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Holomorphic connections on holomorphic bundles on Riemann surfaces

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Abstract We investigate aspects of holomorphic connections on holomorphic principal bundles over a Riemann surface.

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1 Introduction

Holomorphic connections, introduced by Atiyah [2], arise in innumerable contexts in mathematics (see [5, 9, 11, 14, 16] and references therein). Their importance can hardly be overemphasized. In this mostly expository work, we consider holomorphic connections, and also logarithmic connections, on holomorphic principal bundles over compact Riemann surfaces.

A famous theorem of Atiyah and Weil says that a holomorphic vector bundle E over a compact Riemann surface M admits a holomorphic connection if and only if the degree of every holomorphic direct summand of E is zero. We recall that in the context of a compact Riemann surface, the degree of a complex vector bundle coincides with its first Chern class; in particular, the degree of a complex vector bundle coincides with the degree of the complex line bundle given by its top exterior product. In [3], this result was extended to holomorphic principal G -bundles on M , where G is a connected reductive complex algebraic group. More precisely, the following was proved in [3]:

Theorem 1.1 *Let G be a connected reductive affine algebraic group defined over $\mathbb{C}$. A holomorphic principal G -bundle E_G over a compact connected Riemann surface M admits a holomorphic connection if and only if for every triple of the form (H, E_H, λ) , where*

- (1) H is a Levi subgroup of G ,
- (2) $E_H \subset E_G$ is a holomorphic reduction of structure group to H , and
- (3) λ is a holomorphic character of H ,

the associated line bundle $E_H(\lambda) = (E_H \times \mathbb{C})/H$ over M is of degree zero.

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The assumption Theorem 1.1 that G is reductive is rather essential; see Sect. 4.

In the final section, we consider logarithmic connections on a compact Riemann surface. The following theorem was proved in [4]:

Theorem 1.2 *Take a holomorphic vector bundle E on M . Fix a finite subset*

$$D := \{x_1, \dots, x_n\} \subset M.$$

For each point $x \in D$, fix

$$T_x \in \text{End}(E_x)$$

such that $\phi(x) \circ T_x = T_x \circ \phi(x)$ for every holomorphic automorphism ϕ of the holomorphic vector bundle E . There is a logarithmic connection θ on E singular over D such that the residue

$$\text{Res}(\theta, x) = T_x$$

if and only if

$$\text{degree}(E) + \sum_{x \in D} \text{trace}(\text{Res}(\theta, x)|_{E_x}) = 0.$$

When $D = \emptyset$, Theorem 1.2 coincides with the above-mentioned criterion of Atiyah and Weil for the existence of a holomorphic connection on a holomorphic vector bundle over M .

2 Holomorphic connection on principal bundles

Let X be a compact connected complex manifold. Let G be a complex Lie group. A holomorphic principal G -bundle over X consists of

- a complex manifold E_G ,
- a surjective holomorphic map $f : E_G \rightarrow X$, and
- a holomorphic right action $\varphi : E_G \times G \rightarrow E_G$ of G on E_G ,

such that the following two conditions hold:

- (1) $f \circ \varphi = f \circ p_1$, where $p_1 : E_G \times G \rightarrow E_G$ is the natural projection, and
- (2) the map to the fiber product

$$E_G \times G \rightarrow E_G \times_X E_G, (z, g) \mapsto (z, \varphi(z, g))$$

is a biholomorphism.

Note that the first condition implies that $(z, \varphi(z, g)) \in E_G \times_X E_G$.

For notational convenience, $\varphi(z, g)$, where $z \in E_G$ and $g \in G$, will be denoted by zg .

The holomorphic tangent (respectively, cotangent) bundle of a complex manifold Y will be denoted by TY (respectively, Ω_Y^1).

The Lie algebra of G will be denoted by $\mathfrak{g}$. Let

$$f : E_G \rightarrow X$$

be a holomorphic principal G -bundle over X . Consider the holomorphic right action of G on the holomorphic tangent bundle TE_G induced by the right action of G on E . The quotient

$$\text{At}(E_G) := (TE_G)/G \tag{2.1}$$

is a complex manifold; in fact, it is a holomorphic vector bundle over $E_G/G = X$. This holomorphic vector bundle $\text{At}(E_G)$ over X is called the *Atiyah bundle* for E_G [2]. The differential

$$df : TE_G \rightarrow f^*TX$$

of the projection f is evidently G -equivariant for the natural action of G on f^*TX that lifts the action of G on E_G . The action of G on E_G produces a holomorphic homomorphism from the trivial holomorphic bundle $E_G \times \mathfrak{g} \rightarrow E_G$ with fiber $\mathfrak{g}$

$$E_G \times \mathfrak{g} \rightarrow \text{kernel}(df) \quad (2.2)$$

which is in fact an isomorphism. From this it follows that we have a short exact sequence of holomorphic vector bundles on E_G

$$0 \rightarrow \text{kernel}(df) = E_G \times \mathfrak{g} \rightarrow TE_G \xrightarrow{df} f^*TX \rightarrow 0; \quad (2.3)$$

all the homomorphisms in (2.3) are G -equivariant. The quotient $\text{kernel}(df)/G$ is called the adjoint vector bundle, and it is denoted by $\text{ad}(E_G)$. This holomorphic vector bundle $\text{ad}(E_G)$ over X is identified with the holomorphic vector bundle $E_G(\mathfrak{g})$ associated to E_G for the adjoint action of G on the Lie algebra $\mathfrak{g}$; this identification is produced by the isomorphism in (2.2).

Taking quotient of the holomorphic vector bundles in (2.3), by the actions of G , the following short exact sequence of holomorphic vector bundles on X is obtained:

$$0 \rightarrow \text{ad}(E_G) \rightarrow \text{At}(E_G) \xrightarrow{d'f} TX \rightarrow 0, \quad (2.4)$$

where $d'f$ is the descent of the homomorphism df (see [2]). The short exact sequence in (2.4) is known as the Atiyah exact sequence for E_G .

A holomorphic connection on E_G is a holomorphic homomorphism of vector bundles

$$\mathcal{D} : TX \rightarrow \text{At}(E_G)$$

such that

$$(d'f) \circ \mathcal{D} = \text{Id}_{TX},$$

where $d'f$ is the projection in (2.4) (see [2]). Therefore, E_G admits a holomorphic connection if and only if the Atiyah exact sequence in (2.4) splits holomorphically.

Giving a holomorphic connection on E_G is equivalent to giving a $\mathfrak{g}$ -valued holomorphic 1-form

$$\omega \in H^0(E_G, \Omega_{E_G}^1 \otimes_{\mathbb{C}} \mathfrak{g}) \quad (2.5)$$

on E_G such that the following two conditions hold:

- the homomorphism

$$\omega : TE_G \rightarrow \mathfrak{g}$$

is G -equivariant for the adjoint action of G on $\mathfrak{g}$ and the action of G on TE_G given by the right action of G on E_G , and

- the restriction of ω to any fiber of f coincides with the Maurer–Cartan form, equivalently, the restriction of ω to any fiber of f coincides with the inverse of the isomorphism in (2.2).

For a form ω as in (2.5) giving a holomorphic connection on E_G , let

$$\text{kernel}(\omega) \subset TE_G$$

be the kernel of the homomorphism $TE_G \rightarrow E_G \times \mathfrak{g}$ given by ω . The action of G on TE_G preserves this holomorphic subbundle $\text{kernel}(\omega)$. Consequently,

$$\text{kernel}(\omega)/G \subset (TE_G)/G =: \text{At}(E_G)$$

is a holomorphic subbundle. There is a unique holomorphic homomorphism

$$\mathcal{D} : TX \rightarrow \text{At}(E_G)$$

such that

- $\mathcal{D}(TX) \subset \ker(\omega)/G$, and
- $(d'f) \circ \mathcal{D} = \text{Id}_{TX}$, where $d'f$ is the homomorphism in (2.4).

Therefore, $\mathcal{D}$ defines a holomorphic connection on E_G .

Conversely, given a holomorphic homomorphism

$$\mathcal{D} : TX \longrightarrow \text{At}(E_G)$$

such that $(d'f) \circ \mathcal{D} = \text{Id}_{TX}$, let

$$V \subset TE_G \tag{2.6}$$

be the inverse image of $\mathcal{D}(TX) \subset \text{At}(E_G)$ under the quotient map

$$TE_G \longrightarrow (TE_G)/G =: \text{At}(E_G).$$

Then there is a unique holomorphic 1-form $\omega \in H^0(E_G, \Omega_{E_G}^1 \otimes_{\mathbb{C}} \mathfrak{g})$ such that

- the kernel of the homomorphism $TE_G \longrightarrow E_G \times \mathfrak{g}$ given by ω coincides with the subbundle $V \subset TE_G$ in (2.6), and
- the restriction of ω to any fiber of f coincides with the Maurer–Cartan form.

The above homomorphism $\omega : TE_G \longrightarrow \mathfrak{g}$ is G -equivariant, and hence we conclude that ω defines a holomorphic connection on E_G .

For a complex manifold Z , the Lie bracket operation of locally defined holomorphic vector fields on Z makes the coherent analytic sheaf TZ a sheaf of Lie algebras. Note that the Lie bracket operation on locally defined holomorphic vector fields on Z is *not* $\mathcal{O}_Z$ -linear.

The Lie bracket of two G -invariant vector fields on $E_G|_U$, where $U \subset X$ is an open subset, is again G -invariant. Consequently, the Lie algebra structure on the coherent analytic sheaf TE_G makes the coherent analytic sheaf $\text{At}(E_G)$ a sheaf of Lie algebras. Take a holomorphic connection

$$\mathcal{D} : TX \longrightarrow \text{At}(E)$$

on E_G . The *curvature* of $\mathcal{D}$ is the obstruction for the homomorphism $\mathcal{D}$ to be compatible with the Lie algebra structures on the sheaves $\text{At}(E_G)$ and TX . More precisely, given holomorphic vector fields s, t defined on an open subset $U \subset X$, consider

$$\mathcal{K}(\mathcal{D})(s, t) := [\mathcal{D}(s), \mathcal{D}(t)] - \mathcal{D}([s, t]) \in H^0(U, \text{At}(E_G)).$$

Since the projection $d'f$ in (2.4) is Lie algebra structure preserving, it follows that

$$\mathcal{K}(\mathcal{D})(s, t) \in H^0(U, \text{ad}(E_G)).$$

Furthermore, we have

$$\mathcal{K}(\mathcal{D})(s, t) = -\mathcal{K}(\mathcal{D})(t, s),$$

and

$$\mathcal{K}(\mathcal{D})(\phi \cdot s, t) = -\phi \cdot \mathcal{K}(\mathcal{D})(t, s)$$

for all $\phi \in H^0(U, \mathcal{O}_U)$. These properties together imply that

$$\mathcal{K}(\mathcal{D}) \in H^0(X, \text{ad}(E_G) \otimes \Omega_X^2).$$

This $\text{ad}(E_G)$ -valued holomorphic 2-form $\mathcal{K}(\mathcal{D})$ on X is the curvature of D .

The connection D is called flat (or integrable) if $\mathcal{K}(\mathcal{D}) = 0$. A connection D is flat if and only if the distribution on E_G given by the subbundle $V \subset TE_G$ in (2.6) corresponding to $\mathcal{D}$ is integrable [15, p. 78, Corollary 5.3], [10].

If $\dim_{\mathbb{C}} X = 1$, then $\Omega_X^2 = 0$. Therefore, any holomorphic connection on a holomorphic principal G -bundle over a Riemann surface is automatically flat.

3 Holomorphic connections on bundles over a Riemann surface

In this section we will consider holomorphic principal G -bundles on compact Riemann surfaces and holomorphic connections on them.

Let M be a compact connected Riemann surface. As before, G is a complex Lie group. Given a holomorphic principal G -bundle E_G on M , it is natural to ask whether E_G admits a holomorphic connection. A well-known theorem of Atiyah and Weil answers this question when $G = \mathrm{GL}(r, \mathbb{C})$ [2, 17]. This result of [2, 17] will be recalled below.

Using the standard representation of $\mathrm{GL}(r, \mathbb{C})$, holomorphic principal $\mathrm{GL}(r, \mathbb{C})$ -bundles on M are identified with holomorphic vector bundles on M of rank r .

A holomorphic vector bundle W on a compact connected complex manifold X is called *decomposable* if

$$W = W_1 \oplus W_2,$$

where W_1 and W_2 are holomorphic vector bundles of positive rank. A holomorphic vector bundle W on X is called *indecomposable* if it is not decomposable. Every holomorphic vector bundle on X is holomorphically isomorphic to a direct sum of indecomposable holomorphic vector bundles.

The following very important theorem was proved by Atiyah.

Theorem 3.1 ([1, p. 315, Theorem 3]) *Let W be a holomorphic vector bundle over a compact connected complex manifold X . If W is holomorphically isomorphic to both $\bigoplus_{i=1}^m V_i$ and $\bigoplus_{j=1}^n W_j$, where V_i , $1 \leq i \leq m$, and W_j , $1 \leq j \leq n$, are all indecomposable vector bundles on X , then*

- (1) $m = n$, and
- (2) *there is a permutation σ of $\{1, \dots, m\}$ such that V_i is holomorphically isomorphic to $W_{\sigma(i)}$ for all $1 \leq i \leq m$.*

The above-mentioned theorem of [2, 17] says that a holomorphic vector bundle E over a compact connected Riemann surface M admits a holomorphic connection if and only if each direct summand of E is of degree zero. Let

$$E = \bigoplus_{i=1}^m V_i$$

be a decomposition of E into a direct sum of indecomposable vector bundles (as in Theorem 3.1). Then E admits a holomorphic connection if and only if

$$\mathrm{degree}(V_i) = 0$$

for all $1 \leq i \leq m$.

We shall now describe a generalization of the above theorem of [2] and [17] to the context of principal bundles.

Let G be a connected reductive affine algebraic group defined over $\mathbb{C}$. A Zariski closed connected subgroup $P \subset G$ is called a parabolic subgroup if G/P is a projective variety [8, 11.2], [13]. The unipotent radical of a parabolic subgroup $P \subset G$ will be denoted by $R_u(P)$. The quotient group $P/R_u(P)$ is called the *Levi quotient* of P . A *Levi factor* of P is a Zariski closed connected subgroup $L \subset P$ such that the composition

$$L \hookrightarrow P \longrightarrow P/R_u(P)$$

is an isomorphism [13, p. 184]. We note that P admits Levi factors, and any two Levi factors of P are conjugate by an element of $R_u(P)$ [13, § 30.2, p. 185, Theorem]. A Levi factor $L \subset P$ satisfies the following condition: L contains a maximal torus of P , and moreover L is a maximal reductive subgroup of P (see [8, 13]).

Given a holomorphic principal G -bundle E_G on M and a complex Lie subgroup $H \subset G$, a holomorphic reduction of E_G to H is given by a holomorphic section of the holomorphic fiber bundle E_G/H over M . Let

$$q_H : E_G \longrightarrow E_G/H$$

be the quotient map. If $v : M \longrightarrow E_G/H$ is a holomorphic section of the fiber bundle $E_G/H \longrightarrow M$, then note that

$$q_H^{-1}(v(M)) \subset E_G$$

is a holomorphic principal H -bundle on M . Conversely, if a complex submanifold $E_H \subset E_G$ is preserved by the action of H on E_G , and E_H is a principal H -bundle for the action of H , then the natural map

$$M = E_H/H \hookrightarrow E_G/H$$

is a holomorphic section of the projection $E_G/H \rightarrow M$.

If E_H is a holomorphic principal H -bundle on M , and χ is a holomorphic character of H , then the associated holomorphic line bundle

$$E_H(\lambda) = (E_H \times \mathbb{C})/H \rightarrow M \quad (3.1)$$

is the quotient of $E_H \times \mathbb{C}$, where $(z_1, c_1), (z_2, c_2) \in E_H \times \mathbb{C}$ are identified if there is an element $g \in H$ such that

- $z_2 = z_1 g$, and
- $c_2 = \frac{c_1}{\lambda(g)}$.

The following theorem is proved in [3] (see [3, Theorem 4.1]).

Theorem 3.2 *Let G be a connected reductive affine algebraic group defined over $\mathbb{C}$. A holomorphic principal G -bundle E_G over a compact connected Riemann surface M admits a holomorphic connection if and only if for every triple of the form (H, E_H, λ) , where*

- (1) H is a Levi subgroup of G ,
- (2) $E_H \subset E_G$ is a holomorphic reduction of structure group to H , and
- (3) λ is a holomorphic character of H ,

the associated line bundle $E_H(\lambda) = (E_H \times \mathbb{C})/H$ (see (3.1)) over M is of degree zero.

Note that if we set $G = \mathrm{GL}(n, \mathbb{C})$ in Theorem 3.2, then it coincides with the mentioned criterion of Atiyah and Weil for the existence of a holomorphic connection on a holomorphic vector bundle over M .

A sketch of the proof of Theorem 3.2 will be given below.

Let E_G be a holomorphic principal G -bundle over M equipped with a holomorphic connection ∇ . Take any triple (H, E_H, λ) as in Theorem 3.2. We will first show that a holomorphic connection ∇ on E_G produces a holomorphic connection on the principal H -bundle E_H .

Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and H respectively. The group H has adjoint actions on both $\mathfrak{h}$ and $\mathfrak{g}$. To construct the connection on E_H , fix a splitting of the injective homomorphism of H -modules

$$0 \rightarrow \mathfrak{h} \rightarrow \mathfrak{g}.$$

Let

$$\rho : \mathfrak{g} \rightarrow \mathfrak{h}$$

be the chosen splitting. Since a holomorphic connection on E_G is given by a holomorphic splitting of the Atiyah exact sequence for E_G , a holomorphic connection ∇ on E_G produces a $\mathfrak{g}$ -valued holomorphic 1-form ω on E_G satisfying the following two conditions:

- ω is G -equivariant (G acts on $\mathfrak{g}$ by inner automorphism), and
- the restriction of ω to any fiber of E_G is the Maurer–Cartan form on the fiber.

Using the chosen splitting homomorphism

$$\mathfrak{g} \xrightarrow{\rho} \mathfrak{h},$$

the connection form ω on E_G defines a $\mathfrak{h}$ -valued holomorphic one-form ω' on E_G . The restriction of ω' to the complex submanifold $E_H \subset E_G$ satisfies the two conditions needed for a holomorphic $\mathfrak{h}$ -valued 1-form on E_H to define a holomorphic connection on E_H .

Therefore, E_H admits a holomorphic connection. A holomorphic connection on E_H induces a holomorphic connection on the associated line bundle $E_H(\lambda)$. Any line bundle admitting a holomorphic connection must be of degree zero [2]. Therefore, if E_G admits a holomorphic connection then we know that the degree of $E_H(\lambda)$ is zero.

To prove the converse, let E_G be a holomorphic principal G -bundle over M such that

$$\text{degree}(E_H(\lambda)) = 0$$

for all triples (H, E_H, λ) of the above type. We need to show that the Atiyah exact sequence for E_G (see (2.4)) splits holomorphically.

As the first step for it, in [3] the following is proved: it is enough to prove that the Atiyah exact sequence for E_G splits holomorphically under the assumption that E_G does not admit any holomorphic reduction of structure group to any proper Levi subgroup of G . Therefore, we assume that E_G does not admit any holomorphic reduction of structure group to any proper Levi subgroup of G .

Let K_M denote the holomorphic cotangent bundle of M . The obstruction for splitting of the Atiyah exact sequence for E_G is an element

$$\tau(E_G) \in H^1(M, K_M \otimes \text{ad}(E_G)). \quad (3.2)$$

We note that

$$\text{ad}(E_G)^* = \text{ad}(E_G).$$

Indeed, the Lie algebra $\mathfrak{g}$ admits a G -invariant symmetric nondegenerate bilinear form. Such a form on $\mathfrak{g}$ produces an isomorphism of $\text{ad}(E_G)$ with $\text{ad}(E_G)^*$. Consequently, by Serre duality,

$$H^1(M, K_M \otimes \text{ad}(E_G)) = H^0(M, \text{ad}(E_G))^*.$$

So we have

$$\tau(E_G) \in H^0(M, \text{ad}(E_G))^*. \quad (3.3)$$

Any homomorphic section ϕ of the adjoint bundle $\text{ad}(E_G)$ has a Jordan decomposition

$$\phi = f_s + f_n,$$

where f_s is pointwise semisimple and f_n is pointwise nilpotent. From the assumption that E_G does not admit any holomorphic reduction of structure group to any proper Levi subgroup of G it can be deduced that the semisimple section f_s is given by some element of the center of $\mathfrak{g}$. Indeed, the conjugacy class of $f_s(x)$ is independent of $x \in M$; this follows from the fact that there is no nonconstant holomorphic map from M to the affine variety that parametrizes the conjugacy classes of semisimple elements in $\mathfrak{g}$. Fix an element $z_0 \in \mathfrak{g}$ in the conjugacy class determined by $f_s(x)$. Then the centralizer of z_0 in G is a Levi subgroup $L(z_0)$ of some parabolic subgroup of G . Note that for any $y \in E_G$ and $z' \in \mathfrak{g}$, the pair (y, z') represents an element of $\text{ad}(E_G)$. It is straightforward to check that

$$E_{L(z_0)} := \{y \in E_G \mid (y, z_0) \in f_s(M)\} \subset E_G$$

is a holomorphic reduction of structure group of E_G to the subgroup $L(z_0)$ of G . Now from the given assumption that E_G does not admit any holomorphic reduction of structure group to any proper Levi subgroup of G we conclude that $L(z_0) = G$. Consequently, the semisimple section f_s is given by some element of the center of $\mathfrak{g}$.

Since the semisimple section f_s is given by some element of the center of $\mathfrak{g}$, from the given condition on E_G it can be deduced that

$$\tau(E_G)(f_s) = 0,$$

where $\tau(E_G)$ is the element in (3.3).

The nilpotent section f_n of $\text{ad}(E_G)$ gives a holomorphic reduction of structure group

$$E_P \subset E_G$$

of E_G to a proper parabolic subgroup P of G . This reduction E_P has the property that f_n lies in the image

$$H^0(M, \mathcal{R}(\text{ad}(E_P))) \hookrightarrow H^0(M, \text{ad}(E_P)) \hookrightarrow H^0(M, \text{ad}(E_G)),$$

where $\text{ad}(E_P)$ is the adjoint bundle of E_P , and

$$\mathcal{R}(\text{ad}(E_P)) \subset \text{ad}(E_P)$$

is the nilpotent radical bundle of $\text{ad}(E_P)$. On the other hand, the cohomology class $\tau(E_G) \in H^1(M, K_M \otimes \text{ad}(E_G))$ in (3.2) lies in the image of the homomorphism

$$H^1(M, K_M \otimes \text{ad}(E_P)) \longrightarrow H^1(M, K_M \otimes \text{ad}(E_G))$$

induced by the inclusion map $\text{ad}(E_P) \hookrightarrow \text{ad}(E_G)$.

next we observe the following. Let $\mathfrak{p}$ be the Lie algebra of P , and let $\mathcal{R}(\mathfrak{p}) \subset \mathfrak{p}$ be the nilpotent radical of $\mathfrak{p}$. Then for any G -invariant symmetric bilinear form B on $\mathfrak{g}$, we have

$$B(\mathfrak{p}, \mathcal{R}(\mathfrak{p})) = 0.$$

This, and the above observations that

- f_n lies in the image

$$H^0(M, \mathcal{R}(\text{ad}(E_P))) \hookrightarrow H^0(M, \text{ad}(E_G))$$

and

- $\tau(E_G) \in H^1(M, K_M \otimes \text{ad}(E_G))$ lies in the image of the homomorphism

$$H^1(M, K_M \otimes \text{ad}(E_P)) \longrightarrow H^1(M, K_M \otimes \text{ad}(E_G)),$$

combine together to imply that $\tau(E_G)(f_n) = 0$.

Hence $\tau(E_G)(\phi) = 0$ for all $\phi \in H^0(M, \text{ad}(E_G))$, which implies that $\tau(E_G) = 0$. Therefore, the Atiyah exact sequence for E_G splits holomorphically, implying that the holomorphic principal G -bundle E_G admits a holomorphic connection.

3.1 Non-reductive algebraic groups

Now let G be a unipotent affine algebraic group defined over $\mathbb{C}$. It is known that any holomorphic principal G -bundle over a compact connected Riemann surface M admits a holomorphic connection [7, p. 4015, Corollary 3.8].

Let G be a connected affine algebraic group defined over $\mathbb{C}$. The unipotent radical of G will be denoted by $R_u(G)$. Let

$$L(G) := G/R_u(G)$$

be the reductive quotient. Let E_G be a holomorphic principal G -bundle over a compact connected Riemann surface M . Assume that the holomorphic principal $L(G)$ -bundle $E_G/R_u(G)$ admits a holomorphic connection. The following is a natural question to ask:

Does the holomorphic principal G -bundle E_G admit a holomorphic connection?

We will show in Sect. 4 that the above question has a negative answer.

4 A construction

This section is an exposition of [6].

Let M be a compact connected Riemann surface of genus g , with $g \geq 2$. Denote by K_M the holomorphic cotangent bundle of M . The linear equivalence class of K_M can be expressed as

$$K_M = P + D, \quad (4.1)$$

where $P \in M$ is a single point and D is an effective divisor of degree $2g - 3$ such that P is disjoint from the support of D . We note that this can be deduced from the fact that the linear system K_M is globally generated and

$$\dim H^0(M, K_M) = g \geq 2.$$

Indeed, let

$$f : M \longrightarrow \mathbb{P}(H^0(M, K_M))$$

be the map corresponding to K_M . Take a hyperplane $\mathbb{H} \subset \mathbb{P}(H^0(M, K_M))$ such that there is a point $z \in \mathbb{H} \cap f(M)$ satisfying the following two conditions:

- (1) $\mathbb{H}$ and $f(M)$ intersects transversally at z , and
- (2) f is unramified over z .

Now take $P \in f^{-1}(z)$ and $D = f^{-1}(\mathbb{H}) \setminus \{P\}$.

Fix P and D as in (4.1). Let us now split the divisor D into two parts, namely

$$D = D_Q + D_R, \quad (4.2)$$

where D_Q and D_R are effective divisors with

$$\text{degree}(D_Q) + 1 = \text{degree}(D_R) = g - 1. \quad (4.3)$$

By Serre duality,

$$H^1(M, \mathcal{O}_M(D_Q + D_R)) = H^0(M, \mathcal{O}_M(P))^* = \mathbb{C}. \quad (4.4)$$

In particular, we can choose a nonzero element (which is actually unique up to multiplication by a scalar)

$$\theta \in H^1(M, \mathcal{O}_M(D_Q + D_R)) \setminus \{0\}. \quad (4.5)$$

Since

$$\mathcal{O}_M(D_Q + D_R) = \text{Hom}_{\mathcal{O}_M}(\mathcal{O}_M(-D_R), \mathcal{O}_M(D_Q)),$$

the cohomology class θ in (4.5) produces a short exact sequence of holomorphic vector bundles

$$0 \longrightarrow \mathcal{O}_M(D_Q) \longrightarrow V \longrightarrow \mathcal{O}_M(-D_R) \longrightarrow 0 \quad (4.6)$$

on M . This exact sequence does not split holomorphically because $\theta \neq 0$. From (4.3), it follows that $\text{degree}(V) = -1$. Consider the holomorphic vector bundle

$$E := \mathcal{O}_M(P) \oplus V \quad (4.7)$$

on M of rank three and degree zero. We have

$$\mathcal{O}_M(P) \oplus \mathcal{O}_M(D_Q) \subset E \quad (4.8)$$

because $\mathcal{O}_M(D_Q) \subset V$ (see (4.6)). Let s^P (respectively, s^Q) be the holomorphic section of $\mathcal{O}_M(P)$ (respectively, $\mathcal{O}_M(D_Q)$) given by the constant function 1 on M . So s^P (respectively, s^Q) vanishes over P (respectively, the support of D_Q) of order one, and is nonzero everywhere else. Now consider the holomorphic section

$$\sigma_1 : \mathcal{O}_M \longrightarrow \mathcal{O}_M(P) \oplus \mathcal{O}_M(D_Q) \quad (4.9)$$

defined by $x \longmapsto (s^P(x), s^Q(x))$. Note that σ_1 does not vanish anywhere because P is disjoint from the support of D_Q according to (4.1) and (4.2). Let σ be the composition

$$\mathcal{O}_M \xrightarrow{\sigma_1} \mathcal{O}_M(P) \oplus \mathcal{O}_M(D_Q) \hookrightarrow E \quad (4.10)$$

(see (4.8)). Since σ is nowhere vanishing, we get a short exact sequence of holomorphic vector bundles on M

$$0 \longrightarrow F := \mathcal{O}_M \xrightarrow{\sigma} E \longrightarrow \mathcal{Q} := E/\sigma(F) \longrightarrow 0. \quad (4.11)$$

By construction, $\mathcal{O}_M(P)$ is a direct summand of E . Since $\text{degree}(\mathcal{O}_M(P)) \neq 0$, from the criterion of Atiyah–Weil we conclude that E does not admit a holomorphic connection.

We will now prove that both F and $\mathcal{Q}$ in (4.11) admit a holomorphic connection.

Of course, $F := \mathcal{O}_M$ admits the trivial holomorphic connection. So we need to show that $\mathcal{Q}$ in (4.11) admits a holomorphic connection. The following lemma would be needed for that.

Lemma 4.1 *The holomorphic vector bundle $\mathcal{Q}$ in (4.11) is a nontrivial extension of the line bundle $\mathcal{O}_M(-D_R)$ by $\mathcal{O}_M(P + D_Q)$.*

Proof On the one hand, we have

$$\bigwedge^2 (\mathcal{O}_M(P) \oplus \mathcal{O}_M(D_Q)) = \mathcal{O}_M(P + D_Q).$$

On the other hand,

$$\bigwedge^2 (\mathcal{O}_M(P) \oplus \mathcal{O}_M(D_Q)) = (\mathcal{O}_M(P) \oplus \mathcal{O}_M(D_Q)) / \sigma_1(\mathcal{O}_M)$$

(see (4.9)). It follows that

$$(\mathcal{O}_M(P) \oplus \mathcal{O}_M(D_Q)) / \sigma_1(\mathcal{O}_M) = \mathcal{O}_M(P + D_Q). \quad (4.12)$$

The inclusion of $\mathcal{O}_M(P) \oplus \mathcal{O}_M(D_Q)$ in E (see (4.10)) produces an inclusion of the quotient $(\mathcal{O}_M(P) \oplus \mathcal{O}_M(D_Q)) / \sigma_1(\mathcal{O}_M)$ in $E / \sigma(F) = \mathcal{Q}$ (see (4.11)). Therefore, from (4.12), we have

$$\mathcal{O}_M(P + D_Q) \subset \mathcal{Q} \quad (4.13)$$

as a subbundle. Using (4.6), (4.7) we have

$$\bigwedge^2 \mathcal{Q} = \bigwedge^3 E = \mathcal{O}_M(P) \otimes \bigwedge^2 V = \mathcal{O}_M(P + D_Q - D_R).$$

Note that its degree is zero (4.3). Therefore, from (4.13),

$$\mathcal{O}_M(P + D_Q - D_R) = \bigwedge^2 \mathcal{Q} = \mathcal{O}_M(P + D_Q) \otimes (\mathcal{Q} / \mathcal{O}_M(P + D_Q)).$$

So, $\mathcal{Q} / \mathcal{O}_M(P + D_Q) = \mathcal{O}_M(-D_R)$. Consequently, from (4.13), we get a short exact sequence of vector bundles

$$0 \longrightarrow \mathcal{O}_M(P + D_Q) \longrightarrow \mathcal{Q} \longrightarrow \mathcal{O}_M(-D_R) \longrightarrow 0. \quad (4.14)$$

To complete the proof of the lemma, we need to show that the short exact sequence in (4.14) does not split. Let

$$\omega \in H^1(M, \text{Hom}(\mathcal{O}_M(-D_R), \mathcal{O}_M(P + D_Q))) = H^1(M, \mathcal{O}_M(P + D_Q + D_R)) \quad (4.15)$$

be the extension class for the exact sequence in (4.14). We will now compute ω .

From (4.6) and (4.7) we have the short exact sequence

$$0 \longrightarrow \mathcal{O}_M(P) \oplus \mathcal{O}_M(D_Q) \longrightarrow E \longrightarrow \mathcal{O}_M(-D_R) \longrightarrow 0$$

of holomorphic vector bundles on M . Let

$$\begin{aligned} \theta' &\in H^1(M, (\mathcal{O}_M(P) \oplus \mathcal{O}_M(D_Q)) \otimes \mathcal{O}_M(D_R)) \\ &= H^1(M, \mathcal{O}_M(P + D_R)) \oplus H^1(M, \mathcal{O}_M(D_Q + D_R)) \end{aligned}$$

be the cohomology class for this exact sequence. Evidently, θ' coincides with

$$(0, \theta) \in H^1(M, \mathcal{O}_M(P + D_R)) \oplus H^1(M, \mathcal{O}_M(D_Q + D_R)),$$

where θ is the class in (4.5).

Next, consider the homomorphism γ defined by the composition

$$\mathcal{O}_M(D_Q) \hookrightarrow \mathcal{O}_M(P) \oplus \mathcal{O}_M(D_Q) \twoheadrightarrow (\mathcal{O}_M(P) \oplus \mathcal{O}_M(D_Q)) / \sigma_1(F) = \mathcal{O}_M(P + D_Q)$$

(see (4.12)), where the homomorphism

$$\mathcal{O}_M(D_Q) \hookrightarrow \mathcal{O}_M(P) \oplus \mathcal{O}_M(D_Q)$$

is the inclusion of the second factor. Clearly, this composition γ coincides with the natural inclusion of the coherent sheaf $\mathcal{O}_M(D_Q)$ in $\mathcal{O}_M(P + D_Q)$. Therefore, the cohomology classes ω and θ (constructed in (4.15) and (4.5)) satisfy the equation

$$\omega = \rho(\theta), \quad (4.16)$$

where

$$\rho : H^1(M, \mathcal{O}_M(D_Q + D_R)) \longrightarrow H^1(M, \mathcal{O}_M(P + D_Q + D_R))$$

is the homomorphism induced by the natural inclusion map

$$\mathcal{O}_M(D_Q + D_R) \hookrightarrow \mathcal{O}_M(P + D_Q + D_R)$$

of coherent analytic sheaves. Consider the short exact sequence of coherent sheaves

$$0 \longrightarrow \mathcal{O}_M(D_Q + D_R) \longrightarrow \mathcal{O}_M(P + D_Q + D_R) \longrightarrow \mathcal{O}_M(P + D_Q + D_R)_P \longrightarrow 0,$$

where $\mathcal{O}_M(P + D_Q + D_R)_P$ is the torsion sheaf supported at P with its stalk being the fiber of the line bundle $\mathcal{O}_M(P + D_Q + D_R)$ over P . Let

$$\begin{aligned} 0 &\longrightarrow H^0(M, \mathcal{O}_M(D_Q + D_R)) \longrightarrow H^0(M, \mathcal{O}_M(P + D_Q + D_R)) \\ &\xrightarrow{\alpha_1} \mathcal{O}_M(P + D_Q + D_R)_P \xrightarrow{\alpha_2} H^1(M, \mathcal{O}_M(D_Q + D_R)) \\ &\xrightarrow{\rho} H^1(M, \mathcal{O}_M(P + D_Q + D_R)) \end{aligned} \quad (4.17)$$

be the long exact sequence of cohomologies associated with it. We have

$$\dim H^0(M, \mathcal{O}_M(P + D_Q + D_R)) = \dim H^0(M, \mathcal{O}_M(K_M)) = g$$

and, by Riemann–Roch and (4.4),

$$\dim H^0(M, \mathcal{O}_M(D_Q + D_R)) = g - 1.$$

These imply that α_1 in (4.17) is surjective. Therefore, α_2 in (4.17) is the zero homomorphism. This implies that ρ in (4.17) is injective.

Since ρ is injective, from (4.16) it follows that $\omega \neq 0$, because $\theta \neq 0$ (see (4.5)). The exact sequence in (4.14) does not split because $\omega \neq 0$. $\square$

Proposition 4.2 *The holomorphic vector bundle $\mathcal{Q}$ in (4.14) admits a holomorphic connection.*

Proof Assume that $\mathcal{Q}$ does not admit any holomorphic connection. Since $\text{degree}(\mathcal{Q}) = 0$, and $\mathcal{Q}$ does not admit any holomorphic connection, the criterion of Atiyah–Weil says that $\mathcal{Q}$ holomorphically decomposes as

$$\mathcal{Q} = L \oplus \mathcal{M}, \quad (4.18)$$

where $\text{degree}(L) = -\text{degree}(\mathcal{M}) > 0$. Let $p_{\mathcal{M}} : \mathcal{Q} \longrightarrow \mathcal{M}$ be the projection given by the decomposition in (4.18). Let β denote the composition

$$\mathcal{O}_M(P + D_Q) \hookrightarrow \mathcal{Q} \xrightarrow{p_{\mathcal{M}}} \mathcal{M},$$

where the inclusion is constructed in (4.13). Since

$$\text{degree}(\mathcal{O}_M(P + D_Q)) = g - 1 > 0 > \text{degree}(\mathcal{M}),$$

there is no nonzero homomorphism from $\mathcal{O}_M(P + D_Q)$ to $\mathcal{M}$. In particular, $\beta = 0$.

We have $\mathcal{O}_M(P + D_Q) \subset L$ because $\beta = 0$. Since both $\mathcal{O}_M(P + D_Q)$ and L are line subbundles on $\mathcal{Q}$, this implies that the two subbundles $\mathcal{O}_M(P + D_Q)$ and L coincide. Hence,

$$\mathcal{M} = \mathcal{Q}/L = \mathcal{Q}/\mathcal{O}_M(P + D_Q) = \mathcal{O}_M(-D_R)$$

(see Lemma 4.1). Therefore, the decomposition $\mathcal{Q} = L \oplus \mathcal{M}$ in (4.18) produces a splitting of the short exact sequence in (4.14). But we know from Lemma 4.1 that the short exact sequence in (4.14) does not split. In view of the above contradiction, we conclude that $\mathcal{Q}$ admits a holomorphic connection. $\square$

As we have seen, E does not admit a holomorphic connection. On the other hand, consider the short exact sequence in (4.11). The trivial holomorphic line bundle $F = \mathcal{O}_M$ admits the trivial holomorphic connection. The quotient bundle $\mathcal{Q}$ is flat by Proposition 4.2.

Therefore, the question in Sect. 3.1 has a negative answer.

5 Logarithmic connections on a Riemann surface

The multiplicative group $\mathbb{C} \setminus \{0\}$ will be denoted by $\mathbb{G}_m$. A torus is a product of copies of $\mathbb{G}_m$. Any two maximal tori in a complex algebraic group are conjugate [8, p. 158, Proposition 11.23(ii)].

By a homomorphism between algebraic groups or by a character we will always mean a holomorphic homomorphism or a holomorphic character.

Let M be a compact connected Riemann surface. Fix a finite subset

$$D := \{x_1, \dots, x_n\} \subset M.$$

The reduced effective divisor $x_1 + \dots + x_n$ will also be denoted by D . The subsheaf $TM \otimes \mathcal{O}_M(-D)$ of TM will be denoted by $TM(-D)$.

Let G be a complex Lie group.

$$f : E_G \longrightarrow M$$

be a holomorphic principal G -bundle over M . Consider the Atiyah exact sequence in (2.4). Now define

$$\text{At}(E_G, D) := (d'f)^{-1}(TM(-D)) \subset \text{At}(E_G),$$

where $d'f$ is the projection in (2.4). So from (2.4) we have the exact sequence of vector bundles on M

$$0 \longrightarrow \text{ad}(E_G) \xrightarrow{i_0} \text{At}(E_G, D) \xrightarrow{\sigma} TM(-D) \longrightarrow 0, \quad (5.1)$$

where σ is the restriction of $d'f$ to $\text{At}(E_G, D) \subset \text{At}(E_G)$. The exact sequence in (5.1) will be called the *logarithmic Atiyah exact sequence* for E_G .

A logarithmic connection on E_G singular over D is a holomorphic homomorphism

$$\theta : TM(-D) \longrightarrow \text{At}(E_G, D) \quad (5.2)$$

such that $\sigma \circ \theta = \text{Id}_{TM(-D)}$, where σ is the homomorphism in (5.1). Note that giving such a homomorphism θ is equivalent to giving a homomorphism

$$\varpi : \text{At}(E_G, D) \longrightarrow \text{ad}(E_G)$$

such that $\varpi \circ i_0 = \text{Id}_{\text{ad}(E_G)}$, where i_0 is the homomorphism in (5.1).

So, a holomorphic principal G -bundle admits a logarithmic connection singular over D if and only if the logarithmic Atiyah exact sequence in (5.1) splits holomorphically.

5.1 Residue of a logarithmic connection

Given a vector bundle W on M , the fiber of W over any point $x \in M$ will be denoted by W_x . For any $\mathcal{O}_M$ -linear homomorphism $f : W \longrightarrow V$ of holomorphic vector bundles, its restriction $W_x \longrightarrow V_x$ will be denoted by $f(x)$.

From (2.4) and (5.1), we have the commutative diagram of homomorphisms

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{ad}(E_G) & \xrightarrow{i_0} & \text{At}(E_G, D) & \xrightarrow{\sigma} & TM(-D) \longrightarrow 0 \\ & & \parallel & & \downarrow j & & \downarrow \iota \\ 0 & \longrightarrow & \text{ad}(E_G) & \xrightarrow{i} & \text{At}(E_G) & \xrightarrow{d'f} & TM \longrightarrow 0 \end{array} \quad (5.3)$$

on M . So for any point $x \in M$, we have

$$d'f(x) \circ j(x) = \iota(x) \circ \sigma(x) : \text{At}(E_G, D)_x \longrightarrow (TM)_x = T_x M.$$

Note that $\iota(x) = 0$ if $x \in D$; therefore, in that case $d'f(x) \circ j(x) = 0$. Consequently, for every $x \in D$ there is a homomorphism

$$R_x : \text{At}(E_G, D)_x \longrightarrow \text{ad}(E_G)_x \quad (5.4)$$

uniquely defined by the identity $i(x) \circ R_x(v) = j(x)(v)$ for all $v \in \text{At}(E_G, D)_x$. Note that

$$R_x \circ i_0(x) = \text{Id}_{\text{ad}(E_G)_x},$$

where i_0 is the homomorphism in (5.3). Therefore, from (5.1), we have

$$\text{At}(E_G, D)_x = \text{ad}(E_G)_x \oplus \text{kernel}(R_x) = \text{ad}(E_G)_x \oplus TM(-D)_x; \quad (5.5)$$

note that the composition of homomorphisms

$$\text{kernel}(R_x) \hookrightarrow \text{At}(E_G, D)_x \xrightarrow{\sigma(x)} TM(-D)_x$$

is an isomorphism.

For any $x \in D$, the fiber $TM(-D)_x$ is identified with $\mathbb{C}$ using the Poincaré adjunction formula [12, p. 146]. Indeed, for any holomorphic coordinate z around x with $z(x) = 0$, the image of $z \frac{\partial}{\partial z}$ in $TM(-D)_x$ is independent of the choice of the coordinate function z ; the above-mentioned identification between $TM(-D)_x$ and $\mathbb{C}$ sends this independent image to $1 \in \mathbb{C}$. Therefore, from (5.5), we have

$$\text{At}(E_G, D)_x = \text{ad}(E_G)_x \oplus \mathbb{C} \quad (5.6)$$

for all $x \in D$.

For a logarithmic connection $\theta : TM(-D) \rightarrow \text{At}(E_G, D)$ as in (5.2), and any $x \in D$, define

$$\text{Res}(\theta, x) := R_x(\theta(1)) \in \text{ad}(E_G)_x, \quad (5.7)$$

where R_x is the homomorphism in (5.4); in the above definition 1 is considered as an element of $TM(-D)_x$ using the identification of $\mathbb{C}$ with $TM(-D)_x$ mentioned earlier.

The element $\text{Res}(\theta, x)$ in (5.7) is called the *residue*, at x , of the logarithmic connection θ .

Let S be a complex affine algebraic group and

$$\rho : H \rightarrow S$$

a homomorphism. As before, E_G is a holomorphic principal G -bundle on M . Let

$$E_S := E_G \times^\rho S \rightarrow M$$

be the holomorphic principal S -bundle obtained by extending the structure group of E_G using ρ . So E_S is the quotient of $E_G \times S$ obtained by identifying (y, m) and $(yh^{-1}, \rho(h)m)$, where y, m and h run over E_G, S and G , respectively. Therefore, we have a morphism

$$\widehat{\rho} : E_G \rightarrow E_S, \quad y \mapsto \widetilde{(y, e_S)},$$

where $\widetilde{(y, e_S)}$ is the equivalence class of (y, e_S) with e_S being the identity element of S . The homomorphism of Lie algebras $d\rho : \mathfrak{g} \rightarrow \mathfrak{s} := \text{Lie}(S)$ associated with ρ produces a homomorphism of vector bundles

$$\alpha : \text{ad}(E_G) \rightarrow \text{ad}(E_S). \quad (5.8)$$

The maps $\widehat{\rho}$ and $d\rho$ together produce a homomorphism of vector bundles

$$\widetilde{A} : \text{At}(E_G) \rightarrow \text{At}(E_S).$$

This map $\widetilde{A}$ produces a homomorphism

$$A : \text{At}(E_G, D) \rightarrow \text{At}(E_S, D), \quad (5.9)$$

which fits in the following commutative diagram of homomorphisms

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{ad}(E_G) & \xrightarrow{i_0} & \text{At}(E_G, D) & \xrightarrow{\sigma} & TM(-D) \longrightarrow 0 \\ & & \downarrow \alpha & & \downarrow A & & \parallel \\ 0 & \longrightarrow & \text{ad}(E_S) & \longrightarrow & \text{At}(E_S, D) & \longrightarrow & TM(-D) \longrightarrow 0 \end{array}, \quad (5.10)$$

where the top exact sequence is the one in (5.1) and the bottom one is the corresponding sequence for E_S .

If $\theta : TM(-D) \rightarrow \text{At}(E_G, D)$ is a logarithmic connection on E_G as in (5.2), then

$$A \circ \theta : TM(-D) \rightarrow \text{At}(E_S, D) \quad (5.11)$$

is a logarithmic connection on E_S singular over D . From the definition of residue in (5.7) it follows immediately that

$$\alpha(\text{Res}(\theta, x)) = \text{Res}(A \circ \theta, x) \quad (5.12)$$

for all $x \in D$. This proves the following:

Lemma 5.1 *With the above notation, if E_G admits a logarithmic connection θ singular over D with residue $w_x \in \text{ad}(E_G)_x$ at each $x \in D$, then E_S admits a logarithmic connection $\theta' = A \circ \theta$ singular over D with residue $\alpha(w_x)$ at each $x \in D$.*

Now we set $G = \text{GL}(r, \mathbb{C})$.

Take a holomorphic vector bundle E on M . The group of all holomorphic automorphisms of E will be denoted by $\text{Aut}(E)$. Let

$$T \subset \text{End}(E_x)$$

be an endomorphism such that $\phi(x) \circ T = T \circ \phi(x)$ for all $\phi \in \text{Aut}(E)$.

Lemma 5.2 *Let $V \subset E$ be a holomorphic subbundle which is a holomorphic direct summand. Then $T(V_x) \subset V_x$.*

Proof Take a holomorphic subbundle $W \subset E$ such that $E = V \oplus W$. For nonzero complex numbers a, b , we have the automorphism $A(a, b) \in \text{Aut}(E)$ which acts on V (respectively, W) as multiplication by a (respectively, b). Since T commutes with $A(a, b)(x)$ for all $a, b \in \mathbb{C} \setminus \{0\}$, it follows that $T(V_x) \subset V_x$. $\square$

Take a holomorphic vector bundle E on M . Fix a finite subset

$$D := \{x_1, \dots, x_n\} \subset M.$$

For each point $x \in D$, fix

$$T_x \in \text{End}(E_x)$$

such that $\phi(x) \circ T_x = T_x \circ \phi(x)$ for all $\phi \in \text{Aut}(E)$.

The following theorem is proved in [4]:

Theorem 5.3 *There is a logarithmic connection θ on E singular over D such that*

$$\text{Res}(\theta, x) = T_x$$

if and only if

$$\text{degree}(V) + \sum_{x \in D} \text{trace}(\text{Res}(\theta, x)|_{V_x}) = 0.$$

Note that when $D = \emptyset$, then Theorem 5.3 reduces to the criterion of Atiyah and Weil mentioned in Sect. 3.

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