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Practical stability for Riemann–Liouville delay fractional differential equations

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Abstract In this paper, we study a system of nonlinear Riemann–Liouville fractional differential equations with delays. First, we define in an appropriate way initial conditions which are deeply connected with the fractional derivative used. We introduce an appropriate generalization of practical stability which we call practical stability in time. Several sufficient conditions for practical stability in time are obtained using Lyapunov functions and the modified Razumikhin technique. Two types of derivatives of Lyapunov functions are used. Some examples are given to illustrate the introduced definitions and results.

Mathematics Subject Classification 34K37 · 34K20

1 Introduction

Fractional differential equations were recently noted to be a valuable tool in modeling of many phenomena in various fields of engineering, physics, and economics [14, 16, 23]. In particular they provide an excellent instrument for the description of memory and hereditary properties in a model.

When using the Riemann–Liouville (RL) fractional derivative in differential equations the consideration of initial conditions is very important. It is worth mentioning that the physical and geometric interpretations of operations of fractional integration and differentiation was recently considered by Podlubny [25]. Fractional differential equations in terms of the RL derivative require initial conditions expressed in terms of initial values of fractional derivatives of the unknown function [24, 26]. Also, in [13] it was noted that initial conditions for fractional differential equations with RL derivatives expressed in terms of fractional derivatives has physical meaning. It was shown that for any physically realistic model, zero initial conditions will be found for a continuous loading program or even in the case of a step discontinuity. Nonzero conditions will only be found in the case of an impulse and this type of process can be found in physics, chemistry, engineering, biology

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and economics. In the case of zero initial conditions the RL, Grünwald–Letnikov (GL) and Caputo fractional derivatives coincide [25]. For this reason, some authors either study Caputo derivatives, or use RL derivatives but avoid the problem of initial values of fractional derivatives by treating only the case of zero initial conditions. This leads to the consideration of mathematical correct problems, but without taking into account the physical nature of the described process. As mentioned sometimes, such as in the case of the impulse response, nonzero initial conditions appear (see, for example, [13]).

In connection with the main idea of stability properties we will consider in this paper nonzero initial conditions for RL fractional equations and we will define in an appropriate way practical stability properties which are slightly different than those for Caputo fractional differential equations. Note stability properties of delay differential equations can be considered by an application of the Lyapunov–Krasovskii method by functionals or by the Razumikhin method by Lyapunov functions. Both of the above mentioned methods are applied for the stability study of Caputo fractional delay differential equations in the literature (see, for example, [2, 3, 7, 8, 10], respectively).

In the case of delay fractional differential equations with the RL fractional derivative, following the idea of initial conditions in ordinary delay differential equations and the above-mentioned idea concerning the initial condition without any delay for RL fractional differential equations we will set up initial conditions in an appropriate way. Note any solution of the defined initial conditions with RL fractional derivatives is not continuous at zero (the initial point) which is the same as in the case without any delay. Delay RL fractional differential equations are set up and studied in [19] but the initial condition does not correspond to the idea of the case of delay differential equations with ordinary derivatives (the lower bound of the RL fractional derivative coincides with the left end side of the initial interval). The existence of the solution of RL fractional differential equations was studied in [1] but the initial condition as well as the integral presentation does not correspond to the RL fractional derivative.

Asymptotic stability for RL fractional differential equations with delays was studied in [9, 20, 22] but only the autonomous case is considered. Also a Lyapunov functional and its integer order derivative is applied. This functional is similar to the one used in the theory of differential equations with ordinary derivative and delay. On one hand the application of the ordinary derivative of the Lyapunov functional is not similar to the fractional derivatives used in the equation and on the other hand it leads to some restrictions on both the delay and the right-hand side parts of the equation ([21]). Additionally, in [9] the initial condition is not adequately associated with the RL fractional derivative. RL fractional equations with delays were studied recently in [4, 5] but there are unclear parts in the statement of the problem (the lower limit of the RL derivative is different than the initial time point) as well as in the initial condition (the RL fractional integral has no meaning, compare with [24] at the initial time). The Razumikhin method is applied to RL fractional differential equations in [10] but the initial condition is not connected with the RL fractional derivative.

There are many papers in the literature which study various types of stability of solutions of differential equations via Lyapunov functions. One type of stability, useful in real world problems, is the so called practical stability (see the book [18] for the basic definitions and applications to differential equations with ordinary derivatives). In this paper, a modified practical stability is defined and we call it practical stability in time. This stability is studied by Lyapunov functions and the modified Razumikhin technique. In connection with the application of Lyapunov functions to fractional equations it is necessary to define in an appropriate way the derivative of Lyapunov function among the studied fractional differential equations. Two different types of derivatives of Lyapunov functions among the studied fractional differential equations are applied. Several sufficient conditions for practical stability in time are obtained by the application of these derivatives. Some examples illustrating the definitions and the results are provided.

2 Notes on fractional calculus

We will give the main definition of fractional derivatives used in the literature (see, for example, [11, 12, 24]). We will give these definitions for scalar functions. Throughout the paper we will assume $q \in (0, 1)$.

- The Riemann–Liouville (RL) fractional derivative:

$${}^{RL}D_t^q m(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} m(s) ds, \quad t > 0$$

where $\Gamma(\cdot)$ denotes the Gamma function.

– The Grünwald–Letnikov fractional derivative is given by

$${}_{t_0}^{GL} D_t^q m(t) = \lim_{h \rightarrow 0} \frac{1}{h^q} \sum_{r=0}^{\left[\frac{t}{h}\right]} (-1)^r {}_q C_r m(t - rh), \quad t > 0$$

and the Grünwald–Letnikov fractional Dini derivative by

$${}_{t_0}^{GL} D_+^q m(t) = \limsup_{h \rightarrow 0+} \frac{1}{h^q} \sum_{r=0}^{\left[\frac{t}{h}\right]} (-1)^r {}_q C_r m(t - rh), \quad t > 0,$$

where ${}_q C_r = \frac{q(q-1)\dots(q-r+1)}{r!}$ and $\left[\frac{t}{h}\right]$ denotes the integer part of the fraction $\frac{t}{h}$.

Remark 2.1 If the derivative ${}^{RL}_0 D_t^q m(t)$ exists on $(0, T)$ then the equalities ${}^{RL}_0 D_t^q m(t) = {}^{GL}_0 D_t^q m(t) = {}^{GL}_0 D_+^q m(t)$ hold (see [11, 24]).

Note the fractional derivatives for scalar functions could be easily generalized to the vector case by taking fractional derivatives with the same fractional order for all components.

First we note a known result from the literature (the space C_{1-q} will be defined below).

Proposition 2.2 (Lemma 2.3 [27]) *Let $m \in C_{1-q}([0, T], \mathbb{R})$. Suppose that for an arbitrary $t_1 \in (0, T)$, we have $m(t_1) = 0$ and $m(t) < 0$ for $0 \leq t < t_1$. Then it follows that ${}^{RL}_0 D_t^q m(t)|_{t=t_1} \geq 0$.*

The practical definition of the initial condition of fractional differential equations with RL derivatives is based on the following result:

Proposition 2.3 [16] *Let $q \in (0, 1)$ and $b > 0$, $m : [0, b] \rightarrow \mathbb{R}$ be a Lebesgue measurable function.*

(a) *If there exists a.e. a limit $\lim_{t \rightarrow 0+} [t^{1-q} m(t)] = c \in \mathbb{R}$, then there also exists a limit*

$$\begin{aligned} {}_0 I_t^{1-q} m(t)|_{t=0} &:= \lim_{t \rightarrow 0+} \frac{1}{\Gamma(1-q)} \int_0^t \frac{m(s)}{(t-s)^q} ds = c \Gamma(q) \\ &= \Gamma(q) \lim_{t \rightarrow 0+} [t^{1-q} m(t)]. \end{aligned}$$

(b) *If there exists a.e. a limit $\lim_{t \rightarrow 0+} {}_0 I_t^{1-q} m(t) = c \in \mathbb{R}$, and if there exists the limit $\lim_{t \rightarrow 0+} [t^{1-q} m(t)]$, then*

$$\lim_{t \rightarrow 0+} [t^{1-q} m(t)] = \frac{c}{\Gamma(q)} = \frac{1}{\Gamma(q)} \lim_{t \rightarrow 0+} {}_0 I_t^{1-q} m(t).$$

3 Statement of the problem and basic definitions

Let $\tau > 0$ be a given number. Define the classes

$$\begin{aligned} C_{1-q}([0, T], \mathbb{R}) &= \{m : [0, T] \rightarrow \mathbb{R} : t^{1-q} m(t) \in C([0, T], \mathbb{R})\}, \\ &\text{with } 0 < T \leq \infty, \\ C_0 &= C([-\tau, 0], \mathbb{R}^n) \text{ with a norm } \|\phi\|_0 = \max_{t \in [-\tau, 0]} \|\phi(t)\|, \end{aligned}$$

and

$$\mathcal{K} = \{w \in C(\mathbb{R}_+; \mathbb{R}_+) : w(s) \text{ is strictly increasing and } w(0) = 0\}.$$

Similarly, we define $C_{1-q}([0, T], \mathbb{R}^n)$.

Let $\rho > 0$ be a given number. Consider the following set

$$S_\rho = \{x \in \mathbb{R}^n : \|x\| \leq \rho\}.$$

Consider the following system of nonlinear Riemann–Liouville fractional delay differential equations (RLFrDDE) of fractional order $q \in (0, 1)$:

$${}_0^RL D_t^q x(t) = f(t, x_t) \text{ for } t > 0, \quad (1)$$

with initial conditions

$$\begin{aligned} x(t) &= \phi(t), \text{ for } t \in [-\tau, 0], \\ \lim_{t \rightarrow t_0+} [t^{1-q} x(t)] &= \frac{\phi(0)}{\Gamma(q)}, \end{aligned} \quad (2)$$

where $x \in \mathbb{R}^n$, $x_t(\Theta) = x(t + \Theta)$ for $\Theta \in [-\tau, 0]$, $\phi \in C_0$ and $f \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$.

Remark 3.1 According to Proposition 2.3 the second equation in the initial conditions (2) could be replaced by the equality ${}_0I_t^{1-q} x(t)|_{t=0} = \phi(0)$.

We give a definition for various types of practical stability of the zero solution of RLFrDDE (1). In our further considerations below we will assume the existence of the solution of the IVP for RLFrDDE (1), (2) and we will denote it by $x(t; \phi) \in C_{1-q}([0, \infty), \mathbb{R}^n)$ (some existence results for RL fractional differential equations were obtained in [6, 15, 17]).

Note practical stability for differential equations with ordinary derivatives is defined in [18].

Now, using the classical ideas of practical stability and taking into account the meaning of RL fractional derivative, the required initial condition at the initial time, we will introduce a modified practical stability for RLFrDDE (1).

Definition 3.2 Let positive constants $\lambda, A : \lambda < A$ be given. The RLFrDDE (1) is said to be *practically stable in time with respect to* (λ, A) if there exists $T > 0$ such that for any $\phi \in C_0$ inequality $\|\phi\|_0 < \lambda$ implies $\|x(t; \phi)\| < A$ for $t \geq T$.

The practical stability in time is closer to boundedness in time than the stability in time. We will illustrate it on an example without delay because it is easier to obtain the exact solution.

Example 3.3 Consider the RL fractional differential equations

$$\begin{aligned} {}_0^RL D_t^q u(t) &= \left(t^q E_{0.5, 2-q}(-t^2) + \frac{2}{t^q \Gamma(1-q)} \right) \frac{u(t)}{\cos(t) + \frac{u_0}{\Gamma(q)t^{1-q}} + 2}, \\ \lim_{t \rightarrow 0+} t^{1-q} u(t) &= \frac{u_0}{\Gamma(q)}, \end{aligned} \quad (3)$$

where $u \in \mathbb{R}$. The solution of (3) is $u(t) = \cos(t) + \frac{u_0}{\Gamma(q)t^{1-q}} + 2$. It is clear that

$$\lim_{t \rightarrow 0+} t^{1-q} u(t) = \lim_{t \rightarrow 0+} t^{1-q} \cos(t) + \frac{u_0}{\Gamma(q)} + 2 \lim_{t \rightarrow 0+} t^{1-q} = \frac{u_0}{\Gamma(q)}$$

and

$${}_0^RL D_t^q u(t) = {}_0^RL D_t^q \cos(t) + \frac{2}{t^q \Gamma(1-q)} = t^q E_{0.5, 2-q}(-t^2) + \frac{2}{t^q \Gamma(1-q)}.$$

The zero solution is not stable because

$$1 + \frac{u_0}{\Gamma(q)t^{1-q}} \leq u(t) = \cos(t) + \frac{u_0}{\Gamma(q)t^{1-q}} + 2 \leq 3 + \frac{u_0}{\Gamma(q)t^{1-q}}$$

but it is practically stable in time with respect to $(1, 3 + \frac{1}{\Gamma(q)})$ because $|u(t)| \leq 3 + \frac{1}{\Gamma(q)}$ for $t > 1$. $\square$

Remark 3.4 Note if RLFrDDE (1) is practically stable in time with respect to (λ, A) , then it is practically stable in time with respect to (λ, B) with $B > A$, but we are interested in the smallest possible value of A .

4 Lyapunov functions and their derivatives among nonlinear RL delay fractional differential equations

One approach to study practical stability in time of RLFrDDE (1) is based on using Lyapunov functions. The first step is to define a Lyapunov function. The second step is to define its derivative among the fractional equation.

We will use the class Λ of Lyapunov functions.

Definition 4.1 Let $J \in \mathbb{R}_+$ be a given interval, and $\Delta \subset \mathbb{R}^n$ be a given set. We will say that the function $V(t, x) \in C(J \times \Delta, \mathbb{R}_+)$ belongs to the class $\Lambda(J, \Delta)$ if it is locally Lipschitz with respect to its second argument.

In connection with the RL fractional derivative it is necessary to use appropriate derivative of Lyapunov functions among the studied equation. We will use two types of derivatives of Lyapunov functions from the class $\Lambda(J, \Delta)$ to study the practical stability properties of RL fractional differential equations (1):

- first type– the RL fractional derivative of the function $V(t, x(t)) \in \Lambda([0, T], \Delta)$, $T \leq \infty$, defined by

$${}_0^{\text{RL}} D^q V(t, x(t)) = \frac{1}{\Gamma(1-q)} \frac{d}{ds} \int_0^t (t-s)^{-q} V(s, x(s)) ds, \quad t \in (0, T), \quad (4)$$

where $x \in C_{1-q}([0, T], \mathbb{R}^n)$ is a solution of (1).

- second type–Dini fractional derivative of the Lyapunov function $V \in \Lambda([0, T], \mathbb{R}^n)$ among (1): Let $\phi \in C_0$ and $t \in (0, T)$. Then

$$D_{(1)}^+ V(t, \phi) = \limsup_{h \rightarrow 0} \frac{1}{h^q} \left[V(t, \phi(0)) - \sum_{r=1}^{\lfloor \frac{t}{h} \rfloor} (-1)^{r+1} {}_q C_r V(t-rh, \phi(0) - h^q f(t, \phi)) \right]. \quad (5)$$

We will illustrate the above-defined derivatives on some simple examples of Lyapunov functions.

Example 4.2 (scalar case) Let $n = 1$. Consider the Lyapunov function depending directly on the time variable $V(t, x) = m(t) x^2$ where $m \in C(\mathbb{R}_+, \mathbb{R}_+)$, $x \in \mathbb{R}$.

Case 1. RL fractional derivative. Let $x(t)$ be a solution of the IVP for RLFrDDE (1), (2). Then the RL fractional derivative of $V(t, x)$ is given by

$${}_0^{\text{RL}} D^q V(t, x(t)) = {}_0^{\text{RL}} D^q (m(t) x^2(t)) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t \frac{m(s)x(s)}{(t-s)^q} ds.$$

Note it is difficult to obtain the RL fractional derivative in the general case for any solution of (1).

Case 2. Dini fractional derivative. Let $\phi \in C_0$ with $n = 1$ and $t > 0$. Then applying (5) we obtain

$$\begin{aligned} D_{(1)}^+ V(t, \phi) &= \limsup_{h \rightarrow 0} \frac{1}{h^q} \left[m(t) (\phi(0))^2 - \sum_{r=1}^{\lfloor \frac{t}{h} \rfloor} (-1)^{r+1} {}_q C_r m(t-rh) (\phi(0) - h^q f(t, \phi))^2 \right] \\ &= \limsup_{h \rightarrow 0} \frac{1}{h^q} \left[m(t) \left((\phi(0))^2 - (\phi(0) - h^q f(t, \phi))^2 \right) \right. \\ &\quad \left. + (\phi(0) - h^q f(t, \phi))^2 \sum_{r=0}^{\lfloor \frac{t}{h} \rfloor} (-1)^r {}_q C_r m(t-rh) \right] \\ &= \phi(0) m(t) f(t, \phi) + (\phi(0))^2 {}_0^{\text{RL}} D^q (m(t)). \end{aligned} \quad (6)$$

Comparing with the RL fractional derivative of V , the Dini fractional derivative is easier to obtain. □

Example 4.3 (multi-dimensional case) Consider the Lyapunov function $V(t, x) = m(t) \sum_{i=1}^n x_i^2$ where $m \in C(\mathbb{R}_+, \mathbb{R}_+)$, $x \in \mathbb{R}^n$, $x = (x_1, x_2, \dots, x_n)$. Let $\phi \in C_0$, $\phi = (\phi_1, \phi_2, \dots, \phi_n)$ and $t > 0$. Then similar to Example 2 the Dini fractional derivative is

$$D_{(1)}^+ V(t, \phi) = m(t) \sum_{i=1}^n \phi_i(0) f_i(t, \phi) + {}_0^{RL} D^q (m(t)) \sum_{i=1}^n (\phi_i(0))^2.$$

□

5 Main results

We will study practical stability properties of the RL delay fractional nonlinear system (1). We will use the above defined fractional derivatives of Lyapunov functions.

5.1 RL fractional derivative of Lyapunov functions

Theorem 5.1 *Let the following conditions be satisfied:*

1. *There exist a function $V \in \Lambda([0, \infty), \mathbb{R}^n)$ such that:*
 - (i) *there exists a number $T > 0$ such that the inequality*

$$a(\|x\|) \leq V(t, x) \text{ for } t > T, x \in \mathbb{R}^n, \quad (7)$$

holds where $a \in \mathcal{K}$;

- (ii) *there exists a positive number $\lambda > 0$ such that for any function $y \in C_{1-q}([0, \infty), \mathbb{R}^n)$:*
 $\lim_{t \rightarrow t_0} [t^{1-q} y(t)] \in S_\lambda$ *the inequality*

$$t^{1-q} V(t, y(t))|_{t=0+} = \lim_{t \rightarrow 0+} t^{1-q} V(t, y(t)) < b(\lambda)$$

holds with $b \in \mathcal{K}$;

- (iii) *for any initial function $\phi \in S_\lambda$ and the corresponding solution $x(t) = x(t; \phi)$ of (1), (2) defined on $[0, \infty)$ such that if for $t > 0$ the inequality $t^{1-q} V(t, x(t)) \geq (t+s)^{1-q} V(t+s, x(t+s))$ holds for $s \in [-\min\{\tau, t\}, 0]$ then*

$${}_0^{RL} D_t^q V(t, x(t)) \leq 0.$$

Then the RLFrDDE (1) is practically stable in time w.r.t. $(\lambda, a^{-1}(b(\lambda)))$.

Proof Choose the initial function $\phi \in C_0$: $\|\phi\|_0 < \lambda$ and consider the solution $x^*(t) = x(t; \phi)$ of the IVP (1), (2).

From condition 1(ii) we have $\lim_{t \rightarrow 0+} t^{1-q} V(t, x^*(t)) < b(\lambda)$.

Therefore, there exists $\delta_1 > 0$ such that $t^{1-q} V(t, x^*(t)) < b(\lambda)$ for $t \in (0, \delta_1)$.

We will prove

$$V(t, x^*(t)) < b(\lambda) t^{q-1}, \quad t > 0. \quad (8)$$

The inequality (8) holds on $(0, \delta_1)$. Assume there exists a point $\xi \geq \delta_1$ such that

$$V(t, x^*(t)) < b(\lambda) t^{q-1}, \quad t \in (0, \xi), \quad V(\xi, x^*(\xi)) = b(\lambda) \xi^{q-1}. \quad (9)$$

Then $t^{1-q} V(t, x^*(t)) < b(\lambda) = \xi^{1-q} V(\xi, x^*(\xi))$, $t \in (0, \xi)$.

Case 1. Let $\xi > \tau$. Then $\min\{\xi, \tau\} = \tau$. From (9) it follows that $(\xi + \Theta)^{q-1} V(\xi + \Theta, x(\xi + \Theta)) < b(\lambda) = \xi^{q-1} V(\xi, x(\xi))$ for $\Theta \in (-\tau, 0)$.

Case 2. Let $\xi \leq \tau$. Then, $\min\{\xi, \tau\} = \xi$. From (9) it follows that $(\xi + \Theta)^{q-1} V(\xi + \Theta, x(\xi + \Theta)) < b(\lambda) = \xi^{q-1} V(\xi, x(\xi))$ for $\Theta \in (-\xi, 0)$.

According to condition 1(iii)

$${}_0^{RL} D_\xi^q V(\xi, x(\xi)) < 0. \quad (10)$$

From condition 1(i) we get $m(t) - b(\lambda)t^{q-1} \in C_{1-q}([0, \xi], \mathbb{R})$. According to Proposition 2.2 with $t_1 = \xi$ the inequality ${}_0^{RL}D_t^q(m(t) - b(\lambda)t^{q-1})|_{t=\xi} > 0$ holds and

$${}_0^{RL}D_t^q m(t)|_{t=\xi} \geq 0. \quad (11)$$

Inequality (11) contradicts (10).

From $\lim_{t \rightarrow \infty} t^{q-1} = 0$ it follows that $t^{q-1} \leq 1$ for $t \geq 1$.

Denote $A = a^{-1}(b(\lambda))$.

Therefore, from inequalities (8), and condition 1(i) we obtain

$$a(||x^*(t)||) \leq V(t, x^*(t)) < b(\lambda)t^{q-1} \leq a(A) \quad \text{for } t \geq \max\{1, T\}$$

or

$$||x^*(t)|| < A \quad \text{for } t \geq \max\{1, T\}.$$

□

We will prove a comparison result that will be used to obtain another type of sufficient condition for practical stability in time.

Lemma 5.2 Assume the following conditions are satisfied:

1. The function $x^*(t) = x(t; \varphi)$ is a solution of (1), (2) for $t \in [-\tau, T)$, $x^* \in C^{1-q}([0, T], \Delta)$ where $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$, $T \leq \infty$.
2. The function $V \in \Lambda([0, T], \Delta)$ is such that
 - (i) there exists a number $\lambda \geq 0$ such the inequality

$$\lim_{t \rightarrow 0+} t^{1-q} V(t, x^*(t)) < b(\lambda)$$

holds with $b \in \mathcal{K}$;

- (ii) there exists a number $C > 0$ such that for any point $t > 0$ such that $(t + s)^{1-q} V(t + s, x^*(t + s)) \leq t^{1-q} V(t, x^*(t))$ holds for $s \in [-\min\{t, \tau\}, 0)$, the inequality

$${}_0^{RL}D_t^q V(t, x^*(t)) < \frac{C}{t^q \Gamma(1-q)} \quad (12)$$

holds.

Then the inequality $V(t, x^*(t)) \leq C + \frac{b(\lambda)}{t^{1-q}}$ for $t \in (0, T)$ holds.

Proof Define the functions $m(t) = V(t, x(t))$ for $t \in (0, T)$. From condition 2(i) it follows that $m \in C_{1-q}([0, T], \mathbb{R}_+)$.

We will prove that

$$m(t) < C + b(\lambda)t^{q-1}, \quad t \in (0, T). \quad (13)$$

From condition 2(i) it follows that there exists a number $\delta > 0$ such that $t^{1-q}m(t) < b(\lambda)$ for $0 < t < \delta$, or $m(t) \leq b(\lambda)t^{q-1} < C + b(\lambda)t^{q-1}$, i.e. inequality (13) holds for $t \in (0, \delta)$. Assume inequality (13) is not true for all $t \in [\delta, T)$. Therefore, there exists a point $\xi \geq \delta > 0$ such that

$$m(\xi) = C + \frac{b(\lambda)}{\xi^{1-q}}, \quad \text{and} \quad m(t) < C + \frac{b(\lambda)}{t^{1-q}}, \quad t \in (0, \xi). \quad (14)$$

Case 1. Let $\xi > \tau$. Then $\min\{\xi, \tau\} = \tau$. From (14) it follows that $\xi^{1-q}m(\xi) - b(\lambda) = C\xi^{1-q} > C\xi^{1-q} > t^{1-q}m(t) - b(\lambda)$ and therefore, $\xi^{1-q}m(\xi) > t^{1-q}m(t)$ for $t \in (0, \xi)$, i.e. $(\xi + \Theta)^{q-1}V(\xi + \Theta, x(\xi + \Theta)) < \xi^{q-1}V(\xi, x(\xi))$ for $\Theta \in (-\tau, 0)$.

Case 2. Let $\xi \leq \tau$. Then, $\min\{\xi, \tau\} = \xi$. From (14) similar to Case 1 it follows that $(\xi + \Theta)^{q-1}V(\xi + \Theta, x(\xi + \Theta)) < \xi^{q-1}V(\xi, x(\xi))$ for $\Theta \in (-\xi, 0)$.

According to condition 2(ii) for $t = \xi$ the inequality

$${}_0^{RL}D_\xi^q V(\xi, x^*(\xi)) < \frac{C}{\xi^q \Gamma(1-q)} \quad (15)$$

holds.

The inclusion $m(t) - C - \frac{b(\lambda)}{t^{1-q}} \in C_{1-q}([0, \xi], \mathbb{R})$ is valid. According to Proposition 2.2 with $t_1 = \xi$ the inequality ${}_0^{RL}D_t^q \left(m(t) - C - \frac{b(\lambda)}{t^{1-q}} \right) |_{t=\xi} \geq 0$ holds and

$${}_0^{RL}D_\xi^q m(\xi) \geq {}_0^{RL}D_\xi^q C + {}_0^{RL}D_\xi^q \frac{b(\lambda)}{\xi^{1-q}} = \frac{C}{\xi^q \Gamma(1-q)}. \quad (16)$$

The inequality (16) contradicts (15). $\square$

Theorem 5.3 *Let the following conditions be satisfied:*

1. The function $V \in \Lambda([0, T], \mathbb{R}^n)$ is such that
 - (i) there exists a number $T > 0$ such that inequality

$$a(\|x\|) \leq V(t, x) \text{ for } t > T, x \in \mathbb{R}^n, \quad (17)$$

holds where $a \in \mathcal{K}$;

- (ii) there exists a number $\lambda \geq 0$ such that for any function $y \in C_{1-q}([0, \infty), \mathbb{R}^n) : \lim_{t \rightarrow 0+} [t^{1-q} y(t)] \in S_\lambda$ the inequality

$$\lim_{t \rightarrow 0+} t^{1-q} V(t, y(t)) < b(\lambda)$$

holds with $b \in \mathcal{K}$;

- (iii) there exists a number $C > 0$ such that for any initial function $\phi \in S_\lambda$ and the corresponding solution $x(t) = x(t; \phi)$ of (1), (2) defined on $[0, \infty)$ such that if for any point $t > 0$ the inequality $(t + s)^{1-q} V(t + s, x^*(t + s)) \leq t^{1-q} V(t, x^*(t))$ holds for $s \in [-\min\{t, \tau\}, 0)$ then

$${}_0^{RL}D_t^q V(t, x(t)) < \frac{C}{t^q \Gamma(1-q)} \quad (18)$$

holds.

Then the RLFrDDE (1) is practically stable in time w.r.t. $(\lambda, a^{-1}(C + b(\lambda)))$.

Proof Choose the initial function $\phi \in C_0 : \|\phi\|_0 < \lambda$ and consider the solution $x(t) = x(t; \phi)$ of the IVP (1), (2). Then according to Lemma 5.2 applied to the solution $x(t)$ the inequality $V(t, x(t)) < C + b(\lambda)t^{q-1}$ holds for $t > 0$. According to condition 1(i) and $t^{q-1} < 1$ for $t \geq 1$ we get the inequality $\|x(t)\| \leq a^{-1}(C + b(\lambda))$ holds for $t \geq \max\{1, T\}$. $\square$

5.2 Dini fractional derivative of Lyapunov functions

We will use Dini fractional derivative defined by (5) to obtain some sufficient conditions for practical stability in time of the RLFrDDE(1).

Theorem 5.4 *Let the following conditions be satisfied:*

1. There exist a function $V \in \Lambda([0, \infty), \mathbb{R}^n)$ such that:
 - (i) there exists a number $T > 0$ such that the inequality

$$a(\|x\|) \leq V(t, x) \text{ for } t > T, x \in \mathbb{R}^n, \quad (19)$$

holds where $a \in \mathcal{K}$;

- (ii) *there exists a positive number $\lambda > 0$ such that for any function $y \in C_{1-q}([0, \infty), \mathbb{R}^n) : \lim_{t \rightarrow 0+} [t^{1-q} - qy(t)] \in S_\lambda$ the inequality*

$$t^{1-q} V(t, y(t))|_{t=0+} = \lim_{t \rightarrow 0+} t^{1-q} V(t, y(t)) < b(\lambda)$$

holds with $b \in \mathcal{K}$;

- (iii) *for any initial function $\phi \in S_\lambda$ and the corresponding solution $x(t) = x(t; \phi)$ of (1), (2) defined on $[0, \infty)$ such that if for $t \in (\max\{\tau, T\}, \infty)$ the inequality $t^{1-q} V(t, x(t)) \geq (t+s)^{1-q} V(t+s, x(t+s))$ holds for $s \in [0, t]$ then*

$${}^{RL}_0 D_t^q V(t, x(t)) \leq 0. \quad (20)$$

Then the RLFrDDE (1) is practically stable in time w.r.t. $(\lambda, a^{-1}(b(\lambda)))$.

The proof of Theorem 5.4 is similar to the one of Theorem 5.1 where Remark 2.1 is applied and inequality (20) for $t = \xi$ is applied instead of (11).

When the Dini fractional derivative defined by (5) is used then the comparison result is:

Lemma 5.5 *Let the conditions 1 and 2(i) of Lemma 5.2 are satisfied and the derivative in 2(ii) is replaced by Dini fractional derivative defined by (5), i.e.*

- 2(ii)* *. there exists a number $C > 0$ such that for any point $t > 0$ such that $(t+s)^{1-q} V(t+s, \psi(s)) \leq t^{1-q} V(t, \psi(0))$ holds for $s \in [-\min\{t, \tau\}, 0)$, then the inequality*

$$D_{(1)}^+ V(t, \psi) < \frac{C}{t^q \Gamma(1-q)} \quad (21)$$

holds where $\psi(s) = x^(t+s)$, $s \in [-\tau, 0]$.*

Then the inequality $V(t, x^(t)) \leq C + \frac{\lambda}{t^{1-q}}$ for $t \in (0, T)$ holds.*

Proof A part of the proof of Lemma 5.5 is similar to the one of Lemma 5.2 and we will emphasize only on the differences which are connected with the application of the Dini fractional derivative and condition 2(ii)*.

We will prove the inequality (13) where the function $m(t) = V(t, x(t)) \in C_{1-q}([0, T], \mathbb{R}_+)$. Assume the contrary and therefore, there exists a point $\xi \geq \delta > 0$ such that (14) hold. According to Proposition 2.2 and Remark 2.1 the inequality

$${}^{GL}_0 D_+^q m(t)|_{t=\xi} \geq \frac{C}{\xi^q \Gamma(1-q)} \quad (22)$$

holds.

For any $t \in (0, \xi]$ and $h > 0$ we let

$$S(x(t), h) = \sum_{r=1}^{[\frac{t}{h}]} (-1)^{r+1} {}_q C_r x(t - rh).$$

From Remark 2.1 and Eq. (1), it follows the function $x(t)$ satisfies for $t \in [t_0, \xi]$ the equalities ${}^{RL}_0 D_t^q x(t) = {}^{GL}_0 D_+^q x(t) = \limsup_{h \rightarrow 0+} \frac{1}{h^q} [x(t) - S(x(t), h)] = f(t, x_t)$ and

$$\limsup_{h \rightarrow 0+} \frac{1}{h^q} [x(t) - S(x(t), h)] = f(t, x_t).$$

Therefore,

$$S(x(t), h) = x(t) - h^q f(t, x_t) - \Lambda(h^q)$$

or

$$x(t) - h^q f(t, x_t) = S(x(t), h) + \Lambda(h^q) \quad (23)$$

with $\frac{||\Lambda(h^q)||}{h^q} \rightarrow 0$ as $h \rightarrow 0$. Then for any $t \in [0, \xi]$ we obtain

$$\begin{aligned} m(t) - \sum_{r=1}^{\lfloor \frac{t}{h} \rfloor} (-1)^{r+1} {}_q C_r m(t - rh) \\ = \left\{ V(t, x(t)) - \sum_{r=1}^{\lfloor \frac{t}{h} \rfloor} (-1)^{r+1} {}_q C_r \left[V(t - rh, x(t) - h^q f(t, x_t)) \right] \right\} \\ + \sum_{r=1}^{\lfloor \frac{t}{h} \rfloor} (-1)^{r+1} {}_q C_r \left\{ \left[V(t - rh, S(x(t), h) + \Lambda(h^q)) \right] \right. \\ \left. - \left[V(t - rh, x(t - rh)) \right] \right\}. \end{aligned} \quad (24)$$

Since V is locally Lipschitzian in its second argument with a Lipschitz constant $L > 0$ we obtain

$$\begin{aligned} \sum_{r=1}^{\lfloor \frac{t}{h} \rfloor} (-1)^{r+1} {}_q C_r \left\{ V(t - rh, S(x(t), h) + \Lambda(h^q)) - V(t - rh, x(t - rh)) \right\} \\ \leq L \left\| \sum_{r=1}^{\lfloor \frac{t}{h} \rfloor} (-1)^{r+1} {}_q C_r (S(x(t), h) + \Lambda(h^q) - x(t - rh)) \right\| \\ \leq L \left\| \sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} (-1)^{r+1} {}_q C_r \sum_{j=1}^{\lfloor \frac{t}{h} \rfloor} (-1)^{j+1} {}_q C_j x(t - jh) \right. \\ \left. - \sum_{r=1}^{\lfloor \frac{t}{h} \rfloor} (-1)^{r+1} {}_q C_r x(t - rh) \right\| + L ||\Lambda(h^q)|| \left\| \sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} (-1)^{r+1} {}_q C_r \right\| \\ = L \left\| \left(\sum_{r=0}^{\lfloor \frac{t}{h} \rfloor} (-1)^{r+1} {}_q C_r \right) \left(\sum_{j=1}^{\lfloor \frac{t}{h} \rfloor} (-1)^{j+1} {}_q C_j x(t - jh) \right) \right\| \\ + L ||\Lambda(h^q)|| \left\| \sum_{r=1}^{\lfloor \frac{t}{h} \rfloor} (-1)^{r+1} {}_q C_r \right\|. \end{aligned} \quad (25)$$

Substitute (25) in (24), divide both sides by h^q , take the limit as $h \rightarrow 0^+$, use $\sum_{r=0}^{\infty} {}_q C_r z^r = (1+z)^q$ if $|z| \leq 1$, and we obtain for any $t \in (0, t^*]$ the inequality

$$\begin{aligned} {}_0^G D_+^q m(t) \leq \lim_{h \rightarrow 0^+} \frac{1}{h^q} \left\{ V(t, x(t)) \right. \\ \left. - \sum_{r=1}^{\lfloor \frac{t}{h} \rfloor} (-1)^{r+1} {}_q C_r \left[V(t - rh, x(t) - h^q f(t, x^*(t))) \right] \right\} \\ + L \lim_{h \rightarrow 0^+} \frac{||\Lambda(h^q)||}{h^q} \lim_{h \rightarrow 0^+} \left\| \sum_{r=1}^{\lfloor \frac{t}{h} \rfloor} (-1)^{r+1} {}_q C_r \right\| \\ + L \lim_{h \rightarrow 0^+} \sup \left\| \sum_{r=1}^{\lfloor \frac{t}{h} \rfloor} (-1)^{r+1} {}_q C_r \right\| \left\| \frac{1}{h^q} \sum_{j=1}^{\lfloor \frac{t}{h} \rfloor} {}_q C_j x(t - jh) \right\| \\ \leq \lim_{h \rightarrow 0^+} \frac{1}{h^q} \left\{ V(t, x(t)) - \sum_{r=1}^{\lfloor \frac{t}{h} \rfloor} (-1)^{r+1} {}_q C_r V(t - rh, x(t) - h^q f(t, x^*(t))) \right\}. \end{aligned} \quad (26)$$

Let $t = \xi$. Define the function $\psi(\Theta) = x^*(\xi + \Theta)$, $\Theta \in [-\tau, 0]$.

Case 1. Let $\xi > \tau$. Then $\min\{\xi, \tau\} = \tau$. From the choice of the point ξ it follows that $\xi^{1-q}m(\xi) - b(\lambda) = C\xi^{1-q} > C\tau^{1-q} > t^{1-q}m(t) - b(\lambda)$ and therefore, $\xi^{1-q}m(\xi) > t^{1-q}m(t)$ for $t \in (0, \xi)$, i.e. $(\xi + \Theta)^{q-1}V(\xi + \Theta, x(\xi + \Theta)) < \xi^{q-1}V(\xi, x(\xi))$ for $\Theta \in (-\tau, 0)$.

Case 2. Let $\xi \leq \tau$. Then, $\min\{\xi, \tau\} = \xi$. From the choice of the point ξ , similar to Case 1, it follows that $(\xi + \Theta)^{q-1}V(\xi + \Theta, x(\xi + \Theta)) < \xi^{q-1}V(\xi, x(\xi))$ for $\Theta \in (-\xi, 0)$.

Therefore, from condition 2(ii) and inequality (26) for $t = \xi$ we get

$$\begin{aligned} & {}_0^GL D_t^q m(t)|_{t=\xi} \\ & \leq \lim_{h \rightarrow 0+} \frac{1}{h} \left\{ V(\xi, \psi(0)) - \sum_{r=1}^{\lfloor \frac{\xi}{h} \rfloor} (-1)^{r+1} {}_q C_r V(\xi - rh, \psi(0) - h^q f(\xi, \psi)) \right\} \\ & = D_{(1)}^+ V(\xi, \psi) < \frac{C}{\xi^q \Gamma(1-q)}. \end{aligned} \quad (27)$$

Now (27) contradicts (22). Inequality (13) proves the claim. $\square$

Theorem 5.6 *Let the conditions 1(i), 1(ii) of Theorem 5.3 be satisfied and the RL fractional derivative is replaced by Dini fractional derivative, i.e. the following condition is fulfilled:*

- 1.(iii)* *there exists a number $C > 0$ such that for any initial function $\phi \in S_\lambda$ and the corresponding solution $x(t) = x(t; \phi)$ of (1), (2) defined on $[0, \infty)$ such that if for any point $t > 0$ the inequality $(t + s)^{1-q}V(t + s, x^*(t + s)) \leq t^{1-q}V(t, x^*(t))$ holds for $s \in [-\min\{t, \tau\}, 0)$ then*

$$D_{(1)}^+ V(t, \psi) < \frac{C}{t^q \Gamma(1-q)} \quad (28)$$

holds where $\psi(s) = x^(t + s)$, $s \in [-\tau, 0]$.*

Then the RLFrDDE (1) is practically stable in time w.r.t. $(\lambda, a^{-1}(C + b(\lambda)))$.

The proof of Theorem 5.6 is similar to the one of Theorem 5.3 where Lemma 5.5 is applied instead of Lemma 5.2.

5.3 Discussions about the type of Lyapunov function

Note the main condition for Lyapunov functions in all Theorems is connected with the application of RL fractional derivative and the corresponding initial condition, i.e. the condition about $\lim_{t \rightarrow 0+} t^{1-q}V(t, x(t))$.

Let $V(t, x) = \sum_{i=1}^n |x_i|$, $x = (x_1, x_2, \dots, x_n)$. Let $x(t)$ be a solution of (1), (2) with the initial function $\|\phi\|_0 < \lambda$. Then $\lim_{t \rightarrow 0+} t^{1-q}V(t, x(t)) = \sum_{i=1}^n \lim_{t \rightarrow 0+} t^{1-q}|x_i(t)| < \frac{n\lambda}{\Gamma(q)}$, i.e. condition 1(ii) of Theorems 1, 2, 3 is fulfilled with $b(s) = \frac{n}{\Gamma(q)}s$.

Let $V(t, x) = t^{1-q}m(t) \sum_{i=1}^n x_i^2$, where $x \in \mathbb{R}^n$, $x = (x_1, x_2, \dots, x_n)$ and $m \in C([0, T), \mathbb{R}_+)$, $T \leq \infty$. Let $x(t)$ be a solution of (1), (2) with the initial function $\|\phi\|_0 < \lambda$. Then the following

$$\lim_{t \rightarrow 0+} t^{1-q}V(t, x(t)) = \lim_{t \rightarrow 0+} m(t) \sum_{i=1}^n \left(t^{1-q}x_i(t) \right)^2 = m(0) \sum_{i=1}^n \frac{\phi_i(0)^2}{\Gamma^2(q)} < \frac{m(0)}{\Gamma^2(q)} \lambda^2$$

holds. Therefore, condition 1(ii) of Theorems 5.1, 5.3, 5.4 is fulfilled with $b(s) = \frac{m(0)}{\Gamma^2(q)}s^2$.

Let $V(t, x) = \sum_{i=1}^n x_i^2$ and $x(t)$ be a solution of (1), (2) with the initial function $\|\phi\|_0 < \lambda$. Then the following

$$\lim_{t \rightarrow 0+} t^{1-q}V(t, x(t)) = \lim_{t \rightarrow 0+} t^{q-1} \sum_{i=1}^n \left(t^{1-q}x_i(t) \right)^2 = \infty$$

holds, i.e. condition 1(ii) of Theorems 5.1, 5.3, 5.4 is not satisfied for the quadratic Lyapunov function. Note the quadratic Lyapunov function is one of the most applicable functions in studying stability properties of

differential equations with ordinary derivatives or Caputo derivatives. But it is not the same for RL fractional derivatives.

Another condition about Lyapunov functions is $(t+s)^{1-q}V(t+s, x^*(t+s)) \leq t^{1-q}V(t, x^*(t))$ for $s \in [-\min\{t, \tau\}, 0]$.

Let $V(t, x) = t^{1-q}m(t) \sum_{i=1}^n x_i^2$ with $m(t)$ a nonincreasing function. Then the above condition is reduced to $m(t+s) \sum_{i=1}^n \left((t+s)^{1-q}x_i^*(t+s) \right)^2 \leq m(t) \sum_{i=1}^n \left(t^{1-q}x_i^*(t) \right)^2$ for $s \in [-\min\{t, \tau\}, 0]$ or $\left| (t+s)^{1-q}x_i^*(t+s) \right| \leq \left| t^{1-q}x_i^*(t) \right|$, $i = 1, 2, \dots, n$.

The above examples and discussions show that one useful Lyapunov function is $V(t, x) = t^{1-q}m(t) \sum_{i=1}^n x_i^2$.

6 Applications

Example 6.1 Consider the nonlinear RL fractional differential equation with a variable delay

$$\begin{aligned} {}_0^RL D_t^q u(t) &= -\left(\frac{1}{t^q \Gamma(1-q)} + 1 + \left(\frac{t}{t-1} \right)^{1-q} \right) x(t) + 2x(t - \sin^2(t)), \quad t > 0, \\ \lim_{t \rightarrow 0+} t^{1-q} u(t) &= \frac{\phi(0)}{\Gamma(q)}, \\ u(t) &= \phi(t), \quad t \in [-1, 0], \end{aligned} \quad (29)$$

where $u \in \mathbb{R}$.

Consider $V(t, x) = t^{1-q}x^2$. Then there exists a number $T = 1 > 0$ such that $t^{1-q} \geq 1$ for $t \geq T$ and thus the condition 1(i) of Theorem 5.4 is satisfied with $a(s) = s^2$.

Let $y \in C_{1-q}([0, \infty), \mathbb{R}^n) : \lim_{t \rightarrow 0+} [t^{1-q}y(t)] \in S_\lambda$. According to Section 3.3 the inequality $\lim_{t \rightarrow 0+} t^{1-q}V(t, y(t)) = \lim_{t \rightarrow 0+} \left(t^{1-q}y(t) \right)^2 = \frac{y(0)^2}{\Gamma^2(q)} < b(\lambda)$ holds, i.e. condition 1(ii) of Theorem 5.4 is satisfied with $b(s) = \frac{1}{\Gamma^2(q)}s^2$.

In addition, let $x^*(t)$ be a solution of (29) with $\|\phi\|_0 < \lambda$.

Let the point $t > 0$ be such that $\left| (t+s)^{1-q}x^*(t+s) \right| \leq \left| t^{1-q}x^*(t) \right|$ for $s \in [-\min\{1, t\}, 0]$. Let $\psi(s) = x^*(t+s)$, $s \in [-\min\{1, t\}, 0]$. Therefore $|\psi(s)| \leq \left(\frac{t}{t+s} \right)^{1-q} |\psi(0)| \leq \left(\frac{t}{t-1} \right)^{1-q} |\psi(0)|$, $s \in (-\min\{1, t\}, 0]$. Then applying Example 2, Case 2 with $m(t) \equiv 1$ and inequality $2\psi(0)\psi(-1) \leq \psi^2(0) + \psi^2(-1) \leq \psi^2(0) + \left(\frac{t}{t-1} \right)^{1-q} \psi^2(0) = \left(1 + \left(\frac{t}{t-1} \right)^{1-q} \right) \psi^2(0)$ we get

$$\begin{aligned} &D_{(29)}^+ V(t, \psi) \\ &= \psi(0) \left[-\left(\frac{1}{t^q \Gamma(1-q)} + 1 + \left(\frac{t}{t-1} \right)^{1-q} \right) \psi(0) + 2\psi(0 - \sin(0)) \right] + (\psi(0))^2 \frac{1}{t^q \Gamma(1-q)} \\ &\leq 0. \end{aligned} \quad (30)$$

Therefore, the system (29) is practical stable in time w.r.t. $(\lambda, \frac{\lambda}{\Gamma(q)})$. $\square$

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