Y^\top\right\rangle .
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
2\phi \left\langle T_k^\infty W, W\right\rangle &= \left\langle \overline{\nabla}_{T_k^\infty W} Y, W\right\rangle + \left\langle \overline{\nabla}_W Y, T_k^\infty W\right\rangle \\
&= \left\langle \overline{\nabla}_{T_k^\infty W} Y^\top, W\right\rangle + \left\langle \overline{\nabla}_W Y^\top, T_k^\infty W\right\rangle - 2\langle Y, N\rangle \left\langle A T_k^\infty W, W\right\rangle
\end{aligned}$$

which implies

$$\left\langle \overline{\nabla}_{T_k^\infty W} Y^\top, W\right\rangle + \left\langle \overline{\nabla}_W Y^\top, T_k^\infty W\right\rangle = 2\phi \left\langle T_k^\infty W, W\right\rangle + 2\langle Y, N\rangle \left\langle A T_k^\infty W, W\right\rangle .$$

This gives

$$\begin{aligned}
\operatorname{div}_f\left(T_k^\infty Y^\top\right) &= \left\langle \operatorname{div}_f T_k^\infty, Y\right\rangle + \sum_{i=0}^{n} \left(\phi \left\langle T_k^\infty e_i, e_i\right\rangle + \langle Y, N\rangle \left\langle A T_k^\infty e_i, e_i\right\rangle\right) \\
&= \left\langle \operatorname{div}_f T_k^\infty, Y\right\rangle + \phi\, tr T_k^\infty + \langle Y, N\rangle \operatorname{tr}\left(A T_k^\infty\right) . \qquad (3.1)
\end{aligned}$$

And in virtue of formula (2.2) we have

$$\operatorname{trace} T_k^\infty = n\sigma_k^\infty - \operatorname{trace}\left(A T_{k-1}^\infty\right)$$

$$= (n-k)\,\sigma_k^{\infty} + \mu_0 \sigma_{k-1}^{\infty}.$$

So

$$\begin{aligned}\operatorname{div}_f\left(T_k^{\infty} Y^{\mathsf{T}}\right) = &\left\langle \operatorname{div}_f T_k^{\infty}, Y\right\rangle + \phi\left[(n-k)\,\sigma_k^{\infty} + \langle \nabla f, N\rangle\, \sigma_{k-1}^{\infty}\right] \\ &+ \langle Y, N\rangle \left[(k+1)\,\sigma_{k+1}^{\infty} - \langle \nabla f, N\rangle\, \sigma_k^{\infty}\right].\end{aligned}$$

Integrating the two sides of this latter equality and applying the divergence theorem, we obtain for $1 \le k \le n-1$,

$$\begin{aligned}&\int_{M^n} \left\langle \operatorname{div}_f T_k^{\infty}, Y\right\rangle \mathrm{d}v_f + \int_{M^n} \phi\left[(n-k)\,\sigma_k^{\infty} + \langle \nabla f, N\rangle\, \sigma_{k-1}^{\infty}\right] \mathrm{d}v_f \\ &\quad + \int_{M^n} \langle Y, N\rangle \left[(k+1)\,\sigma_{k+1}^{\infty} - \langle \nabla f, N\rangle\, \sigma_k^{\infty}\right] \mathrm{d}v_f = 0.\end{aligned}$$

Consequently, we have the following proposition:

Proposition 3.1 *Let $x : M^n \longrightarrow \overline{M}^{n+1}$ be a closed oriented hypersurface in $\overline{M}^{n+1}$. Denoting by N a global vector fields normal to M^n, then for $1 \le k \le n-1$ and for every closed conformal vector field Y on $\overline{M}^{n+1}$, we have :*

$$\begin{aligned}&\int_{M^n} \left\langle \operatorname{div}_f T_k^{\infty}, Y\right\rangle \mathrm{d}v_f + \int_{M^n} \phi\left[(n-k)\,\sigma_k^{\infty} + \langle \nabla f, N\rangle\, \sigma_{k-1}^{\infty}\right] \mathrm{d}v_f \\ &\quad + \int_{M^n} \langle Y, N\rangle \left[(k+1)\,\sigma_{k+1}^{\infty} - \langle \nabla f, N\rangle\, \sigma_k^{\infty}\right] \mathrm{d}v_f = 0. \qquad (3.2)\end{aligned}$$

This formula generalizes the kth Minkowski formula for the non weighted case [6].

Corollary 3.2 *: Let $\varphi : M^n \longrightarrow \overline{M}^{n+1}$ be a closed oriented hypersurface of $\overline{M}^{n+1}$. Denoting by N the global vector fields normal to M^n, if f is constant, then for $1 \le k \le n-1$ and for every closed conformal vector field Y on $\overline{M}^{n+1}$, we have :*

$$\int_{M^n} \left\langle \operatorname{div}_f T_k, Y\right\rangle \mathrm{d}v + c_k \int_{M^n} \left(\phi H_k + \langle Y, N\rangle\, H_{k+1}\right) \mathrm{d}v = 0$$

where $c_k = (k+1)\binom{n}{k+1}$.

If $\overline{M}^{n+1}$ has constant sectional curvature, then by Corollary (1), we obtain:

Proposition 3.3 *Under the hypothesis of Proposition 3.1, if $\overline{M}^{n+1}$ has constant sectional curvature, then*

$$\begin{aligned}&-\int_{M^n} \left\langle \nabla f, T_k^{\infty} Y\right\rangle \mathrm{d}v_f + \int_{M^n} \left\langle \nabla \mu_0, T_{k-1}^{\infty} Y\right\rangle \mathrm{d}v_f + \int_{M^n} \phi\left[(n-k)\,\sigma_k^{\infty} + \langle \nabla f, N\rangle\, \sigma_{k-1}^{\infty}\right] \mathrm{d}v_f \\ &+ \int_{M^n} \langle Y, N\rangle \left[(k+1)\,\sigma_{k+1}^{\infty} - \langle \nabla f, N\rangle\, \sigma_k^{\infty}\right] \mathrm{d}v_f = 0\end{aligned}$$

with $\mu_0 = \langle \nabla f, N\rangle$.

Formula (3.2) becomes simple when $\overline{M}^{n+1}$ has constant sectional curvature and Y is a Killing vector field, that is $\phi = 0$. In that case we have :

Corollary 3.4 . *If $\overline{M}^{n+1}$ has constant sectional curvature, then for every Killing vector filed Y, we have for $1 \leq k \leq n-1$:*

$$-\int_{M^n} \left\langle \nabla f, T_k^{\infty} Y \right\rangle \mathrm{d}v_f + \int_{M^n} \left\langle \nabla \mu_0, T_{k-1}^{\infty} Y \right\rangle \mathrm{d}v_f + \int_{M^n} \langle Y, N \rangle \left[(k+1)\, \sigma_{k+1}^{\infty} - \langle \triangledown f, N \rangle\, \sigma_k^{\infty} \right] \mathrm{d}v_f = 0. \tag{3.3}$$

4 Examples and applications

Example 4.1 Suppose that the Killing vector field Y never vanishes. If the distribution :

$$p \in M \longrightarrow D(p) = \left\{ V \in T_p M, \langle Y(p), V \rangle = 0 \right\}$$

has constant rank n, and it is integrable, then it determines a codimension 1 Riemannian foliation $F(Y)$ oriented by $\frac{Y}{\|Y\|}$. Set $f = \|Y\|^2$. Since Y is a Killing vector field, we have :

$$\langle \nabla f, Y \rangle = 0$$

Hence, it is easy to see by (5) that each leaf of the foliation $F(Y)$ satisfies :

$$H = 0 = \langle \nabla f, N \rangle$$

where $N = \frac{Y}{\|Y\|}$.

Taking $k = 0$ in (3.1) and applying the divergence theorem,we obtain for every Killing vector filed Y :

$$-\int_{M^n} \langle \nabla f, Y \rangle\, \mathrm{d}v_f + \int_{M^n} \langle Y, N \rangle\, \sigma_1^{\infty} \mathrm{d}v_f - \int_{M^n} \langle Y, N \rangle \langle \triangledown f, N \rangle\, \mathrm{d}v_f = 0. \tag{4.1}$$

If the mean curvature σ_1^{∞} is constant, multiplying by the constant σ_1^{∞}, the last equation allows us to write :

$$-\int_{M^n} \langle \nabla f, Y \rangle\, \sigma_1^{\infty} \mathrm{d}v_f + \int_{M^n} \langle Y, N \rangle \left(\sigma_1^{\infty} \right)^2 \mathrm{d}v_f - \int_{M^n} \langle Y, N \rangle \langle \triangledown f, N \rangle\, \sigma_1^{\infty} \mathrm{d}v_f = 0.$$

On the other hand, for $k = 1$, (3.3) gives :

$$-\int_{M^n} \langle \nabla f, Y \rangle\, \sigma_1^{\infty} \mathrm{d}v_f + \int_{M^n} \langle \nabla f, AY \rangle\, \mathrm{d}v_f + \int_{M^n} \langle \nabla \mu_0, Y \rangle\, \mathrm{d}v_f + \int_{M^n} \langle Y, N \rangle\, 2\sigma_2^{\infty} \mathrm{d}v_f - \int_{M^n} \langle Y, N \rangle \langle \triangledown f, N \rangle\, \sigma_1^{\infty} \mathrm{d}v_f = 0.$$

So that subtracting these two formulae we obtain that :

$$\int_{M^n} \langle Y, N \rangle \left[\left(\sigma_1^{\infty} \right)^2 - 2\sigma_2^{\infty} \right] \mathrm{d}v_f - \int_{M^n} \langle (A \nabla f + \nabla \langle \triangledown f, N \rangle), Y \rangle\, \mathrm{d}v_f = 0.$$

It is not difficult to prove that :

$$\langle (A \nabla f + \nabla \langle \triangledown f, N \rangle), Y \rangle = \langle \nabla_Y \triangledown f, N \rangle$$

and in virtue of (*2.3*), we have that

$$\left(\sigma_1^{\infty} \right)^2 - 2\sigma_2^{\infty} \geq 0$$

with equality if and only if M^n is totally geodesic, or $\langle \triangledown f, N \rangle = 0$ and M^n is totally geodesic.

So in both cases M^n is totally geodesic.

Proposition 4.2 *: Let $x : M^n \longrightarrow \overline{M}^{n+1}$ a close oriented hypersurface in a weighted manifold $\overline{M_f}^{n+1}$ of constant sectional curvature. If the weighted mean curvature σ_1^∞ of M^n is constant and there exists a Killing vector fields Y satisfies $\langle \nabla_Y \nabla f, N\rangle = 0$. Then M^n is totally geodesic.*

If $\overline{M}^{n+1} = \mathbb{R}^{n+1}$, denoting by Ω be the compact domain whose boundary is $x(M^n)$, and N the global vector fields normal to M^n. We have $\operatorname{div}_\Omega Y = (n+1)$ and $\operatorname{div}_f Y = \operatorname{div}_\Omega Y - \langle \nabla f, Y\rangle = (n+1) - \langle \nabla f, Y\rangle$.

By applying the weighted version of the divergence theorem, we have :

$$\int_{M^n} \langle Y, N\rangle \, \mathrm{d}v_f = \int_\Omega \operatorname{div}_f Y \mathrm{d}v_f = (n+1)\, vol_f \Omega - \int_\Omega \langle \nabla f, Y\rangle \, \mathrm{d}v_f.$$

If M^n has constant strictly positive weighted mean curvature, we can choose Y as unit vector field and we obtain :

$$(n+1)\operatorname{vol}_f \Omega = \frac{1}{\sigma_1^\infty}\int_{M^n} \langle \nabla f, Y\rangle \, \mathrm{d}v_f + \int_\Omega \langle \nabla f, Y\rangle \, \mathrm{d}v_f \le \frac{1}{\sigma_1^\infty}\operatorname{vol}_f M + \operatorname{vol}_f \Omega$$

which implies that

$$n\operatorname{vol}_f \Omega \le \frac{1}{\sigma_1^\infty}\operatorname{vol}_f M.$$

Proposition 4.3 *Let $x : M^n \longrightarrow \mathbb{R}^{n+1}$ be a close oriented hypersurface in $\mathbb{R}^{n+1}$ with positive constant weighted mean curvature σ_1^∞. Then we have :*

$$\sigma_1^\infty.\operatorname{vol}_f \Omega \le \frac{1}{n}\operatorname{vol}_f M.$$

Moreover, the equality holds if and only if M^n is part of round sphere with f constant.

This result was also obtained by [7, Corollary 1.3] using a different argument.

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