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Repdigits as products of consecutive Padovan or Perrin numbers

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Abstract A repdigit is a positive integer that has only one distinct digit in its decimal expansion, i.e., a number of the form $a(10^m - 1)/9$, for some $m \geq 1$ and $1 \leq a \leq 9$. Let $(P_n)_{n\geq 0}$ and $(E_n)_{n\geq 0}$ be the sequence of Padovan and Perrin numbers, respectively. This paper deals with repdigits that can be written as the products of consecutive Padovan or/and Perrin numbers.

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1 Introduction

A positive integer is called a *repdigit* if it has only one distinct digit in its decimal expansion. The sequence of numbers with repeated digits is included in Sloane's *On-Line Encyclopedia of Integer Sequences* (OEIS) [13] as the sequence A010785.

Let $(P_n)_{n\geq 0}$ be the Padovan sequence satisfying the recurrence relation $P_{n+3} = P_{n+1} + P_n$ with initial conditions $P_0 = 0$ and $P_1 = P_2 = 1$. Let $(E_n)_{n\geq 0}$ be the Perrin sequence following the same recursive pattern as the Padovan sequence, but with initial conditions $E_0 = 2$, $E_1 = 0$, and $E_2 = 1$. P_n and E_n are called nth Padovan number and nth Perrin number, respectively. The Padovan and Perrin sequences are included in the OEIS [13] as the sequences A000931 and A001608, respectively.

Finding some specific properties of sequences is of big interest since the famous result of Bugeaud, Mignotte, and Siksek [2]. One can also see [1–9,11,12]. Marques and the second author [9] studied repdigits as products of consecutive Fibonacci numbers. Irmak and the second author [5] studied repdigits as products of consecutive Lucas numbers. Rayaguru and Panda [11] studied repdigits as products of consecutive Balancing and Lucas-Balancing numbers. It is natural to ask what will happen if we consider Padovan and Perrin numbers. This is the aim of this paper.

Therefore, in this paper, we investigate repdigits which can be written as the product of consecutive Padovan or/and Perrin numbers. More precisely, we prove the following results.

Theorem 1.1 *The Diophantine equation*

$$P_n \cdots P_{n+(\ell-1)} = a\left(\frac{10^m - 1}{9}\right), \tag{1.1}$$

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has no solution in positive integers n, ℓ, m, a, *with* $1 \le a \le 9$ *and* $m \ge 2$.

Theorem 1.2 *The only solution of the Diophantine equation:*

$$E_n \cdots E_{n+(\ell-1)} = a\left(\frac{10^m - 1}{9}\right), \tag{1.2}$$

in positive integers n, ℓ, m, a, *with* $1 \le a \le 9$ *and* $m \ge 2$ *is* $(n, \ell, m, a) = (11, 1, 2, 2)$, *i.e.,* $E_{11} = 22$.

Theorem 1.3 *The Diophantine equation*

$$E_n \cdots E_{n+(k-1)} P_{n+k} \cdots P_{n+k+(\ell-1)} = a\left(\frac{10^m - 1}{9}\right) \tag{1.3}$$

has no solution in positive integers n, k, ℓ, m, a, *with* $1 \le a \le 9$ *and* $m \ge 2$.

Theorem 1.4 *The Diophantine equation*

$$P_n \cdots P_{n+(k-1)} E_{n+k} \cdots E_{n+k+(\ell-1)} = a\left(\frac{10^m - 1}{9}\right) \tag{1.4}$$

has no solution in positive integers n, k, ℓ, m, a, *with* $1 \le a \le 9$ *and* $m \ge 2$.

Here is the outline of this paper. In Sect. 2, we will recall the results that will be used to prove Theorems 1.1, 1.2, 1.3, and 1.4. In Sect. 3, first, we will use Baker's method and 2-adic valuation of Padovan numbers to obtain a bound for n that is too high to completely solve Eq. (1.1). We will then need to apply twice the reduction method of de Weger to find a very low bound for n, which enables to run a program to find the small solutions of Eq. (1.1). We will use the same method in the next sections to prove the remaining theorems. Computations are done with the help of a computer program in `Maple`.

2 The tools

We start by recalling some useful properties of Padovan and Perrin sequences. The characteristic equation of $\{P_n\}_{n\ge0}$ and $\{E_n\}_{n\ge0}$ is $z^3 - z - 1 = 0$ and has one real root α and two complex roots β and $\gamma = \overline{\beta}$. The Binet formulae for the Padovan and Perrin numbers are respectively:

$$P_s = c_\alpha \alpha^s + c_\beta \beta^s + c_\gamma \gamma^s, \quad \text{for all} \quad s \ge 0 \tag{2.1}$$

and

$$E_s = \alpha^s + \beta^s + \gamma^s, \quad \text{for all} \quad s \ge 0, \tag{2.2}$$

where

$$c_\alpha = \frac{1+\alpha}{-\alpha^2 + 3\alpha + 1}, \quad c_\beta = \frac{1+\beta}{-\beta^2 + 3\beta + 1}, \quad c_\gamma = \frac{1+\gamma}{-\gamma^2 + 3\gamma + 1} = \overline{c_\beta}.$$

It is easy to see that $\alpha \in (1.32, 1.33)$, $|\beta| = |\gamma| \in (0.86, 0.87)$, $c_\alpha \in (0.72, 0.73)$ and $|c_\beta| = |c_\gamma| \in (0.24, 0.25)$.

By the facts that $\beta = \alpha^{-1/2} e^{i\theta}$ and $\gamma = \alpha^{-1/2} e^{-i\theta}$, for some $\theta \in (0, 2\pi)$, we can show that:

$$P_s = c_\alpha \alpha^s + e_s, \qquad \text{with} \quad |e_s| < \frac{1}{\alpha^{s/2}}, \quad \text{for all} \ \ s \ge 1 \tag{2.3}$$

and

$$E_s = \alpha^s + e'_s, \qquad \text{with} \quad |e'_s| < \frac{2}{\alpha^{s/2}}, \quad \text{for all} \ \ s \ge 1. \tag{2.4}$$

Further, we have:

$$\alpha^{s-2} \le P_s \le \alpha^{s-1}, \quad \text{for all} \quad s \ge 4 \tag{2.5}$$

and

$$\alpha^{s-2} \le E_s \le \alpha^{s+1}, \quad \text{for all} \quad s \ge 2. \tag{2.6}$$

For a prime number p and a non-zero integer r, the p-adic order $\upsilon_p(r)$ is the exponent of the highest power of a prime p which divides r. The following two results, due to Irmak [4], characterize the 2-adic order of Padovan and Perrin numbers, respectively.

Lemma 2.1 *For $n \ge 1$, we have:*

$$\upsilon_2(P_n) = \begin{cases} 0 & \text{if } n \equiv 0, 1, 2, 5 \pmod 7, \\ \upsilon_2(n+4)+1 & \text{if } n \equiv 3 \pmod 7, \\ \upsilon_2((n+3)(n+17))+1 & \text{if } n \equiv 4 \pmod 7, \\ \upsilon_2((n+1)(n+8))+1 & \text{if } n \equiv 6 \pmod 7. \end{cases}$$

Lemma 2.2 *For $n \ge 1$, we have:*

$$\upsilon_2(E_n) = \begin{cases} 0 & \text{if } n \equiv 0, 3, 5, 6 \pmod 7, \\ 1 & \text{if } n \equiv 2 \pmod{14}, \\ 2 & \text{if } n \equiv 9 \pmod{14}, \\ \upsilon_2(n-1)+1 & \text{if } n \equiv 1 \pmod 7, \\ 1 & \text{if } n \equiv 4 \pmod 7. \end{cases}$$

The next tools are related to the transcendental approach to solve Diophantine equations. For any non-zero algebraic number γ of degree d over $\mathbb{Q}$, whose minimal polynomial over $\mathbb{Z}$ is $a \prod_{j=1}^{d} \left(X - \gamma^{(j)}\right)$, we denote by:

$$h(\gamma) = \frac{1}{d}\left(\log|a| + \sum_{j=1}^{d} \log\max\left(1, \left|\gamma^{(j)}\right|\right)\right)$$

the usual absolute logarithmic height of γ.

To prove our main results, we use lower bounds for linear forms in logarithms to bound the index n appearing in Eqs. (1.1), (1.2), (1.3), and (1.4). We need the following result of Bugeaud, Mignotte, and Siksek [2], which is a modified version of the result of Matveev [10].

Lemma 2.3 *Let $\gamma_1, \ldots, \gamma_s$ be real algebraic numbers and let $b_1, \ldots, b_s$ be non-zero rational integer numbers. Let D be the degree of the number field $\mathbb{Q}(\gamma_1, \ldots, \gamma_s)$ over $\mathbb{Q}$ and let A_j be a positive real number satisfying:*

$$A_j = \max\{Dh(\gamma), |\log\gamma|, 0.16\} \quad \text{for } j = 1, \ldots, s.$$

Assume that:

$$B \ge \max\{|b_1|, \ldots, |b_s|\}.$$

If $\gamma_1^{b_1} \cdots \gamma_s^{b_s} \ne 1$, then:

$$|\gamma_1^{b_1} \cdots \gamma_s^{b_s} - 1| \ge \exp(-C(s, D)(1 + \log B)A_1 \cdots A_s),$$

where $C(s, D) := 1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot D^2(1 + \log D)$.

After getting the upper bound of n, which is generally too large, the next step is to reduce it. For this reduction purpose, we present a variant of the reduction method of Baker and Davenport due to de Weger [14]).

Let $\vartheta_1, \vartheta_2, \beta \in \mathbb{R}$ be given, and let $x_1, x_2 \in \mathbb{Z}$ be unknowns. Let:

$$\Lambda = \beta + x_1\vartheta_1 + x_2\vartheta_2. \tag{2.7}$$

Let c, δ be positive constants. Set $X = \max\{|x_1|, |x_2|\}$. Let X_0, Y be positive numbers. Assume that:

$$|\Lambda| < c \cdot \exp(-\delta \cdot Y), \tag{2.8}$$
$$Y \le X \le X_0. \tag{2.9}$$

When $\beta = 0$ in (2.7), we get

$$\Lambda = x_1\vartheta_1 + x_2\vartheta_2.$$

Put $\vartheta = -\vartheta_1/\vartheta_2$. We assume that x_1 and x_2 are coprime. Let the continued fraction expansion of ϑ be given by

$$[a_0, a_1, a_2, \ldots],$$

and let the kth convergent of ϑ be p_k/q_k for $k = 0, 1, 2, \ldots$. We may assume without loss of generality that $|\vartheta_1| < |\vartheta_2|$ and that $x_1 > 0$. We have the following results.

Lemma 2.4 [14, Lemma 3.2] *Let:*

$$A = \max_{0 \le k \le Y_0} a_{k+1},$$

where

$$Y_0 = -1 + \frac{\log(\sqrt{5}X_0 + 1)}{\log\left(\frac{1+\sqrt{5}}{2}\right)}.$$

If (2.8) *and* (2.9) *hold for* x_1, x_2 *and* $\beta = 0$, *then:*

$$Y < \frac{1}{\delta}\log\left(\frac{c(A+2)X_0}{|\vartheta_2|}\right). \tag{2.10}$$

When $\beta \neq 0$ in (2.7), put $\vartheta = -\vartheta_1/\vartheta_2$ and $\psi = \beta/\vartheta_2$. Then, we have

$$\frac{\Lambda}{\vartheta_2} = \psi - x_1\vartheta + x_2.$$

Let p/q be a convergent of ϑ with $q > X_0$. For a real number x, we let $\|x\| = \min\{|x-n|, n \in \mathbb{Z}\}$ be the distance from x to the nearest integer. We have the following result.

Lemma 2.5 [14, Lemma 3.3] *Suppose that:*

$$\| q\psi \| > \frac{2X_0}{q}.$$

Then, the solutions of (2.8) *and* (2.9) *satisfy:*

$$Y < \frac{1}{\delta}\log\left(\frac{q^2 c}{|\vartheta_2| X_0}\right).$$

We conclude this section by recalling the following lemma that we need in the sequel:

Lemma 2.6 [14, Lemma 2.2, page 31] *Let* $a, x \in \mathbb{R}$ *and* $0 < a < 1$. *If* $|x| < a$, *then:*

$$|\log(1+x)| < \frac{-\log(1-a)}{a}|x|$$

and

$$|x| < \frac{a}{1-e^{-a}}\left|e^x - 1\right|.$$

3 Proof of Theorem 1.1

In this section, we will use Baker's method and the p-adic valuation to completely prove Theorem 1.1.

Table 1 2-adic order of product of consecutive Padovan numbers

x	0	1	2	3	4	5	6
ℓ	7	6	5	4	7	7	6
$\upsilon_2(P_n P_{n+1} \cdots P_{n+(\ell-1)}) \geq$	4	4	4	4	5	4	4

3.1 Absolute bounds on the variables

We start by giving the number of factors ℓ in the Diophantine equation (1.1).

Lemma 3.1 *If Diophantine equation (1.1) has solutions, then $\ell \leq 6$.*

Proof Note that, for all $1 \leq a \leq 9$, we have:

$$\upsilon_2\left(a\left(\frac{10^m - 1}{9}\right)\right) = \upsilon_2(a) \leq 3.$$

Therefore, if $\upsilon_2(P_n P_{n+1} \cdots P_{n+(\ell-1)}) \geq 4$, then Diophantine equation (1.1) has no solution.

Let $x \in \{0, 1, 2, \ldots, 6\}$, such that $n \equiv x \pmod 7$. Suppose that $x = 2$, and hence, $n + 1 \equiv 3 \pmod 7$, $n + 2 \equiv 4 \pmod 7$, and $n + 4 \equiv 6 \pmod 7$. Therefore, by Lemma 2.1, we get:

$$\upsilon_2(P_n P_{n+1} \cdots P_{n+4}) = 3\upsilon_2(n + 5) + \upsilon_2(n + 12) + \upsilon_2(n + 19) + 3.$$

Clearly, if n is odd, then $n+5$ and $n+19$ are even; otherwise, $n+12$ is odd. Thus, we get $\upsilon_2(P_n P_{n+1} \cdots P_{n+4}) \geq 4$. Therefore, Diophantine equation (1.1) has no solution if $\ell \geq 5$ in this case.

The other cases can be treated using a similar method. As a conclusion, we get Table 1. Thus, we deduce that $\ell \leq 6$.

□

Now, we give an upper bound for n and m.

Lemma 3.2 *If (n, ℓ, m, a) is a positive integer solution of (1.1) with $n \geq 15$, $m \geq 2$, $1 \leq a \leq 9$, and $1 \leq \ell \leq 6$, then:*

$$m \leq \ell n + \ell(\ell - 3)/2 \quad and \quad n < 1.8 \times 10^{16}.$$

Proof By (1.1) and (2.5), we have:

$$10^{m-1} < a\left(\frac{10^m - 1}{9}\right) = P_n P_{n+1} \cdots P_{n+(\ell-1)} < \alpha^{\ell n + \frac{\ell(\ell-3)}{2}}.$$

Thus, we get:

$$m \leq \ell n + \ell(\ell - 3)/2. \tag{3.1}$$

Now, by (2.3), we obtain:

$$\begin{aligned} P_n \cdots P_{n+(\ell-1)} &= (c_\alpha \alpha^n + e_n) \cdots (c_\alpha \alpha^{n+(\ell-1)} + e_{n+\ell-1}) \\ &= c_\alpha^\ell \alpha^{\ell n + \ell(\ell-1)/2} + r_1(c_\alpha, \alpha, n, \ell), \end{aligned} \tag{3.2}$$

where $r_1(c_\alpha, \alpha, n, \ell)$ involves the part of the expansion of the previous line that contains the product of powers of c_α, α and the errors e_i, for $i = n, \ldots n + (\ell - 1)$. Moreover, $r_1(c_\alpha, \alpha, n, \ell)$ is the sum of 63 terms with maximum absolute value $c_\alpha^{\ell-1} \alpha^{(\ell-1)n + \ell(\ell-1)/2} \alpha^{-n/2}$.

The equality (3.2) enables us to express (1.1) into the form:

$$\frac{a}{9} 10^m - c_\alpha^\ell \alpha^{\ell n + \ell(\ell-1)/2} = \frac{a}{9} + r_1(c_\alpha, \alpha, n, \ell).$$

Dividing through by $c_\alpha^\ell \alpha^{\ell n+\ell(\ell-1)/2}$ and taking the absolute value, we deduce that:

$$|\Gamma_1| \le \left(\frac{a}{9} + |r_1(c_\alpha, \alpha, n, \ell)|\right) \cdot c_\alpha^{-\ell} \alpha^{-(\ell n+\ell(\ell-1)/2)} \quad (3.3)$$
$$< (1 + 63 c_\alpha^{\ell-1} \alpha^{(\ell-1)n+\ell(\ell-1)/2} \alpha^{-n/2}) \cdot c_\alpha^{-\ell} \alpha^{-(\ell n+\ell(\ell-1)/2)}$$
$$\le 64 c_\alpha^{-1} \alpha^{-3n/2} < 89 \alpha^{-3n/2},$$

where::

$$\Gamma_1 = \frac{a}{9 c_\alpha^\ell} \alpha^{-(\ell n+\ell(\ell-1)/2)} 10^m - 1. \quad (3.4)$$

To find a lower bound for Γ_1, we take the parameters $s := 3$:

$$(\gamma_1, b_1) := ((a/9)c_\alpha^{-\ell}, 1), \ (\gamma_2, b_2) := (\alpha, -(\ell n + \ell(\ell-1)/2)) \text{ and } (\gamma_3, b_3) := (10, m),$$

in Lemma 2.3. For our choices, we have $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{Q}(\alpha)$, with degree $D := 3$. To apply Lemma 2.3 it is necessary to show that $\Gamma_1 \neq 0$. If we assume the contrary, thus we get:

$$a \cdot 10^m/9 = c_\alpha^\ell \alpha^{\ell n+\ell(\ell-1)/2}.$$

Conjugating the above relation by the Galois automorphism $\sigma := (\alpha\beta)$, and then taking absolute values on both sides of the resulting equality, we obtain:

$$1 < a \cdot 10^m/9 = |c_\beta|^\ell |\beta|^{\ell n+\ell(\ell-1)/2} < 1.$$

This is a contradiction. Thus, $\Gamma_1 \neq 0$. Next, we give estimates to A_i for $i = 1, 2, 3$. By the properties of the absolute logarithmic height, we have:

$$h(\gamma_1) \le h(d) + h(9) + \ell h(c_\alpha) \le 2\log 9 + \ell h(c_\alpha).$$

Now, we need to estimate $h(c_\alpha)$. For that, the minimal polynomial of c_α is $23X^3 - 23x^2 + 6X - 1$. Therefore, $h(c_\alpha) = \frac{1}{3}\log 23$, and thus:

$$h(\gamma_1) \le 2\log 9 + 2\log 23.$$

On the other hand, $h(\gamma_2) = \frac{1}{3}\log\alpha$ and $h(\gamma_3) = \log 10$. Therefore, we take $A_1 := 32$, $A_2 := 0.3$ and $A_3 := 7$. Finally, by (3.1) and the fact that $\ell \le 6$, we take $B := 6n + 15$. Applying Lemma 2.3, we get a lower bound for $|\Gamma_1|$, which by comparing it to (3.3) leads to:

$$\frac{3n}{2}\log\alpha - \log 89 < 1.82 \times 10^{14}(1 + \log(6n + 15)).$$

Hence, we get:

$$n < 4.4 \times 10^{14}(1 + \log(6n + 15)).$$

Therefore, we obtain $n < 1.8 \times 10^{16}$. □

3.2 Reducing n

To lower the bound of n, we will use Lemma 2.5.
Let:

$$\Lambda_1 := m \log 10 - (\ell n + \ell(\ell-1)/2) \log \alpha + \log(a/9c_\alpha^\ell).$$

Therefore, (3.3) can be rewritten as $|e^{\Lambda_1} - 1| < 89/\alpha^{3n/2}$. Furthermore, if $n \geq 15$ $|\Gamma_1| < 0.16$. Therefore, by applying Lemma 2.6, we deduce that:

$$|\Lambda_1| < -\frac{\log(1-0.16)}{0.16}|\Gamma_1| < 97 \exp(-0.42n). \tag{3.5}$$

Put:

$$\vartheta_1 := -\log \alpha, \quad \vartheta_2 := \log 10, \quad \psi := \log\left(\frac{a}{9c_\alpha}\right), \quad c := 97, \quad \delta := 0.42.$$

Furthermore, as $\max\{m, \ell n + \ell(\ell-1)/2\} < 1.1 \times 10^{17}$, then we take $X_0 = 1.1 \times 10^{17}$. Using Maple, one can see that:

$$q_{43} = 762002911251177096225$$

satisfies the conditions of Lemma 2.5 for all $1 \leq a \leq 9$ and $1 \leq \ell \leq 6$. Therefore, Lemma 2.5 implies that if the Diophantine equation (1.1) has solutions, then:

$$n \leq \frac{1}{0.42} \times \log\left(\frac{762002911251177096225^2 \times 97}{\log 10 \times 1.1 \times 10^{17}}\right) < 134.$$

Now, we reduce again this new bound of n. In this application of Lemma 2.5, we take $X_0 = 813$ and see that $q_{10} = 869219$ satisfies the conditions of Lemma 2.5. Thus, we obtain:

$$n \leq \frac{1}{0.42} \times \log\left(\frac{869219^2 \times 97}{\log 10 \times 813}\right) < 59.$$

Hence, it remains to check Eq. (1.1) for $1 \leq n \leq 58$, $1 \leq \ell \leq 6$, $2 \leq m \leq 363$ and $1 \leq a \leq 9$. A quick inspection using Maple reveals that Diophantine equation (1.1) has no solutions. This completes the proof of Theorem 1.1.

4 Proof of Theorem 1.2

In this section, we will use the same method for the proof of Theorem 1.1 to completely prove Theorem 1.2. However, for the sake of completeness, we will give some details.

4.1 Absolute bounds on the variables

First of all, we give the number of factors in the Diophantine equation (1.2).

Lemma 4.1 *The Diophantine equation* (1.2) *has a solution if* $\ell \leq 7$*:*

Proof Let $x \in \{0, 1, 2, \ldots, 13\}$, such that $n \equiv x \pmod{14}$. Assume that $x = 8$, and hence, $n \equiv 1 \pmod 7$, $n+1 \equiv 9 \pmod{14}$, and $n+3 \equiv 4 \pmod 7$. Thus, Lemma 2.2 gives:

$$\upsilon_2(E_n E_{n+1} \cdots E_{n+3}) = \upsilon_2(n-1) + 4 = 4,$$

because $n \equiv 8 \pmod{14}$ leads to $\upsilon_2(n-1) = 0$. We give the other results in Table 2.
As $\upsilon_2\left(a\left(\frac{10^m-1}{9}\right)\right) = \upsilon_2(a) \leq 3$, for all $1 \leq a \leq 9$, then it follows from Table 2 that $\ell \leq 7$. □

Now, we will show the following lemma.

Table 2 2-adic order of product of consecutive Perrin numbers

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13
ℓ	5	4	8	7	6	7	6	5	4	7	7	6	7	6
$\upsilon_2(E_nE_{n+1}\cdots E_{n+(\ell-1)})\geq$	4	4	5	4	4	4	4	4	4	5	4	4	4	4

Lemma 4.2 *If* (n, ℓ, m, a) *is a positive integer solution of* (1.2) *with* $n \geq 20$, $m \geq 2$, $1 \leq a \leq 9$, *and* $1 \leq \ell \leq 10$, *then:*

$$m \leq \ell n + \ell(\ell+1)/2 \quad and \quad n < 7.1 \times 10^{15}.$$

Proof First, we assume that $n \geq 20$. Combining (1.2) and (2.6), we obtain:

$$10^{m-1} < \frac{a(10^m-1)}{9} = E_nE_{n+1}\cdots E_{n+(\ell-1)} < \alpha^{\ell n+\frac{\ell(\ell+1)}{2}}.$$

Thus, we have:

$$m \leq \ell n + \ell(\ell+1)/2. \tag{4.1}$$

Now, by (2.4), we get:

$$\begin{aligned} E_n\cdots E_{n+(\ell-1)} &= (\alpha^n+e_n)\cdots(\alpha^{n+(\ell-1)}+e_{n+\ell-1}) \\ &= \alpha^{\ell n+\ell(\ell-1)/2} + r_2(\alpha,n,\ell), \end{aligned} \tag{4.2}$$

where $r_2(\alpha, n, \ell)$ involves the part of the expansion of the previous line that contains the product of powers of α and the errors e_i', for $i = n, \ldots n+(\ell-1)$. Moreover, $r_2(\alpha, n, \ell)$ is the sum of 127 terms with maximum absolute value $2^\ell\alpha^{(\ell-1)n+\ell(\ell-1)/2}\alpha^{-n/2}$.

The equality (4.2) enables us to express (1.2) as:

$$\frac{a}{9}10^m - \alpha^{\ell n+\ell(\ell-1)/2} = \frac{a}{9} + r_2(\alpha,n,\ell).$$

Multiplying both sides by $\alpha^{-(\ell n+\ell(\ell-1)/2)}$ and taking the absolute value, we conclude that:

$$\begin{aligned} |\Gamma_2| &\leq \left(\frac{a}{9} + |r_2(\alpha,n,\ell)|\right)\cdot\alpha^{-(\ell n+\ell(\ell-1)/2)} \\ &< (1+127\cdot 2^\ell\alpha^{(\ell-1)n+\ell(\ell-1)/2}\alpha^{-n/2})\cdot\alpha^{-(\ell n+\ell(\ell-1)/2)} \\ &\leq 128\cdot 2^\ell\alpha^{-3n/2} < 16384\alpha^{-3n/2}, \end{aligned} \tag{4.3}$$

where:

$$\Gamma_2 = \frac{a}{9}\alpha^{-(\ell n+\ell(\ell-1)/2)}10^m - 1. \tag{4.4}$$

Now, we use Lemma 2.3 to find a lower bound for Γ_2, with the parameters $s := 3$:

$$(\gamma_1, b_1) := ((a/9), 1), \quad (\gamma_2, b_2) := (\alpha, -(\ell n+\ell(\ell-1)/2)), \quad \text{and} \quad (\gamma_3, b_3) := (10, m).$$

The number field containing $\gamma_1, \gamma_2, \gamma_3$ is $\mathbb{Q}(\alpha)$, which degree is $D := 3$.

We next justify that $\Gamma_2 \neq 0$. Indeed, if this were zero, we would then get:

$$\alpha^{\ell n+\ell(\ell-1)/2} = a\cdot 10^m/9.$$

Conjugating the above relation by the Galois automorphism $\sigma := (\alpha\beta)$, and then taking absolute values on both sides of the resulting equality, we obtain:

$$|\beta|^{\ell n+\ell(\ell-1)/2} = a\cdot 10^m/9,$$

which is not possible, because $|\beta|^{\ell n+\ell(\ell-1)/2} < 1$ and $a \cdot 10^m/9 > 10$. Thus, $\Gamma_2 \neq 0$. Next, $h(\gamma_1) \leq h(d) + h(9) \leq 2\log 9$, $h(\gamma_2) = \frac{1}{3}\log\alpha$ and $h(\gamma_3) = \log 10$. Thus, we take $A_1 := 13.2$, $A_2 := 0.3$ and $A_3 := 7$. According to (4.1) and the fact that $\ell \leq 10$, we take $B := 7n + 28$. Applying Lemma 2.3, we get a lower bound for $|\Gamma_2|$, and taking into account inequality (4.3), we obtain:

$$\exp\left(-7.5 \times 10^{13}(1 + \log(7n + 28))\right) < \frac{16384}{\alpha^{3n/2}}.$$

Taking the logarithm of both sides of the above inequality, we obtain:

$$n < 1.78 \times 10^{14}(1 + \log(7n + 28)).$$

With the help of `Maple`, we get $n < 7.1 \times 10^{15}$. □

4.2 Reducing n

To lower the bound of n, we will use Lemmas 2.4 and 2.5. We will proceed with three successive reductions.

Let:

$$\Lambda_2 := m\log 10 - (\ell n + \ell(\ell-1)/2)\log\alpha + \log(a/9).$$

Therefore, (4.3) can be rewritten as $|e^{\Lambda_2} - 1| < 16384\alpha^{-3n/2}$. Furthermore, if $n \geq 30$, then $|\Gamma_1| < 0.06$. Therefore, Lemma 2.6 leads to:

$$|\Lambda_2| < -\frac{\log(1-0.06)}{0.06}|\Gamma_2| < 16897\exp\left(-0.42n\right).$$

If $a \neq 9$, then we take:

$$\vartheta_1 := -\log\alpha, \quad \vartheta_2 := \log 10, \quad \psi := \log\left(\frac{a}{9}\right), \quad c := 16897, \quad \delta := 0.42$$

in Lemma 2.5. Furthermore, as $\max\{m, \ell n + \ell(\ell-1)/2\} < 5 \times 10^{16}$, we can take $X_0 = 5 \times 10^{16}$. One can use `Maple` to see that:

$$q_{43} = 762002911251770096225$$

satisfies the conditions of Lemma 2.5, for all $1 \leq a \leq 8$ and $1 \leq \ell \leq 10$. Therefore, from Lemma 2.5, we deduce that if Diophantine equation (1.2) has solutions, then:

$$n \leq \frac{1}{0.42} \times \log\left(\frac{762002911251770096225^2 \times 16897}{\log 10 \times 5 \times 10^{16}}\right) < 148.$$

Now, we again reduce this new bound of n. In this application of Lemma 2.5, we take $X_0 = 1057$ and see that $q_9 = 139228$ satisfies the conditions of Lemma 2.5. Thus, we obtain:

$$n \leq \frac{1}{0.42} \times \log\left(\frac{139228^2 \times 16897}{\log 10 \times 1057}\right) < 62.$$

If $a = 9$, then Λ_2 becomes:

$$\Lambda_2 = m\log 10 - (\ell n + \ell(\ell-1)/2)\log\alpha,$$

so we apply Lemma 2.4 in this case by choosing

$$c := 16897, \quad \delta := 0.42, \quad X_0 = 5 \times 10^{16}, \quad Y_0 = 80.5763\ldots.$$

`Maple` gives us $\max_{0\leq k\leq 81} a_{k+1} = 49$, and thus, by Lemma 2.4, we have:

$$n \leq \frac{1}{0.42} \times \log\left(\frac{16897 \times 51 \times 5 \times 10^{16}}{\log 10}\right) < 123.$$

Now, we proceed to the third reduction of the bound of n. To apply Lemma 2.4, we take $X_0 = 882$ and find that $\max_{0 \le k \le 15} a_{k+1} = 8$. Therefore, we get:

$$n \le \frac{1}{0.42} \times \log\left(\frac{16897 \times 10 \times 882}{\log 10}\right) < 43.$$

Therefore, $n \le 62$ holds in all cases. Hence, it remains to check Eq. (1.2) for $1 \le n \le 62$, $1 \le \ell \le 10$, $2 \le m \le 462$, and $1 \le a \le 9$. By a fast computation with `Maple` in these ranges, we conclude that the quadruple $(n, \ell, m, d) = (11, 1, 2, 2)$ is the only solution of Diophantine equation (1.2). This completes the proof of Theorem 1.2.

5 Proofs of Theorems 1.3 and 1.4

We will use the same method as above to only show Theorem 1.3 as the proof of Theorem 1.4 is similar.

5.1 Absolute bounds on the variables

By Lemmas 3.1 and 4.1, we claim that $k \le 7$ and $\ell \le 6$. Indeed, one has:

$$\upsilon_2(E_n \cdots E_{n+(k-1)} P_{n+k} \cdots P_{n+k+(\ell-1)}) > 3,$$

and then, Diophantine equation (1.3) has no solution.

Now, we will prove the following lemma.

Lemma 5.1 *If (n, k, ℓ, m, a) is a positive integer solution of (1.3) with $n \ge 30$, $m \ge 2$, $1 \le a \le 9$, $1 \le k \le 7$, and $1 \le \ell \le 6$, then:*

$$m \le (k+\ell)n + \frac{k(k+1)}{2} + \frac{\ell(2k+\ell-3)}{2} \quad \text{and} \quad n < 1.88 \times 10^{17}.$$

Proof First, assume that $n \ge 30$. Combining (1.3), (2.5), and (2.6), we get:

$$\begin{aligned} 10^{m-1} < \frac{a(10^m - 1)}{9} &= E_n E_{n+1} \cdots E_{n+(k-1)} P_{n+k} \cdots P_{n+k+(\ell-1)} \\ &< \alpha^{(k+\ell)n + \frac{k(k+1)}{2} + \frac{\ell(2k+\ell-3)}{2}}. \end{aligned}$$

Taking the logarithm of both sides of the above inequality leads to:

$$m \le (k+\ell)n + \frac{k(k+1)}{2} + \frac{\ell(2k+\ell-3)}{2}. \tag{5.1}$$

Now, by (2.3) and (2.4), we get:

$$\begin{aligned} &E_n E_{n+1} \cdots E_{n+(k-1)} P_{n+k} \cdots P_{n+k+(\ell-1)} \\ &\quad = \alpha^{(k+\ell)n + k(k-1)/2 + \ell(2k+\ell-1)/2} + r_3(c_\alpha, \alpha, n, k, \ell), \end{aligned} \tag{5.2}$$

where $r_3(c_\alpha, \alpha, n, k, \ell)$ involves the part of the expansion of the previous line that contains the product of powers of c_α α and the errors e_{k+j} and e'_i, for $i = n, \ldots, n+(k-1)$ and $j = n, \ldots, n+(\ell-1)$. Moreover, $r_3(c_\alpha, \alpha, n, k, \ell)$ is the sum of 8191 terms with maximum absolute value $2^k c_\alpha^\ell \alpha^{(k+\ell-1)n+k(k-1)/2+\ell(2k+\ell-1)/2}$ $\alpha^{-n/2}$.

The equality (5.2) enables us to express (1.3) as:

$$\frac{a}{9} 10^m - c_\alpha^\ell \alpha^{(k+\ell)n + k(k-1)/2 + \ell(2k+\ell-1)/2} = \frac{a}{9} + r_3(c_\alpha, \alpha, n, k, \ell).$$

Multiplying through by $c_\alpha^{-\ell}\alpha^{-((k+\ell)n+k(k-1)/2+\ell(2k+\ell-1)/2)}$ and taking the absolute value, we deduce that:

$$\begin{aligned}|\Gamma_3| &\le \left(\frac{a}{9}+|r_3(c_\alpha,\alpha,n,k,\ell)|\right)\cdot\alpha^{-((k+\ell)n+k(k-1)/2+\ell(2k+\ell-1)/2)} \\ &< (1+8191\cdot 2^k c_\alpha^\ell \alpha^{(k+\ell-1)n+k(k-1)/2+\ell(2k+\ell-1)/2}\alpha^{-n/2})\cdot\alpha^{-((k+\ell)n+k(k-1)/2+\ell(2k+\ell-1)/2)} \\ &\le 8192\cdot 2^k c_\alpha^{-1}\alpha^{-3n/2} < 1452072\alpha^{-3n/2},\end{aligned} \tag{5.3}$$

where:

$$\Gamma_3 = \frac{a}{9c_\alpha^\ell}\alpha^{(k+\ell)n+k(k-1)/2+\ell(2k+\ell-1)/2}10^m - 1. \tag{5.4}$$

Now, we use Lemma 2.3 to find a lower bound for Γ_3, with the parameters $s := 3$ and:

$$(\gamma_1,b_1) := ((d/9c_\alpha^\ell),1),\quad (\gamma_2,b_2) := (\alpha,-((k+\ell)n+k(k-1)/2+\ell(2k+\ell-1)/2)),\quad (\gamma_3,b_3) := (10,m).$$

The algebraic numbers $\gamma_1,\gamma_2,\gamma_3$ belong to the number field $\mathbb{Q}(\alpha)$, which degree is $D := 3$. We claim that $\Gamma_3 \neq 0$. Otherwise, we get:

$$c_\alpha^\ell \alpha^{(k+\ell)n+k(k-1)/2+\ell(2k+\ell-1)/2} = a\cdot 10^m/9.$$

Conjugating the above relation by the automorphism $\sigma := (\alpha\beta)$, and then taking absolute values on both sides of the resulting equality, we obtain:

$$|c_\beta^\ell||\beta|^{(k+\ell)n+k(k-1)/2+\ell(2k+\ell-1)/2} = a\cdot 10^m/9,$$

which is not possible, because $|c_\beta^\ell||\beta|^{(k+\ell)n+k(k-1)/2+\ell(2k+\ell-1)/2} < 1$ and $a\cdot 10^m/9 > 10$. Thus, $\Gamma_3 \neq 0$. Next, $h(\gamma_1) \le h(d)+h(9) \le 2\log 9$, $h(\gamma_2) = \frac{1}{3}\log\alpha$ and $h(\gamma_3) = \log 10$. Thus, we can take $A_1 := 13.2$, $A_2 := 0.3$ and $A_3 := 7$. According to (5.1) and the facts $k \le 7$ and $\ell \le 6$, we take $B := 13n+85$. Applying Lemma 2.3 we get a lower bound for $|\Gamma_3|$, which, by comparing it to (5.3), leads to:

$$\exp\left(-1.82\times 10^{14}(1+\log(13n+85)\right) < \frac{1452072}{\alpha^{3n/2}}.$$

Hence, we get:

$$n < 4.32\times 10^{14}(1+\log(13n+85)).$$

Therefore, we obtain $n < 1.88\times 10^{17}$. □

5.2 Reducing n

To lower the bound of n, we will use Lemma 2.5.

Let

$$\Lambda_3 := m\log 10 - ((k+\ell)n+k(k-1)/2+\ell(2k+\ell-1)/2)\log\alpha + \log(a/9c_\alpha^\ell).$$

Therefore, (5.3) can be rewritten as $|e^{\Lambda_3}-1| < 1452072\alpha^{-3n/2}$. Furthermore, if $n \ge 40$ then $|\Gamma_3| < 0.07$. Therefore, by applying Lemma 2.6, we deduce that:

$$|\Lambda_3| < -\frac{\log(1-0.07)}{0.07}|\Gamma_3| < 1505399\exp(-0.42n).$$

Put:

$$\vartheta_1 := -\log\alpha,\quad \vartheta_2 := \log 10,\quad \psi := \log\left(\frac{a}{9c_\alpha}\right),\quad c := 1505399,\quad \delta := 0.42.$$

Furthermore, as $\max\{m, (k+\ell)n+k(k-1)/2+\ell(2k+\ell-1)/2\} < 2.45\times 10^{18}$, then we take $X_0 = 2.45\times 10^{18}$. We use Maple to find that:

$$q_{45} = 2067412494302352454860 5$$

satisfies the conditions of Lemma 2.5 for all $1 \le a \le 9$, $1 \le k \le 10$ and $1 \le \ell \le 6$. Therefore, by Lemma 2.5, if Diophantine equation (1.3) has solutions, then:

$$n \le \frac{1}{0.42} \times \log\left(\frac{20674124943023524548605^2 \times 1505399}{\log 10 \times 2.45 \times 10^{18}}\right) < 176.$$

Now, we reduce one more time this new bound of n. In this application of Lemma 2.5, we take $X_0 = 2360$ and observe that $q_{10} = 869219$ is a good candidate to verify the conditions of Lemma 2.5. Thus, we obtain:

$$n \le \frac{1}{0.42} \times \log\left(\frac{869219^2 \times 1505399}{\log 10 \times 2360}\right) < 79.$$

Hence, it remains to check Eq. (1.1) for $1 \le n \le 78$, $1 \le k \le 7$, $1 \le \ell \le 6$, $2 \le m \le 1216$, and $1 \le a \le 9$. For this, we use a simple routine written in Maple which (in a few minutes) does not return any solution of Diophantine equation (1.3) in these ranges. This completes the proof of Theorem 1.3.

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