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Orders of automorphisms of smooth plane curves for the automorphism groups to be cyclic

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Abstract For a fixed integer $d \geq 4$, the list of groups that appear as automorphism groups of smooth plane curves whose degree is d is unknown, except for $d = 4$ or 5 . Harui showed a certain characteristic about structures of automorphism groups of smooth plane curves. Badr and Bars began to study for certain orders of automorphisms and try to obtain exact structures of automorphism groups of smooth plane curves. In this paper, based on the result of T. Harui, we extend Badr–Bars study for different and new cases, mainly for the cases of cyclic groups that appear as automorphism groups.

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1 Introduction

In this paper, we work over $\mathbb{C}$. Let C be a smooth plane curve of degree $d \geq 4$, and $\text{Aut}(C)$ be the automorphism group of C . The group $\text{Aut}(C)$ is a finite subgroup of the group $\text{PGL}(3, \mathbb{C}) = \text{Aut}(\mathbb{P}^2)$ [17]. For $d = 4$, Henn [11] and Kuribayashi–Komiya [14] independently give a list of precise automorphism groups that appear in a smooth plane curve of degree 4, see [1] for a survey of both results for quartics. For $d = 5$, Badr and Bars give a precise list [2]. In [6], Harui provides a list of group structures that may be displayed as automorphism groups of smooth plane curves of degree $d \geq 4$. Part of his list are expressed using exact sequences of groups, and exact structures are unknown. Especially when the degree d is fixed, his list does not clearly state group structures which are actually realized as automorphism groups of smooth plane curves of degree d .

For an automorphism $\sigma \in \text{Aut}(C)$, we write the order of σ as $\text{ord}(\sigma)$. In this paper, we study the group structure of $\text{Aut}(C)$ from $\text{ord}(\sigma)$. In [3], Badr and Bars shows that $\text{ord}(\sigma)$ divides one of $(d-2)(d-1)+1$, $(d-2)d$, $(d-1)^2$, and $(d-1)d$. In particular, when $\text{ord}(\sigma)$ is one of $(d-2)(d-1)+1$, $(d-2)d$, $(d-1)^2$, and $(d-1)d$, they show that the pair C and $\text{Aut}(C)$ are uniquely determined.

We fix $d \geq 4$, and let $\mathcal{G}_d$ be the set consisting of cyclic groups Cyc_m , dihedral groups D_m , the tetrahedral group A_4 , the octahedral group S_4 , and the icosahedral group A_5 where $m \leq d-1$.

Let F_d be the Fermat curve defined by $X^d + Y^d + Z^d = 0$, and K_d be the Klein curve defined by $ZX^{d-1} + XY^{d-1} + YZ^{d-1} = 0$. The group structures of the automorphism groups F_d and K_d are known [4, 19].

In [3], Badr and Bars study the group structure of $\text{Aut}(C)$ when $\text{ord}(\sigma) = k(d-2)$, $k(d-1)$, or kd for $k \geq 2$, and show the following theorem.

Theorem 1.1 (Please see [3] for details) *Let C be a smooth plane curve of degree $d \geq 4$, $\text{Aut}(C)$ be the automorphism group of C , $\sigma \in \text{Aut}(C)$ be an automorphism, and $k \geq 2$.*

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1. If $\text{ord}(\sigma) = k(d-1)$, then $\text{Aut}(C)$ is a cyclic group.
2. If $\text{ord}(\sigma) = kd$, then $\text{Aut}(C)$ is a subgroup of $\text{Aut}(F_d)$ or there is a normal subgroup H of $\text{Aut}(C)$ such that the quotient group $\text{Aut}(C)/H$ is one of $\mathcal{G}_d$.
3. If $\text{ord}(\sigma) = k(d-2)$, $d > 6$, and $d \neq 10$, then there is a normal cyclic group H such that the quotient group $\text{Aut}(C)/H$ is a dihedral group.

Our first main theorem is the following Theorem 1.2 which is a continuation work of Theorem 1.1 based on [6]. We write the greatest common divisor of positive integers s and t as $\gcd(s, t)$. Let C be a smooth plane curve of degree $d \geq 4$, and σ be the automorphism group of C such that $\text{ord}(\sigma) \geq 4$. Notice that if $\gcd(\text{ord}(\sigma), d-1) \geq 2$, then $\text{ord}(\sigma)$ divides $(d-1)d$ or $(d-1)^2$, if $\gcd(\text{ord}(\sigma), d) \geq 2$, then $\text{ord}(\sigma)$ divides $(d-1)d$ or $(d-2)d$, and if $\gcd(\text{ord}(\sigma), d-2) \geq 2$, then $\text{ord}(\sigma)$ divides $(d-2)d$.

Theorem 1.2 *Let C be a smooth plane curve of degree $d \geq 4$, and $\text{Aut}(C)$ be the automorphism group of C . We assume that there is an automorphism $\sigma \in \text{Aut}(C)$ such that $\text{ord}(\sigma) \geq 8$.*

1. If $\text{ord}(\sigma)$ divides $(d-1)^2$ but does not divide $d-1$, then $\text{Aut}(C)$ is a cyclic group.
2. If $\gcd(\text{ord}(\sigma), d) \geq 2$, and $\gcd(\text{ord}(\sigma), d-1) \geq 3$, then $\text{Aut}(C)$ is a cyclic group or there is a normal subgroup H of $\text{Aut}(C)$ such that the quotient group $\text{Aut}(C)/H$ is the tetrahedral group A_4 . In particular, if $\gcd(\text{ord}(\sigma), d) \geq 2$ and $\gcd(\text{ord}(\sigma), d-1) \geq 4$, then $\text{Aut}(C)$ is a cyclic group.
3. If $\gcd(\text{ord}(\sigma), d-2) \geq 12$, then $\text{Aut}(C)$ is a cyclic group, or there is a normal cyclic group H of $\text{Aut}(C)$ such that the quotient group $\text{Aut}(C)/H$ is a dihedral group.

Notice that if the order $\text{ord}(\sigma)$ of σ divides $(d-1)^2$ but not $d-1$, then $\text{ord}(\sigma)$ is not the product of different prime numbers, i.e. $\text{ord}(\sigma) \neq p_1 \times \cdots \times p_r$ where p_i is a prime number, and $p_i \neq p_j$ for $1 \leq i < j \leq r$. In particular, $\text{ord}(\sigma) \neq 2, 3, 5, 6, 7$. For $d \geq 6$, the (1) and (2) of Theorem 1.2 uses smaller orders than the (1) of Theorem 1.1 to give sufficient conditions for $\text{Aut}(C)$ to be a cyclic group. Please see Examples 3.4 and 3.6. When $\text{ord}(\sigma)$ is not a multiple of $d-2$, either case of the (3) of Theorem 1.2 is occurred. Please see Examples 3.9 and 3.10. For the case $\text{ord}(\sigma)$ divides $(d-2)(d-1)+1$, please see Theorem 3.11.

Among smooth plane curves of degree $d \geq 4$, characterizations of the Fermat curve F_d using the structure or order of the automorphism group is known [6, 18]. Our second main theorem is a characterization of the Fermat curve F_d not by the order or structure of the automorphism group, but by the orders of two automorphisms and an automorphism obtained by their composition.

Theorem 1.3 *Let C be a smooth plane curve of degree $d \geq 4$, and $\text{Aut}(C)$ be the automorphism group of C . The curve C has two automorphism σ and ς such that $\text{ord}(\sigma) = 2d$, $\text{ord}(\varsigma) = 3$, and $\text{ord}(\varsigma\sigma^2) = 3$ if and only if the curve C is projectively equivalent to the Fermat curve F_d .*

Section 2 is preliminary. We will explain the basic facts of the automorphism groups of smooth plane curves and the Galois point. To show Theorem 1.3, we will use the Galois point as a crucial step. Let C be a smooth plane curve of degree $d \geq 4$, and $\sigma \in \text{Aut}(C)$. Based on [3], we will explain the relationship between the order of σ and the defining equation of C . To show Theorem 1.2, we will use the relationship. In Sect. 3, we will show Theorems 1.2 and 1.3.

2 Preliminary

Let C be a smooth plane curve of degree $d \geq 4$, and $\text{Aut}(C)$ be the automorphism group of the curve C . Recall that the group $\text{Aut}(C)$ is a finite subgroup of the group $\text{PGL}(3, \mathbb{C})$.

Definition 2.1 For $d \geq 4$, the Fermat curve F_d is a smooth plane curve defined by

$$X^d + Y^d + Z^d = 0,$$

and the Klein curve K_d is a smooth plane curve defined by

$$ZX^{d-1} + XY^{d-1} + YZ^{d-1} = 0.$$

From here, we will explain the automorphism groups of the Klein curve K_d and the Fermat curve F_d which are the famous and important smooth plane curves [4, 19]. First, we explain the automorphism group $\text{Aut}(F_d)$ of the Fermat curve F_d . Let N_{F_d} be a subgroup of the group $\text{Aut}(F_d)$ generated by

$$\begin{pmatrix} e_d & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ 0 & e_d & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $\mathcal{S}_3$ be the symmetric group of degree 3.

Theorem 2.2 [6, 19] *Let F_d be the Fermat curve.*

1. *The group N_{F_d} is a normal subgroup of $\text{Aut}(F_d)$, and $\text{Aut}(F_d)$ is generated by*

$$\begin{pmatrix} e_d & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & e_d & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

2. *There is a split short exact sequence of groups:*

$$1 \rightarrow N_{F_d} \rightarrow \text{Aut}(F_d) \rightarrow \mathcal{S}_3 \rightarrow 1.$$

Notice that $\left\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle \cong \mathcal{S}_3$.

Corollary 2.3 *Let F_d be the Fermat curve for $d \geq 4$. The order of an automorphism σ of F_d is 3 or divides $2d$.*

Proof Let σ be an automorphism of F_d . Let $q : \text{Aut}(F_d) \rightarrow \text{Aut}(F_d)/N_{F_d}$ be the quotient morphism. Since the quotient group $\text{Aut}(F_d)/N_{F_d}$ is $\mathcal{S}_3$, the order of $q(\sigma)$ is 2 or 3 as an element of $\text{Aut}(F_d)/N_{F_d}$. If the order of $q(\sigma)$ is 2, then $\sigma^2 \in N_{F_d}$, and hence the order of σ divides $2d$. We assume that the order of $q(\sigma)$ is 3. By

Theorem 2.2, σ is defined by a matrix $A = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{pmatrix}$. Since $A^3 = \begin{pmatrix} abc & a & 0 \\ 0 & abc & 0 \\ 0 & 0 & abc \end{pmatrix}$, $\sigma^3 = \text{id}_{F_d}$. Then the

order of σ is 3. $\square$

The characterization of the Fermat curve F_d using the order of the automorphism group is known as follows.

Theorem 2.4 [18] *Let C be a smooth plane curve of degree $d \geq 4$, and $\text{Aut}(C)$ be the automorphism group of C . If the order of $\text{Aut}(C)$ is $6d^2$, then C is projectively equivalent to the Fermat curve F_d .*

We explain the automorphism group $\text{Aut}(K_d)$ of the Klein curve K_d . Let M_{K_d} be a subgroup of $\text{Aut}(K_d)$ generated by

$$\begin{pmatrix} e_{d^2-3d+3} & 0 & 0 \\ 0 & e_{d^2-3d+3}^{d-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Theorem 2.5 [4, 6] *Let K_d be the Klein curve defined by $ZX^{d-1} + XY^{d-1} + YZ^{d-1} = 0$.*

1. *If $d = 4$, then the group $\text{Aut}(K_4)$ is called the Klein group $\text{PSL}(2, \mathbb{F}_7)$.*
2. *If $d \geq 5$, then the group M_{K_d} is a normal subgroup of $\text{Aut}(K_d)$, and the group $\text{Aut}(K_d)$ is generated by*

$$\begin{pmatrix} e_{d^2-3d+3} & 0 & 0 \\ 0 & e_{d^2-3d+3}^{d-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

and there exists a split short exact sequence of groups:

$$1 \rightarrow M_{K_d} \rightarrow \text{Aut}(K_d) \rightarrow \text{Cyc}_3 \rightarrow 1.$$

Corollary 2.6 Let K_d be the Klein curve for $d \geq 5$. The order of an automorphism σ of K_d is 3 or divides $(d-2)(d-1)+1$.

Proof Let σ be an automorphism of K_d . If $\sigma \in M_{K_d}$, then $\text{ord}(\sigma)$ divides $(d-2)(d-1)+1$. We assume that $\sigma \notin M_{K_d}$. By Theorem 2.5, σ is defined by a matrix of the form $\begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{pmatrix}$. Then $\text{ord}(\sigma)$ is 3. $\square$

The characterization of the Klein curve K_d using the order of an automorphism is known as follows.

Theorem 2.7 [3,6] Let C be a smooth plane curve of degree $d \geq 4$. If C has an automorphism of order $(d-2)(d-1)+1$, then C is projectively equivalent to the Klein curve K_d .

Definition 2.8 [6] For a monomial $X^i Y^j Z^k$, let

$$\text{ex}(X^i Y^j Z^k) := \max\{i, j, k\}.$$

For a homogeneous polynomial $F := \sum_{i+j+k=d} a_{i,j,k} X^i Y^j Z^k$ of degree d , we set $l := \max\{\text{ex}(X^i Y^j Z^k) : a_{i,j,k} \neq 0\}$. The core of F is defined as

$$\sum_{\text{ex}(X^i Y^j Z^k)=l} a_{i,j,k} X^i Y^j Z^k.$$

Definition 2.9 [6] Let C_0 be a smooth plane curve of degree $d \geq 4$. A pair (C, G) of a smooth plane curve C and a subgroup G of the group $\text{Aut}(C)$ is said to be a descendant of C_0 if C is defined by a homogeneous polynomial whose core is a defining polynomial of C_0 , and G is a subgroup of the group $\text{Aut}(C_0)$ as a subgroup of the group $\text{PGL}(3, \mathbb{C})$ under a suitable coordinate system.

For a positive integer $m \geq 2$, let e_m be a primitive m -th root of unity.

Definition 2.10 Let $\text{PBD}(2, 1)$ be the subgroup of $\text{PGL}(3, \mathbb{C})$ that consists of all elements representable by a matrix of the form $\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and H_l be a cyclic group of order l generated by $\begin{pmatrix} e_l & 0 & 0 \\ 0 & e_l & 0 \\ 0 & 0 & 1 \end{pmatrix}$ for $l \geq 1$.

Notice that the group H_l is contained in the center of $\text{PBD}(2, 1)$. Harui provides a list of group structures that may be displayed as automorphism groups of smooth plane curves as follows.

Theorem 2.11 [6] Let C be a smooth plane curve of degree $d \geq 4$, $\text{Aut}(C)$ be the automorphism group of C , and G a subgroup of $\text{Aut}(C)$. Then one of the following holds.

- (a-i) The group G is a cyclic group.
- (a-ii) The group G fixes a point not lying on the curve C , by replacing the local coordinate system if necessary, $G \subset \text{PBD}(2, 1)$ and $H_l \subset G$ for some l such that l divides d , H_l is contained in the center of G , and the quotient group G/H_l is one of $\mathcal{G}_d$. In addition, if $G/H_l \cong D_m$ and H_l is non-trivial, then m divides $d-2$.
- (b-i) The pair (C, G) is a descendant of the Fermat curve F_d .
- (b-ii) The pair (C, G) is a descendant of the Klein curve K_d .
- (c) The group G is primitive and conjugate to the icosahedral group A_5 , the Klein group $\text{PSL}(2, \mathbb{F}_7)$, the alternating group A_6 , the Hessian group H_{216} of order 216, or its subgroup of order 36 or 72.

The order of an element of the groups A_5 , $\text{PSL}(2, \mathbb{F}_7)$, A_6 , and H_{216} is 7 or less. Therefore, if the automorphism group $\text{Aut}(C)$ of a smooth plane curve C of degree $d \geq 4$ has an element of order 8 or more, the pair $(C, \text{Aut}(C))$ is not one of the case (c) of Theorem 2.11.

From here, based on [3], we will explain the relationship between the order of an automorphism and the defining equation of a smooth plane curve (see also [8]). Let C be a smooth plane curve of degree $d \geq 4$, and σ be an automorphism of C . We set $\text{Fix}(\sigma) := \{x \in C \mid g(x) = x\}$. By replacing the local coordinate system,

we may assume that σ is defined by a diagonal matrix $A := \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Let

$$n(\sigma) := \#\{\text{Fix}(\sigma) \cap \{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\}\}.$$

Table 1 Cyclic groups of smooth plane curves of degree $d \geq 4$

No.	$n(\sigma)$	F_*	A	$\text{ord}(\sigma)$ divides
1	0	$X^d + Y^d + Z^d$	$\begin{pmatrix} e_l^s & 0 & 0 \\ 0 & e_l^t & 0 \\ 0 & 0 & 1 \end{pmatrix}$	d
2	1	$(Y + Z)X^{d-1} + Y^d + Z^d$	$\begin{pmatrix} e_l & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$d - 1$
3	1	$YX^{d-1} + Y^d + Z^d$	$\begin{pmatrix} e_l & 0 & 0 \\ 0 & e_l^{1-d} & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$(d - 1)d$
4	2	$(Y + Z)X^{d-1} + ZY^{d-1} + Z^d$	$\begin{pmatrix} e_l & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$d - 1$
5	2	$ZX^{d-1} + ZY^{d-1} + Z^d$	$\begin{pmatrix} e_l^s & 0 & 0 \\ 0 & e_l^t & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$d - 1$
6	2	$ZX^{d-1} + XY^{d-1} + Z^d$	$\begin{pmatrix} e_l^{1-d} & 0 & 0 \\ 0 & e_l & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$(d - 1)^2$
7	2	$YX^{d-1} + XY^{d-1} + Z^d$	$\begin{pmatrix} e_l & 0 & 0 \\ 0 & e_l^{1-d} & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$(d - 2)d$
8	3	$(Y + Z)X^{d-1} + ZY^{d-1} + YZ^{d-1}$	$\begin{pmatrix} e_l & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$d - 1$
9	3	$ZX^{d-1} + ZY^{d-1} + YZ^{d-1}$	$\begin{pmatrix} e_l & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$d - 1$
10	3	$ZX^{d-1} + XY^{d-1} + YZ^{d-1}$	$\begin{pmatrix} e_l & 0 & 0 \\ 0 & e_l^{d-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$d^2 - 3d + 3$
11	3	$ZX^{d-1} + XY^{d-1} + XZ^{d-1}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & e_l & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$d - 1$

Notice that $n(\sigma) = 0, 1, 2$, or 3 . Since σ is defined by a diagonal matrix, $n(\sigma) = C \cap \{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\}$. We write the defining equation $F(X, Y, Z)$ of the curve C as follows.

$$F(X, Y, Z) = \sum_{0 \leq i, j \leq d} a_{ij} X^i Y^j Z^{d-i-j}.$$

For $F(X, Y, Z) = \sum_{0 \leq i, j \leq d} a_{ij} X^i Y^j Z^{d-i-j}$, we set

$$F_* := \sum_{\min\{i, j, d-i-j\} \geq d-1} a_{ij} X^i Y^j Z^{d-i-j}.$$

The following theorem holds.

Theorem 2.12 [3] *By replacing the local coordinate system, the number $n(\sigma)$, the polynomial F_* , the diagonal matrix A , and the order $\text{ord}(\sigma)$ are one of Table 1. In particular, $\text{ord}(\sigma)$ divides one of $(d - 2)(d - 1) + 1$, $(d - 2)d$, $(d - 1)^2$, and $(d - 1)d$.*

Theorem 2.12 determines the orders of automorphisms, that is, the cyclic groups acting on smooth plane curves of degree $d \geq 4$. In [10], the structures of abelian groups which are not cyclic groups and act on smooth plane curves of degree $d \geq 4$ are determined. Besides, in [9], the structures of groups which act freely on smooth plane curves of degree $d \geq 4$ are determined.

Definition 2.13 Let $C_{(d-1)d}$ be a smooth plane curve defined by

$$YX^{d-1} + Y^d + Z^d.$$

Let $C_{(d-1)^2}$ be a smooth plane curve defined by

$$ZX^{d-1} + XY^{d-1} + Z^d = 0.$$

Let $C_{(d-2)d}$ be a smooth plane curve defined by

$$YX^{d-1} + XY^{d-1} + Z^d = 0.$$

The automorphism groups of the curves $C_{(d-1)d}$, $C_{(d-1)^2}$, and $C_{(d-2)d}$ are as follows.

Theorem 2.14 [3,6] *We have the following.*

1. For $d \geq 5$, the automorphism group $\text{Aut}(C_{(d-1)d})$ is a cyclic group of order $(d-1)d$ which is generated by an automorphism $\sigma := \begin{pmatrix} e_{(d-1)d} & 0 & 0 \\ 0 & e_{(d-1)d}^{1-d} & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
2. The automorphism group $\text{Aut}(C_{(d-1)^2})$ is a cyclic group of order $(d-1)^2$ which is generated by an automorphism $\sigma := \begin{pmatrix} e_{(d-1)^2}^{1-d} & 0 & 0 \\ 0 & e_{(d-1)^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
3. The curve $C_{(d-2)d}$ has an automorphism $\sigma := \begin{pmatrix} e_{(d-2)d} & 0 & 0 \\ 0 & e_{(d-2)d}^{1-d} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ of order $(d-2)d$.
4. The automorphism group $\text{Aut}(C_{(d-2)d})$ is generated by two automorphism σ and $\varsigma := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. In particular, the cyclic group $\langle \sigma^{d-2} \rangle$ is a normal subgroup of $\text{Aut}(C_{(d-2)d})$ such that the quotient group $\text{Aut}(C_{(d-2)d}) / \langle \sigma^{d-2} \rangle$ is D_{d-2} .

The characterization of the curves $C_{(d-1)d}$, $C_{(d-1)^2}$, and $C_{(d-2)d}$ using the order of an automorphism is known as follows.

Theorem 2.15 [3] *Let C be a smooth plane curve of degree $d \geq 4$.*

1. If C has an automorphism of order $(d-1)d$, then C is projectively equivalent to the curve $C_{(d-1)d}$.
2. If C has an automorphism of order $(d-1)^2$, then C is projectively equivalent to the curve $C_{(d-1)^2}$.
3. If C has an automorphism of order $(d-2)d$, then C is projectively equivalent to the curve $C_{(d-2)d}$.

Let C be a smooth plane curve of degree $d \geq 4$, $\text{Aut}(C)$ be the automorphism group of C , and σ be an automorphism of C . Recall that $\text{ord}(\sigma)$ divides $(d-2)(d-1)+1$, $(d-2)d$, $(d-1)^2$, or $(d-1)d$. Theorems 2.7 and 2.15 show that if $\text{ord}(\sigma) = (d-2)(d-1)+1$, $(d-2)d$, $(d-1)^2$, or $(d-1)d$, then C is uniquely determined, and hence the structure of the group $\text{Aut}(C)$ is also uniquely determined.

From here, we will explain the Galois point. Let $\mathbb{P}^1$ be the projective line. The gonality of a curve C is $\text{gon}(C) := \min\{\deg \varphi \mid \varphi : C \rightarrow \mathbb{P}^1\}$. If C is a smooth plane curve of degree $d \geq 3$, then $\text{gon}(C) = d-1$, and the gonality of C is given by a projection $\pi_p : C \rightarrow \mathbb{P}^1$ at a point $p \in C$ [17].

Definition 2.16 [16,20] Let C be an irreducible plane curve of degree $d \geq 4$. A point $p \in \mathbb{P}^1$ is called a Galois point of C if the projection $\pi_p : C \rightarrow \mathbb{P}^1$ is a Galois cover. Moreover, if $p \in C$ (resp. $p \in \mathbb{P}^2 \setminus C$), then the point is said to be an inner (resp. outer) Galois point.

Theorem 2.17 [16,20] *Let C is a smooth plane curve of degree $d \geq 4$. If $p \in \mathbb{P}^2$ is an inner (resp. outer) Galois point of C , then the Galois group of the Galois cover $\pi_p : C \rightarrow \mathbb{P}^1$ is a cyclic group of order $d-1$ (resp. d) which is generated by an automorphism σ of C .*

Definition 2.18 Let C be an irreducible plane curve of degree $d \geq 4$. We denote the number of inner (resp. outer) Galois points of C by $\delta(C)$ (resp. $\delta'(C)$).

Theorem 2.19 [16,20] *Let C is a smooth plane curve of degree $d \geq 4$. We have the following.*

1. The number $\delta(C) = 0, 1$, or 4, and the number $\delta'(C) = 0, 1$, or 3.

2. If $\delta(C) \geq 1$ and $\delta'(C) \geq 1$, then $\delta(C) = 4$ and $\delta'(C) = 1$.
3. The number $\delta(C) = 4$ if and only if C is projectively equivalent to the curve $XZ^3 + X^4 + Y^4 = 0$.
4. The number $\delta'(C) = 3$ if and only if C is projectively equivalent to the Fermat curve F_d .
5. If $\delta(C) = 1$, then $\text{Aut}(C)$ is a cyclic group.
6. If $\delta'(C) = 1$, then the pair $(C, \text{Aut}(C))$ is the case (a-ii) of Theorem 2.11. In particular, let G be the Galois group of the Galois cover $\pi_p : C \rightarrow \mathbb{P}^1$ where $p \in \mathbb{P}^2$ is the Galois point of C . The quotient group $\text{Aut}(C)/G$ is one of $\mathcal{G}_d$.

Theorem 2.20 [3, 10] Let C be a smooth plane curve in of degree $d \geq 4$, and σ be an automorphism of C .

1. If $\text{ord}(g) = d - 1$, then $\sharp|\text{Fix}(g)| \neq 2$ if and only if C has an inner Galois point p , and the group $\langle g \rangle$ is the Galois group of the Galois cover $\pi_p : C \rightarrow \mathbb{P}^1$.
2. If $\text{ord}(\sigma) = k(d - 1)$ for $k \geq 2$, then C has an inner Galois point p , and the group $\langle \sigma^k \rangle$ is the Galois group of the Galois cover $\pi_p : C \rightarrow \mathbb{P}^1$.
3. If $\text{ord}(g) = d$, then $\sharp|\text{Fix}(g)| \neq 0$ if and only if C has an outer Galois point p , and the group $\langle g \rangle$ is the Galois group of the Galois cover $\pi_p : C \rightarrow \mathbb{P}^1$.
4. If $\text{ord}(\sigma) = kd$ for $k \geq 2$, then C has an outer Galois point p , and the group $\langle \sigma^k \rangle$ is the Galois group of the Galois cover $\pi_p : C \rightarrow \mathbb{P}^1$.

Other characterizations of smooth plane curves with Galois points are known [12, 13].

Recall that for $d \geq 4$, $\mathcal{G}_d$ is a set consisting of cyclic groups Cyc_m , dihedral groups D_m , the tetrahedral group A_4 , the octahedral group S_4 , and the icosahedral group A_5 where $m \leq d - 1$. By Theorems 2.19 and 2.20, we have the following.

Theorem 2.21 [3] Let C be a smooth plane curve of degree d , and $\text{Aut}(C)$ be the automorphism group of C . We take an automorphism $\sigma \in \text{Aut}(C)$.

1. If $d \geq 5$, and $\text{ord}(\sigma) = k(d - 1)$ for $k \geq 2$, then $\text{Aut}(C)$ is a cyclic group.
2. If $\text{ord}(\sigma) = kd$ for $k \geq 2$, then the quotient group $\text{Aut}(C)/\langle \sigma^k \rangle$ is one of $\mathcal{G}_d$, or $\text{Aut}(C) \cong \text{Aut}(F_d)$ as a group.

There are studies on the automorphism groups of plane curves using Galois points [3, 5, 7, 15]. In [3], for the case where $\text{ord}(\sigma) = k(d - 2)$, $k(d - 1)$, or kd for $k \geq 2$, the structure of $\text{Aut}(C)$ is classified in detail.

3 Proof of Theorems 1.2, and 1.3

First, we will show the case (1) of Theorem 1.2. We will establish Lemma 3.1 and 3.2 in preparation for that. Recall that $\text{PSL}(2, \mathbb{F}_7)$ is the Klein group, A_6 is the alternating group, and H_{216} is the Hessian group of order 216.

Lemma 3.1 Let C be a smooth plane curve of degree $d \geq 4$, and $\text{Aut}(C)$ be the automorphism group of C . If there is an automorphism $\sigma \in \text{Aut}(C)$ such that $\text{ord}(\sigma) \geq 8$, and $\text{ord}(\sigma)$ divides $(d - 1)^2$ but does not divide $d - 1$, then the pair $(C, \text{Aut}(C))$ is the case (a-i) or (a-ii) of Theorem 2.11.

Proof Let $\sigma \in \text{Aut}(C)$ be an automorphism such that $\text{ord}(\sigma) \geq 8$, and $\text{ord}(\sigma)$ divides $(d - 1)^2$ but does not divide $d - 1$. Since orders of the elements of the groups A_5 , $\text{PSL}(2, \mathbb{F}_7)$, A_6 , and H_{216} is 7 or less, we get that the pair $(C, \text{Aut}(C))$ is not the case (c) of Theorem 2.11. Since $\text{ord}(\sigma) \geq 5$, and $\text{ord}(\sigma)$ divides $(d - 1)^2$ but does not divide $d - 1$ Corollary 2.6, we get that the pair $(C, \text{Aut}(C))$ is not the case (b-ii) of Theorem 2.11. Same, since Corollary 2.3, the pair $(C, \text{Aut}(C))$ is not the case (b-i) of Theorem 2.11. $\square$

Lemma 3.2 Let C be a smooth plane curve of degree $d \geq 4$, and $\text{Aut}(C)$ be the automorphism group of C . We assume that the pair $(C, \text{Aut}(C))$ is the case (a-ii) of Theorem 2.11. We take an automorphism σ . Let F be the defining equation of C , and A be a matrix defining σ . By replacing the local coordinate system, we get that F_* and A are one of Table 1, $[0 : 0 : 1] \notin C$, and $\text{Aut}(C) \subset \text{PBD}(2, 1)$.

Proof Since the pair $(C, \text{Aut}(C))$ is the case (a-ii) of Theorem 2.11, by replacing the local coordinate system,

we get that $[0 : 0 : 1] \notin C$, and $\text{Aut}(C) \subset \text{PBD}(2, 1)$. Let $A = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ be a defining matrix of σ . From

the form of the matrix A , there is a matrix $B = \begin{pmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ such that $B^{-1}AB$ is a diagonal matrix. Since

$B([0 : 0 : 1]) = [0 : 0 : 1]$, and $B^{-1}(\text{PBD}(2, 1))B = \text{PBD}(2, 1)$. Therefore, by replacing the local coordinate system, we get that σ is defined by a diagonal matrix, $[0 : 0 : 1] \notin C$, and $\text{Aut}(C) \subset \text{PBD}(2, 1)$.

Let F be the defining equation of C . Since the automorphism σ is defined by a diagonal matrix, and Theorem 2.12, by multiplying X, Y , and Z by a constant or permuting X and Y , we get that the pair (C, σ) is one of Table 1, i.e. F_* and A are one of Table 1. Multiplying X, Y , and Z by a constant and permuting X and Y do not change the situation where $[0 : 0 : 1] \notin C$ and $\text{Aut}(C) \subset \text{PBD}(2, 1)$. Therefore, we get this lemma. $\square$

The following Theorem 3.3 corresponds to the (1) of Theorem 1.2.

Theorem 3.3 *Let C be a smooth plane curve of degree $d \geq 4$, and $\text{Aut}(C)$ be the automorphism group of C . If there is an automorphism $\sigma \in \text{Aut}(C)$ such that $\text{ord}(\sigma) \geq 8$, and $\text{ord}(\sigma)$ divides $(d - 1)^2$ but does not divide $d - 1$, then the group $\text{Aut}(C)$ is a cyclic group.*

Proof Let $\sigma \in \text{Aut}(C)$ be an automorphism such that $\text{ord}(\sigma) \geq 5$, and $\text{ord}(\sigma)$ divides $(d - 1)^2$ but does not divide $d - 1$. By Lemma 3.1, the pair $(C, \text{Aut}(C))$ is the case (a-i) or (a-ii) of Theorem 2.11.

We assume that the pair $(C, \text{Aut}(C))$ is the case (a-ii) of Theorem 2.11. From here, we will show that the group $\text{Aut}(C)$ is a cyclic group. By the case (a-ii) of Theorem 2.11, there is a normal cyclic subgroup H of $\text{Aut}(C)$ such that H is contained in the center of $\text{Aut}(C)$, the order of H divides d , and the quotient group $\text{Aut}(C)/H$ is one of $\mathcal{G}_d$. Since $\gcd(d, (d - 1)^2) = 1$, the group $\langle \sigma \rangle \cap H = \{\text{id}_C\}$. Since H is contained in the center of $\text{Aut}(C)$, $\langle \sigma, H \rangle = \langle \sigma \rangle \oplus H$. Since $\gcd(d, (d - 1)^2) = 1$, the group $\langle \sigma, H \rangle$ is a cyclic group. Let ς be a generator of $\langle \sigma, H \rangle$, and $v := \sharp|H|$. Then $\text{ord}(\varsigma) = v \times \text{ord}(\sigma)$. By Theorem 2.12, we get that $v = 1$, and hence the subgroup H is trivial. Therefore, the group $\text{Aut}(C)$ is one of $\mathcal{G}_d$. Then the cyclic group $\langle \sigma \rangle$ is isomorphic to a subgroup of the groups $\mathcal{C}_{ycm}$, D_m , A_4 , S_4 , or A_5 where $m \leq d - 1$. Since $\text{ord}(\sigma) \geq 8$, and orders of the elements of the groups A_4 , S_4 , and A_5 is 5 or less, we get that the group $\text{Aut}(C)$ is not one of the groups A_4 , S_4 , and A_5 .

We assume that the group $\text{Aut}(C)$ is a dihedral group D_m where $m \leq d - 1$. Since $\text{ord}(\sigma) \geq 8$, there is an automorphism $\tau \in \text{Aut}(C)$ of order 2 such that $\tau\sigma\tau = \sigma^{-1}$. Since Lemma 3.2, $\text{ord}(\sigma)$, and Theorem 2.12, by replacing the local coordinate system, we get that $[0 : 0 : 1] \notin C$, and $\text{Aut}(C) \subset \text{PBD}(2, 1)$, and the pair (C, σ) is No. 6 of Table 1, i.e. C and σ are defined as follows:

$$ZX^{d-1} + XY^{d-1} + Z^d + \sum_{i=0}^{d-2} H_{d-i}(Y, Z)X^i = 0 \quad \text{and} \quad \begin{pmatrix} e_l^{1-d} & 0 & 0 \\ 0 & e_l & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $l := \text{ord}(g)$, $H_{d-i}(Y, Z)$ is a homogeneous polynomial of degree $d - i$ for $0 \leq i \leq d - 2$ such that $H_{d-1}(Y, Z)$ does not have a Y^{d-1} term, and $H_d(Y, Z)$ does not have Y^d , Z^d , and $Y^{d-1}Z$ terms.

Since $\text{ord}(\sigma)$ divides $(d - 1)^2$, and $\text{ord}(\sigma)$ does not divide $d - 1$, we get that $\text{Fix}(\sigma)$ is $\{[1 : 0 : 0], [0 : 1 : 0]\}$.

Recall that the automorphism $\tau \in \text{Aut}(C)$ is defined by a matrix of the form $\begin{pmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Since $\tau\sigma\tau = \sigma^{-1}$

and $\text{ord}(\tau) = 2$, we get that $\tau(\text{Fix}(\sigma)) = \text{Fix}(\sigma)$. Since $\text{Fix}(\sigma) = \{[1 : 0 : 0], [0 : 1 : 0]\}$, we get that $\tau(\{[1 : 0 : 0], [0 : 1 : 0]\}) = \{[1 : 0 : 0], [0 : 1 : 0]\}$. We assume that $\tau([1 : 0 : 0]) = [1 : 0 : 0]$ or $\tau([0 : 1 : 0]) = [0 : 1 : 0]$. Then $\text{Fix}(\tau) \cap \text{Fix}(\sigma) \neq \emptyset$. Since C is smooth, the subgroup $\langle \tau, \sigma \rangle$ of $\text{Aut}(C)$ is a cyclic group. This contradicts $\tau\sigma\tau = \sigma^{-1}$ and $\text{ord}(\sigma) \geq 8$. Then $\tau([1 : 0 : 0]) = [0 : 1 : 0]$ and

$\tau([0 : 1 : 0]) = [1 : 0 : 0]$. This implies that $c_{11} = c_{22} = 0$, i.e. $k = \begin{pmatrix} 0 & c_{12} & 0 \\ c_{21} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. However, this contradicts

that the defining equation of the curve C has a ZX^{d-1} term but no ZY^{d-1} term. Therefore, the group $\text{Aut}(C)$ is a cyclic group. $\square$

We give an example of a smooth plane curve which satisfies the condition of the (1) of Theorem 1.2 but not the condition of the (1) of Theorem 1.1.

Example 3.4 Let C be a smooth plane curve of degree $d := 3(2k + 1)$ defined by

$$YX^{d-1} + Y^d + Z^d + X^{\frac{d-1}{2}}Y^{\frac{d+3}{6}}Z^{\frac{d}{3}} = 0$$

where $k \geq 1$. Let $l := \frac{d-1}{2} \frac{d}{3}$. The curve C has an automorphism σ of order l :

$$\sigma := \begin{pmatrix} e_l & 0 & 0 \\ 0 & e_l^{1-d} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By Theorem 3.3, the automorphism group $\text{Aut}(C)$ is a cyclic group. From here, we will show that the order of $\text{Aut}(C)$ is l . Let ς be a generator of $\text{Aut}(C)$, and $t := \text{ord}(\varsigma)$. Since σ and ς are commutative, the automorphism ς is defined by a diagonal matrix. Since $\frac{d-1}{2} \frac{d}{3}$ divides t , by Theorem 2.12, we get that

$$\varsigma = \begin{pmatrix} e_t & 0 & 0 \\ 0 & e_t^{1-d} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and t divides $(d-1)d$. Let $u := \gcd(t, d-1)$ and $v := \gcd(t, d)$. Then $t = uv$,

$$\varsigma^u = \begin{pmatrix} e_v & 0 & 0 \\ 0 & e_v & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } \varsigma^v = \begin{pmatrix} e_u & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By the defining equation of C , we get that $\text{ord}(\varsigma^u) = v$ divides $\frac{d}{3}$, and $\text{ord}(\varsigma^v) = u$ divides $\frac{d-1}{2}$. Therefore, the order of $\text{Aut}(C)$ is $l = \frac{d-1}{2} \frac{d}{3}$.

Next, we will show the (2) of Theorem 1.2.

Theorem 3.5 *Let C be a smooth plane curve of degree $d \geq 4$, and $\text{Aut}(C)$ be the automorphism group of C . We assume that C has an automorphism σ such that $\text{ord}(\sigma) \geq 8$ and $\gcd(\text{ord}(\sigma), d) \geq 2$.*

1. *If $\gcd(\text{ord}(\sigma), d-1) \geq 3$, then $\text{Aut}(C)$ is a cyclic group, or there is a subgroup H of $\text{Aut}(C)$ such that the quotient group $\text{Aut}(C)/H$ is the tetrahedral group A_4 .*
2. *If $\gcd(\text{ord}(\sigma), d-1) \geq 4$, then $\text{Aut}(C)$ is a cyclic group.*

Proof Let $\sigma \in \text{Aut}(C)$ be an automorphism such that $\text{ord}(\sigma) \geq 8$, $\gcd(\text{ord}(\sigma), d) \geq 2$, and $\gcd(\text{ord}(\sigma), d-1) \geq 3$. By Lemmas 2.3 and 2.6, the pair $(C, \text{Aut}(C))$ is not the case (b-i) or (b-ii) of Theorem 2.11. Since $\text{ord}(\sigma) \geq 8$, the pair $(C, \text{Aut}(C))$ is not the case (c) of Theorem 2.11. Then we get that the pair $(C, \text{Aut}(C))$ is the case (a-i) or (a-ii) of Theorem 2.11.

We assume that the pair $(C, \text{Aut}(C))$ is the case (a-ii) of Theorem 2.11. We assume that the quotient group $\text{Aut}(C)/H$ is a cyclic group. Let $\varsigma \in \text{Aut}(C)$ be a generator of the cyclic group $\text{Aut}(C)/H$. Then $\varsigma^i = \sigma$ as an element of $\text{Aut}(C)/H$ for some i . Since the order of H divides d , and $\gcd(\text{ord}(\sigma), d-1) \geq 3$, we get that $\gcd(\text{ord}(\varsigma), d-1) \geq 3$. By Theorem 2.12 and Lemma 3.2, by replacing the local coordinate system, $[0 : 0 : 1] \notin C$, $\text{Aut}(C) \subset \text{PBD}(2, 1)$, the automorphism ς is defined by a diagonal matrix, and $n(\varsigma) \neq 0$, and hence $C \cap \{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\} \neq \emptyset$. Recall that in this local coordinate system,

$$H = \text{Aut}(C) \cap \left\langle \begin{pmatrix} e_d & 0 & 0 \\ 0 & e_d & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle. \text{ Then } C \cap \{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\} \subset \text{Fix}(\tau) \text{ for any } \tau \in H. \text{ Since } C$$

is a smooth curve, $\text{Aut}(C)$ is a cyclic group.

From here, we will show that the quotient group $\text{Aut}(C)/H$ is not one of the groups D_m , S_4 , and A_5 where $m \leq d-1$. Since the order of g and Lemma 3.2, by replacing the local coordinate system, $[0 : 0 : 1] \notin C$,

and $\text{Aut}(C) \subset \text{PBD}(2, 1)$, the pair (C, σ) is No. 3 of Table 1, and $H = \text{Aut}(C) \cap \left\langle \begin{pmatrix} e_d & 0 & 0 \\ 0 & e_d & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$. Then the

automorphism $\sigma = \begin{pmatrix} e_l & 0 & 0 \\ 0 & e_l^{1-d} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $C \cap \{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\} = \{[1 : 0 : 0]\}$ where $l := \text{ord}(\sigma)$

divides $(d-1)d$. Since $\text{ord}(\sigma)$ does not divide $d-1$ and d , $\text{Fix}(\sigma) = \{[1 : 0 : 0]\}$. We assume that the quotient group $\text{Aut}(C)/H$ is a dihedral group D_m where $m \leq d-1$. Let $\varphi : \text{Aut}(C) \rightarrow \text{Aut}(C)/H$ be the quotient morphism. Since $\gcd(\text{ord}(\sigma), d) \geq 2$, $\gcd(l, d-1) \geq 3$, and the order of the group H divides d , we get that the order of an element $\varphi(\sigma)$ of $\text{Aut}(C)/H$ is 3 or more. Then there is an automorphism $k \in \text{Aut}(C)$

such that the order of an element $\varphi(\zeta)$ is 2 and $\varphi(\sigma)\varphi(\zeta) = \varphi(\zeta)\varphi(\sigma)^{-1}$. Then for some $\tau \in H$, we get that $\sigma\zeta = \zeta\sigma^{-1}\tau$. Since τ is defined by a diagonal matrix and $\text{Fix}(\sigma) = \{[1 : 0 : 0]\}$, we get that $\text{Fix}(\sigma) \subset \text{Fix}(\tau)$. By the equation $\sigma\zeta = \zeta\sigma^{-1}\tau$, we get that $\sigma(\zeta([1 : 0 : 0])) = \zeta([1 : 0 : 0])$. Since $\text{Fix}(\sigma) = \{[1 : 0 : 0]\}$, we get that $\zeta([1 : 0 : 0]) = [1 : 0 : 0]$, and hence $[1 : 0 : 0] \in \text{Fix}(\zeta)$. Since C is a smooth curve and $[1 : 0 : 0] \in \text{Fix}(\zeta) \cap \text{Fix}(\sigma)$, the group $\langle \zeta, \sigma \rangle$ is a cyclic group, and hence $\zeta\sigma = \sigma\zeta$. Since the order of an element $\varphi(\sigma)$ is 3 or more, it contradicts the equation $\varphi(\sigma)\varphi(\zeta) = \varphi(\zeta)\varphi(\sigma)^{-1}$. Therefore, the quotient group $\text{Aut}(C)/H$ is not a dihedral group D_m for $m \leq d - 1$.

Since the order of an element of $\mathcal{S}_4$, A_4 , and A_5 is at most 5, if $\gcd(l, d - 1) \geq 6$, then the quotient group $\text{Aut}(C)/H$ is a cyclic group, and hence $\text{Aut}(C)$ is a cyclic group. Let s be an element of $\mathcal{S}_4$, and l be the order of s . Then l is 2, 3, or 4. If $l = 3$ or 4, then there is a subgroup G_l of $\mathcal{S}_4$ such that $s \in G_l$ and G_l is the dihedral group of degree l . Therefore, if $\gcd(l, d - 1) \geq 3$, then the quotient group $\text{Aut}(C)/H$ is not $\mathcal{S}_4$. In the same way, we get that if $\gcd(l, d - 1) \geq 3$, then the quotient group $\text{Aut}(C)/H$ is not A_5 . In addition, since the order of an element of the group A_4 is 2 or 3, if $\gcd(l, d - 1) \geq 4$, then the group $\text{Aut}(C)$ is a cyclic group. $\square$

We give an example of a smooth plane curve which satisfies the condition of the (2) of Theorem 1.2 but not the condition of the (1) of Theorem 1.1.

Example 3.6 Let C be a smooth plane curve of degree $d := 6(3k + 1)$ defined by

$$ZX^{d-1} + XY^{d-1} + Z^d + X^2Y^{\frac{d-1}{2}}Z^{\frac{d-3}{2}} = 0$$

where $k \geq 1$. Let $l := 3\frac{d-1}{2} = 3^2(3k + 1)$. The curve C has an automorphism σ of order l :

$$\sigma := \begin{pmatrix} e_l^{1-d} & 0 & 0 \\ 0 & e_l & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By Theorem 3.5, the automorphism group $\text{Aut}(C)$ is a cyclic group. From here, we will show that the order of $\text{Aut}(C)$ is l . Let ζ be a generator of $\text{Aut}(C)$, and $t := \text{ord}(\zeta)$. Since σ and ζ are commutative, the automorphism ζ is defined by a diagonal matrix. Since $3\frac{d-1}{2}$ divides t , by Theorem 2.12, we get that

$$\zeta = \begin{pmatrix} e_t^{1-d} & 0 & 0 \\ 0 & e_t & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and t divides $(d - 1)^2$. By the defining equation of C , we get that $\text{ord}(\zeta) = t$ divides $3\frac{d-1}{2}$. Therefore, the order of $\text{Aut}(C)$ is $l = 3\frac{d-1}{2}$.

From here, we show the case (3) of Theorem 1.2.

Lemma 3.7 Let C be a smooth plane curve of degree $d \geq 4$, and $\text{Aut}(C)$ be the automorphism group of C . If there is an automorphism $\sigma \in \text{Aut}(C)$ such that $\gcd(\text{ord}(\sigma), d - 2) \geq 12$, then the pair $(C, \text{Aut}(C))$ is the case (a-i) or (a-ii) of Theorem 2.11.

Proof Let $\sigma \in \text{Aut}(C)$ be an automorphism such that $\gcd(\text{ord}(\sigma), d - 2) \geq 12$. Since $\text{ord}(\sigma) \geq 8$, the pair $(C, \text{Aut}(C))$ is not the case (c) of Theorem 2.11. Since Lemmas 2.3 and 2.6, the pair $(C, \text{Aut}(C))$ is neither the case (b-i) nor the case (b-ii) of Theorem 2.11. Therefore, the pair $(C, \text{Aut}(C))$ is the case (a-i) or (a-ii) of Theorem 2.11. $\square$

By Theorem 2.12, if $\gcd(\text{ord}(\sigma), d - 2) \geq 12$, then $\text{ord}(\sigma)$ divides $(d - 2)d$. The following Theorem 3.8 corresponds to the (3) of Theorem 1.2.

Theorem 3.8 Let C be a smooth plane curve of degree $d \geq 4$, and $\text{Aut}(C)$ be the automorphism group of C . If there is an automorphism $\sigma \in \text{Aut}(C)$ such that $\gcd(\text{ord}(\sigma), d - 2) \geq 12$, then $\text{Aut}(C)$ is a cyclic group, or there is a normal cyclic group H of $\text{Aut}(C)$ such that the quotient group $\text{Aut}(C)/H$ is D_m , and the order of H divides d where m divides $d - 2$.

Proof Let $\sigma \in \text{Aut}(C)$ be an automorphism such that $\gcd(\text{ord}(\sigma), d-2) \geq 12$. By Lemma 3.7, the pair $(C, \text{Aut}(C))$ is the case (a-i) or (a-ii) of Theorem 2.11. We assume that the pair $(C, \text{Aut}(C))$ is the case (a-ii) of Theorem 2.11. Since Lemma 3.2, $\text{ord}(\sigma)$ and Theorem 2.12, by replacing the local coordinate system, $[0 : 0 : 1] \notin C$, and $\text{Aut}(C) \subset \text{PBD}(2, 1)$, and the pair (C, σ) is No. 7 of Table 1. Then the automorphism σ is defined

by a diagonal matrix $\begin{pmatrix} e_l & 0 & 0 \\ 0 & e_l^{1-d} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and $C \cap \{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\} = \{[1 : 0 : 0], [0 : 1 : 0]\}$. Let

$H = \text{Aut}(C) \cap \left\langle \begin{pmatrix} e_d & 0 & 0 \\ 0 & e_d & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$. Then the quotient group $\text{Aut}(C)/H$ is one of $\mathcal{G}_d$. Let $q : \text{Aut}(C) \rightarrow \text{Aut}(C)/H$

be the quotient morphism. Since $|H|$ divides d , $\gcd(d-2, d) \leq 2$ and $\gcd(\text{ord}(\sigma), d-2) \geq 12$, we get that the order of $q(\sigma)$ is at least 6 as an element of $\text{Aut}(C)/H$. Then, the quotient group $\text{Aut}(C)/H$ is $\mathcal{C}_{yc_m}$ or D_m . Since $C \cap \{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\} \neq \emptyset$, if the quotient group $\text{Aut}(C)/H$ is $\mathcal{C}_{yc_m}$, then $\text{Aut}(C)$ is a cyclic group. $\square$

As be seen from the following two Examples 3.9 and 3.10, either case of Theorem 3.8 is occurred. Therefore, if $\text{ord}(\sigma)$ divides $(d-2)d$, then the structure of $\text{Aut}(C)$ is may not determined by $\text{ord}(\sigma)$.

Example 3.9 Let C_1 be a smooth plane curve of degree $d := 8k + 6$ defined by

$$YX^{d-1} + XY^{d-1} + Z^d + X^{\frac{3d-2}{8}}Y^{\frac{d+2}{8}}Z^{\frac{d}{2}} = 0.$$

Let $l := (\frac{d-2}{4})(\frac{d}{2})$. The curve C_1 has an automorphism

$$\sigma := \begin{pmatrix} e_l & 0 & 0 \\ 0 & e_l^{1-d} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

From here, we will show that the automorphism group $\text{Aut}(C_1)$ is a cyclic group of order l . We assume that $\text{Aut}(C_1)$ is not a cyclic group. Since $\gcd(\text{ord}(\sigma), d-2) \geq 12$, by Theorem 3.8, there is a normal cyclic group H such that H is contained in the center of $\text{Aut}(C_1)$, the order of H divides d , and the quotient group $\text{Aut}(C_1)/H$ is D_m where m divides $d-2$. Since $\sigma^{\frac{d-2}{4}} \in H$, and $\text{Aut}(C_1)/H$ is D_m , there is an automorphism ς such that $\varsigma \notin H$, $\varsigma^2 \in H$, and $\varsigma\sigma\varsigma\sigma \in H$. Recall that since the pair $(C_1, \text{Aut}(C_1))$ is the case (a-ii) of Theorem 2.11, there is one invariant point $p \in \mathbb{P}^2 \setminus C_1$ under the action of $\text{Aut}(C_1)$. From the defining matrix of the automorphism σ , the above-mentioned p is $[0 : 0 : 1]$. Since $\varsigma([0 : 0 : 1]) = [0 : 0 : 1]$, the automorphism ς is defined by a matrix of the form

$$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Since $\varsigma\sigma^{\frac{d-2}{4}} = \sigma^{\frac{d-2}{4}}\varsigma$ and the automorphism $\sigma^{\frac{d-2}{4}}$ is defined by a matrix of the form

$$\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we get that $a_{31} = a_{32} = 0$. Since $\varsigma\sigma\varsigma\sigma \in H$, $\sigma^2 \notin H$, and $\text{Fix}(\sigma) = \{[1 : 0 : 0], [0 : 1 : 0]\}$, we get that $a_{11} = a_{22} = 0$. This contradicts that ς is an automorphism of C_1 from the form of the defining equation of C_1 . Therefore, the group $\text{Aut}(C_1)$ is a cyclic group.

Next, we show that the order of $\text{Aut}(C_1)$ is $l = (\frac{d-2}{4})(\frac{d}{2})$. Let $\tau \in \text{Aut}(C_1)$ be a generator. Since the group $\text{Aut}(C_1)$ is a cyclic group, $\tau^i = \sigma$ for some i . In particular, $\text{ord}(\sigma)$ divides $\text{ord}(\tau)$. By Theorem 2.12, the pair (C_1, τ) is No. 7 of Table 1. Then $\text{Fix}(\tau)$ is a set of two points, and there is only one point of $\mathbb{P}^2 \setminus C_1$ that is invariant under the action of τ . Since $\tau^i = \sigma$ for some i , we get that $\text{Fix}(\tau) = \{[1 : 0 : 0], [0 : 1 : 0]\}$, and

$\tau([0 : 0 : 1]) = [0 : 0 : 1]$. Then the automorphism τ is defined by a diagonal matrix. In addition, by the core of the defining equation of C_1 , the automorphism τ is defined by a matrix of the form

$$\begin{pmatrix} e_m & 0 & 0 \\ 0 & e_m^{1-d} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $m := \text{ord}(\tau)$. Since the defining equation of C_1 has a term $X^{\frac{3d-2}{8}} Y^{\frac{d+2}{8}} Z^{\frac{d}{2}}$,

$$d - 1 \equiv \frac{3d-2}{8} + (1-d)\frac{d+2}{8} \pmod{m}.$$

Then

$$0 \equiv \left(\frac{d-2}{4}\right)\left(\frac{d}{2}\right) \pmod{m}.$$

This implies that m divides $l = \left(\frac{d-2}{4}\right)\left(\frac{d}{2}\right)$. Therefore, $\text{Aut}(C_1)$ is a cyclic group of order $l = \left(\frac{d-2}{4}\right)\left(\frac{d}{2}\right)$.

Example 3.10 Let C_2 be a smooth plane curve of degree $d := 8k + 6$ defined by

$$YX^{d-1} + XY^{d-1} + Z^d + X^{\frac{3d-2}{8}} Y^{\frac{d+2}{8}} Z^{\frac{d}{2}} + X^{\frac{d+2}{8}} Y^{\frac{3d-2}{8}} Z^{\frac{d}{2}} = 0.$$

Let $l := \left(\frac{d-2}{4}\right)\left(\frac{d}{2}\right)$. The curve C_2 has two automorphism

$$\sigma := \begin{pmatrix} e_l & 0 & 0 \\ 0 & e_l^{1-d} & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \varsigma := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

As like Example 3.9, we get that the quotient group $\text{Aut}(C_2)/\langle \sigma^{\frac{d-2}{4}} \rangle$ is $D_{\frac{d-2}{4}}$.

We give a sufficient condition for the pair $(C, \text{Aut}(C))$ to be a descendant of the Klein curve K_d .

Theorem 3.11 Let C be a smooth plane curve of degree $d \geq 4$, and $\text{Aut}(C)$ be the automorphism group of C . If there is an automorphism $\sigma \in \text{Aut}(C)$ such that $\text{ord}(\sigma)$ divides $(d-2)(d-1)+1$, and $\text{ord}(\sigma) \geq 8$, then the pair $(C, \text{Aut}(C))$ is a descendant of the Klein curve K_d .

Proof Let $\sigma \in \text{Aut}(C)$ be an automorphism such that $\text{ord}(\sigma)$ divides $(d-2)(d-1)+1$, and $\text{ord}(\sigma) \geq 8$.

First, we assume that $\text{Aut}(C)$ is a cyclic group. Let ς be a generator of the group $\text{Aut}(C)$. Then $\text{ord}(\varsigma)$ is a multiple of $\text{ord}(\sigma)$. Since $\gcd((d-2)(d-1)+1, d-2) = 1$, $\gcd((d-2)(d-1)+1, d-1) = 1$, $\gcd((d-2)(d-1)+1, d) = 1$ or 3 , and Theorem 2.12, we get that the pair (C, ς) is No. 10 of Table 1. Then the pair $(C, \text{Aut}(C))$ is a descendant of the Klein curve K_d .

Next, we assume that $\text{Aut}(C)$ is not a cyclic group. Since the group $\text{Aut}(C)$ has an element whose order $l \geq 8$, the pair $(C, \text{Aut}(C))$ is not the case (c) of Theorem 2.11. By Corollary 2.3, the pair $(C, \text{Aut}(C))$ is not the case (b-i) of Theorem 2.11. We assume that the pair $(C, \text{Aut}(C))$ is the case (a-ii) of Theorem 2.11. Since Lemma 3.2, by replacing the local coordinate system if necessary, we may assume that $[0 : 0 : 1] \notin C$, and the automorphism σ is defined by a diagonal matrix. Since $\text{ord}(\sigma)$ divides $(d-2)(d-1)+1$, and $\text{ord}(\sigma) > 3$, by Theorem 2.12, the pair (C, σ) is No. 10 of Table 1, and hence $n(\sigma) = 3$. This contradicts that $[0 : 0 : 1] \notin C$. Therefore, the pair $(C, \text{Aut}(C))$ is not the case (a-ii) of Theorem 2.11. As a result, the pair $(C, \text{Aut}(C))$ is the case (b-ii) of Theorem 2.11, i.e. the pair $(C, \text{Aut}(C))$ is a descendant of the Klein curve K_d . $\square$

By the following two Examples 3.12 and 3.13, if $\text{ord}(\sigma)$ divides $(d-2)(d-1)+1$, then the structure of $\text{Aut}(C)$ is may not determined by $\text{ord}(\sigma)$.

Example 3.12 Let C_3 be a smooth plane curve of degree $d := 7k + 6$ defined by

$$ZX^{d-1} + XY^{d-1} + YZ^{d-1} + X^{k+1}Y^{2k+2}Z^{4k+3} = 0.$$

Let $l := \frac{(d-2)(d-1)+1}{7} = 7k^2 + 9k + 3$. The curve C_3 has an automorphism

$$\sigma := \begin{pmatrix} e_l & 0 & 0 \\ 0 & e_l^{d-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

From here, we show that the automorphism group $\text{Aut}(C_3)$ is a cyclic group of order l . We assume that $\text{Aut}(C_3)$ is not a cyclic group. Since $\text{ord}(\sigma)$ divides $(d-2)(d-1)+1$, $\text{ord}(\sigma) \geq 8$, and Theorem 3.11, there is an automorphism ς such that $\text{ord}(\varsigma) = 3$, and $\langle \varsigma \rangle \cap \langle \sigma \rangle = \{\text{id}_{C_3}\}$. Let $\tau \in \text{Aut}(C_3)$ such that $\text{ord}(\tau)$ is the largest order among the automorphisms of $\text{Aut}(C_3)$. Since $\text{ord}(\sigma) \neq 3$, and Theorem 2.5, $\varsigma^i = \sigma$ for some i . Then $\text{ord}(\sigma)$ divides $\text{ord}(\tau)$. By Theorem 2.12, the pair (C, h) is No. 10 of Table 1. Then $\text{Fix}(\tau)$ is a set of three points. Since $\tau^i = \sigma$ for some i , $\text{Fix}(\tau) = \text{Fix}(\sigma) = \{[1:0:0], [0:1:0], [0:0:1]\}$. Since $\langle \tau \rangle$ is a normal group of $\text{Aut}(C_3)$, we may assume that the automorphism ς is defined by a matrix of the form

$$\begin{pmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{pmatrix}.$$

This contradicts that the form of the defining equation of C_3 . Therefore, $\text{Aut}(C_3)$ is a cyclic group.

Next, we show that the order of the group $\text{Aut}(C_3)$ is $l = \frac{(d-2)(d-1)+1}{7} = 7k^2 + 9k + 3$. Let $\tau \in \text{Aut}(C_3)$ be a generator. Since $\text{Aut}(C_3)$ is a cyclic group, $\tau^i = \sigma$ for some i . In particular, $\text{ord}(\sigma)$ divides $\text{ord}(\tau)$. By Theorem 2.12, the pair (C_3, τ) is No. 10 of Table 1. Then $\text{Fix}(\tau)$ is a set of three points. Since $\tau^i = \sigma$ for some i , we get that $\text{Fix}(\tau) = \{[1:0:0], [0:1:0], [0:0:1]\}$. Then the automorphism τ is defined by a diagonal matrix. In addition, by the core of the defining equation of C_3 , the automorphism τ is defined by a matrix of the form

$$\begin{pmatrix} e_m & 0 & 0 \\ 0 & e_m^{d-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $m := \text{ord}(\tau)$. Since the defining equation of C_3 has a term $X^{k+1}Y^{2k+2}Z^{4k+3}$,

$$d-1 \equiv (k+1) + (d-1)(2k+2) \pmod{m}.$$

Then

$$0 \equiv 2(7k^2 + 9k + 3) \pmod{m}.$$

This implies that m divides $\frac{2}{7}((d-2)(d-1)+1)$. Therefore, $\text{Aut}(C_3)$ is a cyclic group of order $l = \frac{1}{7}((d-2)(d-1)+1)$.

Example 3.13 Let C_4 be a smooth plane curve of degree $d := 7k + 6$ defined by

$$ZX^{d-1} + XY^{d-1} + YZ^{d-1} + X^{k+1}Y^{2k+2}Z^{4k+3} + X^{4k+3}Y^{k+1}Z^{2k+2} + X^{2k+2}Y^{4k+3}Z^{k+1} = 0.$$

Let $l := \frac{(d-2)(d-1)+1}{7} = 7k^2 + 9k + 3$. The curve C_4 has two automorphism

$$\sigma := \begin{pmatrix} e_l & 0 & 0 \\ 0 & e_l^{d-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \tau := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then $\text{Aut}(C_4)$ is not a cyclic group, and the quotient group $\text{Aut}(C_4)/\langle g \rangle$ is $\mathcal{C}yc_3$.

Lastly, we will characterize the Fermat curve F_d using an automorphism of order $2d$ and an automorphism of order 3 for a smooth plane curve of degree d .

Theorem 3.14 Let C be a smooth plane curve whose degree $d \geq 4$, and $\text{Aut}(C)$ be the automorphism group of C . If C has two automorphism σ and ς such that $\text{ord}(\sigma) = 2d$, $\text{ord}(\varsigma) = 3$, and $\text{ord}(\varsigma\sigma^2) = 3$, then C is projectively equivalent to the Fermat curve F_d .

Proof Let $\sigma \in \text{Aut}(C)$ be an automorphism such that $\text{ord}(\sigma) = 2d$. By Theorem 2.20, C has an outer Galois point. Let $\delta'(C)$ be the number of outer Galois points of C . By Theorem 2.19, $\delta'(C) = 1$ or 3, and if $\delta'(C) = 3$, then C is projectively equivalent to the Fermat curve F_d . We assume that $\delta'(C) = 1$. By Theorem 2.19, the pair $(C, \text{Aut}(C))$ is the case $(a-ii)$. Since the order of σ and Lemma 3.2, by replacing the local coordinate system, $[0:0:1] \notin C$, and $\text{Aut}(C) \subset \text{PBD}(2, 1)$, and the pair (C, σ) is No. 3 of Table 1. Then the automorphism σ

is defined by a diagonal matrix $\begin{pmatrix} e_{2d} & 0 & 0 \\ 0 & e_{2d}^{1-d} & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then the automorphism σ^2 is defined by a matrix $\begin{pmatrix} e_d & 0 & 0 \\ 0 & e_d & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Since $\text{Aut}(C) \subset \text{PBD}(2, 1)$, the automorphism ς is defined by a matrix of the form $\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$. This contradicts

that $\text{ord}(\sigma) = 2d$, $\text{ord}(\varsigma) = 3$, and $\text{ord}(\varsigma\sigma^2) = 3$. Therefore, $\delta'(C) = 3$, and hence the curve C is projectively equivalent to the Fermat curve F_d . $\square$

By Theorem 2.2, $\sigma := \begin{pmatrix} 0 & e_d & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\varsigma := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ are automorphisms of the Fermat curve F_d such that $\text{ord}(\sigma) = 2d$, $\text{ord}(\varsigma) = 3$, and $\text{ord}(\varsigma\sigma^2) = 3$.

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