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Quasi-dual Baer modules

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Abstract Let R be a ring and let M be an R -module with $S = \text{End}_R(M)$. The module M is called quasi-dual Baer if for every fully invariant submodule N of M , $\{\phi \in S \mid \text{Im} \phi \subseteq N\} = eS$ for some idempotent e in S . We show that M is quasi-dual Baer if and only if $\sum_{\phi \in \mathfrak{J}} \phi(M)$ is a direct summand of M for every left ideal $\mathfrak{J}$ of S . The R -module R_R is quasi-dual Baer if and only if R is a finite product of simple rings. Other characterizations of quasi-dual Baer modules are obtained. Examples which delineate the concepts and results are provided.

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1 Introduction

Throughout this paper, R is an associative ring with identity, and all the modules are unital right R -modules. Let M be an R -module. By $\text{End}_R(M)$ and $E(M)$, we denote the endomorphism ring of M and the injective hull of M , respectively. The notation $N \subseteq M$ means that N is a subset of M and $N \leq M$ means that N is a submodule of M . By $\mathbb{Q}$ and $\mathbb{Z}$, we denote the ring of rational numbers and integer numbers, respectively. For any prime number p , $\mathbb{Z}(p^\infty)$ denotes the Prüfer p -group.

A submodule K of M is called fully invariant if $f(K) \subseteq K$ for all $f \in \text{End}_R(M)$. In 1967, Clark introduced the concept of quasi-Baer rings. A ring R is called right quasi-Baer if the right annihilator of any right ideal of R is generated as a right ideal by an idempotent. Rizvi and Roman generalized the notion of right quasi-Baer rings to a module theoretic version. A module M is called quasi-Baer if the right annihilator in M of any two-sided ideal of $\text{End}_R(M)$ is a direct summand of M . This notion was dualized by Amouzegar–Talebi in 2013 [1]. A module M is said to be quasi-dual Baer if for every fully invariant submodule N of M , there exists an idempotent e in S such that $\{\phi \in S \mid \text{Im} \phi \subseteq N\} = eS$. In Sect. 2, we continue the study of quasi-dual Baer modules. We show that an R -module M is quasi-dual Baer if and only if $\sum_{\phi \in \mathfrak{J}} \phi(M)$ is a direct summand of M for every left ideal $\mathfrak{J}$ of $\text{End}_R(M)$. We provide examples to separate the properties of quasi-dual Baer modules and dual Baer modules. We prove that for a ring R , the R -module R_R is quasi-dual Baer if and only if R is

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a finite product of simple rings. We obtain conditions under which an indecomposable module is quasi-dual Baer. It is also shown that if M is a quasi-dual Baer module over a commutative ring R , then $M\mathfrak{a}$ is a direct summand of M for any ideal $\mathfrak{a}$ of R . We give a characterization of when a direct sum of a module M_1 with $\text{Rad}(M_1) = M_1$ and a semisimple module M_2 is quasi-dual Baer. It is also shown that over a commutative perfect ring, every quasi-dual Baer module is semisimple.

2 Some results on quasi-dual Baer modules

Let M be an R -module with $S = \text{End}_R(M)$. For a submodule N of M , the right ideal $\{\phi \in S \mid \text{Im}\phi \subseteq N\}$ of S will be denoted by $\mathfrak{D}(N)$. For a subset $\mathfrak{X}$ of S and a submodule N of M , let $\mathfrak{X}(N)$ denote the submodule $\sum_{\phi \in \mathfrak{X}} \phi(N)$ of M . It is easily seen that if N is a fully invariant submodule of M , then $\mathfrak{D}(N)$ is a two-sided ideal of S and if $\mathfrak{J}$ is a two-sided ideal of S , then $\mathfrak{J}(M)$ is a fully invariant submodule of M . Therefore, we have the following two maps:

$$\begin{aligned} \mathfrak{D} : \{\text{fully invariant submodules of } M\} &\longrightarrow \{\text{two-sided ideals of } S\} \text{ and} \\ \mathfrak{T} : \{\text{two-sided ideals of } S\} &\longrightarrow \{\text{fully invariant submodules of } M\} \end{aligned}$$

with $\mathfrak{T}(\mathfrak{J}) = \mathfrak{J}(M)$ for any two-sided ideal $\mathfrak{J}$ of S .

Following [1], [4], the module M is called dual Baer (quasi-dual Baer) if for every submodule (fully invariant submodule) N of M , there exists an idempotent e in S such that $\mathfrak{D}(N) = eS$. In this section, we continue the study of quasi-dual Baer modules. We begin with the following characterization of this class of modules.

Proposition 2.1 *Let M be an R -module with $S = \text{End}_R(M)$. Then, the following statements are equivalent:*

- (i) M is a quasi-dual Baer module;
- (ii) For any fully invariant submodule N of M , there exists a decomposition $M = D_1 \oplus D_2$ with $D_1 \subseteq N$ and $\text{Hom}_R(M, N \cap D_2) = 0$.

Proof (i) $\Rightarrow$ (ii) Let N be a fully invariant submodule of M . Then $\mathfrak{D}(N) = eS$ for some idempotent e of S . Set $D_1 = e(M)$ and $D_2 = (1 - e)(M)$. Therefore, $\mathfrak{T}(\mathfrak{D}(N)) = D_1 \subseteq N$. Since $M = D_1 \oplus D_2$, we have $N = D_1 \oplus (N \cap D_2)$. Now, let h be an endomorphism of M such that $h(M) \subseteq N \cap D_2$. Hence, $h \in \mathfrak{D}(N)$ and so $h = eg$ for some $g \in S$. Thus, $h(M) \subseteq e(M)$. But $h(M) \subseteq (1 - e)(M)$. Then, $h = 0$. It follows that $\text{Hom}_R(M, N \cap D_2) = 0$.

(ii) $\Rightarrow$ (i) Let N be a fully invariant submodule of M . So there exists a decomposition $M = D_1 \oplus D_2$ with $D_1 \subseteq N$ and $\text{Hom}_R(M, N \cap D_2) = 0$. Let e be an idempotent of S with $D_1 = e(M)$. Let us prove that $\mathfrak{D}(N) = eS$. It is clear that $eS \subseteq \mathfrak{D}(N)$. Now, let $f \in \mathfrak{D}(N)$. Let $\pi : N = D_1 \oplus (N \cap D_2) \rightarrow N \cap D_2$ be the projection map. Then, $\pi f \in \text{Hom}_R(M, N \cap D_2) = 0$. This implies that $f(M) \subseteq e(M)$. Thus, $(1 - e)f = 0$. Since $f = ef + (1 - e)f$, we obtain $f = ef \in eS$. Therefore, $\mathfrak{D}(N) = eS$ and so M is quasi-dual Baer. $\square$

The following corollary follows directly from Proposition 2.1.

Corollary 2.2 *Let M be an R -module. Assume that every fully invariant submodule of M is a direct summand of M . Then, M is quasi-dual Baer.*

- Example 2.3* (i) Clearly, every dual Baer module is quasi-dual Baer. From [9, Proposition 4.16], it follows that every torsion-free injective module over a commutative domain is quasi-dual Baer.
- (ii) Let M be an R -module such that $\text{Hom}_R(M, N) = 0$ for every fully invariant proper submodule N of M . Then, clearly M is quasi-dual Baer (see Proposition 2.1). For example, the $\mathbb{Z}$ -modules $\mathbb{Q}$ and $\mathbb{Z}(p^\infty)$ (where p is a prime number) are quasi-dual Baer modules.
- (iii) Recall that a nonzero ring R is called a simple ring if (0) and R are the only two-sided ideals in R . Let M be a module and assume that M has no nonzero proper fully invariant submodules. Then, M is quasi-dual Baer. For example, for any simple ring S , the right S -module S_S is quasi-dual Baer.

Next, we present another characterization of quasi-dual Baer modules.

Proposition 2.4 *Let M be an R -module with $S = \text{End}_R(M)$. Then, the following are equivalent:*

- (i) M is a quasi-dual Baer R -module;
- (ii) For every left ideal $\mathfrak{J}$ of S , $\mathfrak{J}(M)$ is a direct summand of M ;
- (iii) For every two-sided ideal $\mathfrak{J}$ of S , $\mathfrak{J}(M)$ is a direct summand of M .

Proof (i) $\Rightarrow$ (ii) Let $\mathfrak{J}$ be a left ideal of S . Put $N = \mathfrak{J}(M)$. Clearly, N is a fully invariant submodule of M . By hypothesis, $\mathfrak{D}(N) = eS$ for some idempotent $e \in S$. Therefore, $e(M) \subseteq N$. Now, let $\varphi \in \mathfrak{J}$. Then $Im\varphi \subseteq \mathfrak{J}(M)$. Hence, $\varphi \in \mathfrak{D}(N)$. This implies that $\varphi = e\psi$ for some $\psi \in S$. Thus $\varphi(M) \subseteq e(M)$. It follows that $N = \mathfrak{J}(M) \subseteq e(M)$. Consequently, $N = e(M)$ is a direct summand of M .

(ii) $\Rightarrow$ (iii) This is obvious.

(iii) $\Rightarrow$ (i) Let L be a fully invariant submodule of M . Note that $\mathfrak{J} = \mathfrak{D}(L)$ is a two-sided ideal of S . By (iii), $\mathfrak{J}(M) = e(M)$ for some $e^2 = e \in S$. Let $\phi \in \mathfrak{J}$. Then, $Im\phi \subseteq e(M)$. Hence, $(1_S - e)\phi = 0$. Since $1_S = e + (1_S - e)$, we have $\phi = e\phi \in eS$. Therefore, $\mathfrak{J} \subseteq eS$. Now let $\psi \in S$. Then, $e\psi(M) \subseteq e(M) \subseteq \mathfrak{J}(M) \subseteq L$. So $e\psi \in \mathfrak{J}$. This yields $eS \subseteq \mathfrak{J}$. It follows that $\mathfrak{D}(L) = eS$. This completes the proof. $\square$

As an application of Proposition 2.4, we provide another way of proving [1, Theorem 2.1].

Corollary 2.5 *Let M be a quasi-dual Baer module with $S = End_R(M)$. Then every direct summand of M is quasi-dual Baer.*

Proof Let N be a direct summand of M . Let $\pi : M \rightarrow N$ be the projection map and $\mu : N \rightarrow M$ be the inclusion map. For every $f \in T = End_R(N)$, let $\bar{f} = \mu f \pi \in S$. Let I be a left ideal of T . Set $\bar{I} = \{\bar{f} \mid f \in I\}$ and $\mathfrak{J} = S\bar{I}$. By Proposition 2.4, $\mathfrak{J}(M) = \mathfrak{J}(N)$ is a direct summand of M . Let $e = \mu\pi \in S$. Then, $e = e^2$ and $N = e(M)$. Since $\mathfrak{J}(M)$ is a fully invariant submodule of M , we have $\mathfrak{J}(M) = e(\mathfrak{J}(M)) \oplus (1 - e)(\mathfrak{J}(M))$. Therefore $e(\mathfrak{J}(M)) = e(\mathfrak{J}(N))$ is a direct summand of N . Since $e^2 = e$, we have $e(x) = x$ for any $x \in N$. Thus $\bar{I}(N) = e(\bar{I}(N)) \subseteq e(S\bar{I}(N))$. Now let $g \in S$ and $f \in I$. Then, $eg\bar{f}(N) = \mu\pi g\mu f\pi(N) = \mu(\pi g\mu f)\pi(N)$. Since I is a left ideal of T , it follows that $\pi g\mu f \in I$. This yields $eg\bar{f}(N) = \pi g\mu f(N) \subseteq \bar{I}(N)$. Hence, $e(S\bar{I}(N)) \subseteq \bar{I}(N)$. Thus, $\bar{I}(N) = e(\mathfrak{J}(N))$. Moreover, it is easily seen that $I(N) = \bar{I}(N)$. It follows that $I(N) = e(\mathfrak{J}(N))$ is a direct summand of N . Therefore, N is a quasi-dual Baer module by Proposition 2.4. $\square$

Corollary 2.6 *Let M be a quasi-dual Baer module and let N be a fully invariant submodule of M . Then, the following are equivalent:*

- (i) N is a direct summand of M ;
- (ii) $\mathfrak{J}(M) = N$, where $\mathfrak{J} = \mathfrak{D}(N)$.

Proof (i) $\Rightarrow$ (ii) Let $\pi : M \rightarrow N$ be the projection map and $\mu : N \rightarrow M$ be the inclusion map. Then $\mu\pi \in \mathfrak{D}(N)$ and $\mu\pi(M) = N$. Hence $\mathfrak{J}(M) = N$.

(ii) $\Rightarrow$ (i) This follows directly from Proposition 2.4. $\square$

Next, we present more examples of quasi-dual Baer modules.

Example 2.7 (i) By Proposition 2.4, the notion of being quasi-dual Baer only involves two-sided ideals of the endomorphism ring. It follows that any R -module M for which $End_R(M)$ is a simple ring is quasi-dual Baer.

(ii) To construct an explicit example, let S be a simple ring and let ${}_S N_S$ be an S - S -bimodule. As in [7], consider the generalized matrix ring $R = \begin{pmatrix} S & N \\ N & S \end{pmatrix}$ with multiplication given by

$$\begin{pmatrix} a & x \\ y & b \end{pmatrix} \begin{pmatrix} a' & x' \\ y' & b' \end{pmatrix} = \begin{pmatrix} aa' & ax' + xb' \\ ya' + by' & bb' \end{pmatrix}$$

for all $a, a', b, b' \in S$ and $x, x', y, y' \in N$. Then, $M = N \oplus S$ is a right R -module by the following action:

$$(n, s) \begin{pmatrix} a & x \\ y & b \end{pmatrix} = (na + sy, sb).$$

It is shown in [7, p. 1278] that the rings $End_R(M)$ and S are isomorphic. Therefore M is a quasi-dual Baer R -module.

Next, we compare quasi-dual Baer modules and dual Baer modules. Recall that a module M is called a duo module provided every submodule of M is fully invariant. A ring R is called a right duo ring if the right R -module R_R is a duo module; equivalently, every right ideal of R is a two-sided ideal.

Remark 2.8 Let M be an R -module such that $S = \text{End}_R(M)$ is a right duo ring (e.g., S is a commutative ring). From Proposition 2.4 and [4, Theorem 2.1], it follows that M is a quasi-dual Baer module if and only if M is a dual Baer module. In particular, if M is an indecomposable injective R -module over a commutative noetherian ring R , then M is quasi-dual Baer if and only if M is dual Baer (see [8, Proposition 5.13]).

In [1], there are no examples showing that the class of quasi-dual Baer modules contains properly that of dual Baer modules. In the next example, we provide some quasi-dual Baer modules which are not dual Baer.

Example 2.9 (i) Let R be a ring. It is shown in [4, Corollary 2.9] that the R -module R_R is dual Baer if and only if R is a semisimple ring. It follows that for any simple ring R which is not semisimple, the R -module R_R is quasi-dual Baer but R_R is not dual Baer (see Example 2.3(iii)). For example, we can take the following rings:

- (a) The differential polynomial ring $R = k[x; \delta]$, where k is a simple ring of characteristic 0 and δ is a non-inner derivation on k (see [5, Corollary 3.16]).
- (b) The Weyl algebra $R = A_n(k_0)$, where k_0 is a simple ring of characteristic 0 and n is a positive integer (see [5, Corollary 3.17]).
- (c) The ring of skew Laurent polynomials $R = k[x, x^{-1}; \sigma]$, where k is a field with an automorphism σ of infinite order (see [5, Example 1.8 and Corollary 3.19]).
- (d) See also other examples in [5, p. 40].

(ii) Let S be a simple ring and let ${}_S N_S$ be an S - S -bimodule. As in Example 2.7(ii), consider the ring $R = \begin{pmatrix} S & N \\ N & S \end{pmatrix}$ and the right R -module $M = N \oplus S$. Assume that S is a domain that is not a division ring (we can take $S = A_n(k)$, the n -th Weyl algebra over a field k of characteristic zero). Let $t \in S$ be a nonzero non-invertible element. Consider the map $f_t : M \rightarrow M$ defined by $f_t((n, s)) = (tn, ts)$ for all $(n, s) \in M$. Then f_t is an R -endomorphism of M . As S is a domain, we have $tS \neq S$ (see, e.g., [5, Chapter 1, §1, p. 22, Exercise 1.4]). It follows that f_t is not an epimorphism. Note that the rings $\text{End}_R(M)$ and S are isomorphic (see [7, p. 1278]). Then $\text{End}_R(M)$ is a domain. This implies that M is an indecomposable R -module. Using [4, Corollary 2.2], we conclude that M is not a dual Baer R -module. However, M is a quasi-dual Baer R -module by Proposition 2.4.

(iii) Let S be a simple Ore domain which is not a division ring. Consider the ring $R = \begin{pmatrix} S & K/S \\ 0 & S \end{pmatrix}$, where

K is the classical right ring of quotients of S . Take the right R -module $M = \begin{pmatrix} S & K/S \\ 0 & 0 \end{pmatrix}$. By similar arguments as in (ii), we prove that M is a quasi-dual Baer module which is not dual Baer. We include this proof for the convenience of the reader: By [6, Example 12], we have $\text{End}_R(M) \cong S$. So $\text{End}_R(M)$ is a simple domain. Hence M is an indecomposable R -module which is quasi-dual Baer.

On the other hand, S contains a nonzero non-invertible element t as S is not a division ring. Set $r = \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}$ and consider the R -endomorphism f_t of M defined by $f_t(x) = rx$ for all $x \in M$. It is easily seen that

$$f_t(M) = \left\{ \begin{pmatrix} ts & t\bar{k} \\ 0 & 0 \end{pmatrix} \mid (s, k) \in S \times K \right\}.$$

Since t is non-invertible in S and S is a domain, we have $ts \neq 1$ for every $s \in S$ (see [5, Chapter 1, §1, p. 22, Exercise 1.4]). Hence f_t is not an epimorphism. As M is indecomposable, M is not a dual Baer R -module by [4, Corollary 2.2].

It is clear that for a duo module M , M is quasi-dual Baer if and only if M is a dual Baer module. In particular, for a right duo ring (e.g., R is a commutative ring), the R -module R_R is quasi-dual Baer if and only if R_R is a dual Baer R -module if and only if R is a semisimple ring by [4, Corollary 2.9]. Next, we characterize the class of rings R for which the right R -module R_R is quasi-dual Baer. A ring R is called a *right quasi-dual Baer ring* if the right R -module R_R is a quasi-dual Baer R -module. Left quasi-dual Baer rings are defined similarly.

Proposition 2.10 *The following are equivalent for a ring R :*

- (i) R is a right quasi-dual Baer ring;
- (ii) For every left ideal I of R , the right ideal IR is a direct summand of the right R -module R_R ;

- (iii) For every two-sided ideal I of R , the right ideal I is a direct summand of the right R -module R_R ;
- (iv) R is a finite product of simple rings.

Proof Given $a \in R$, let φ_a denote the R -endomorphism of the R -module R_R defined by $\varphi_a(x) = ax$ for all $x \in R$.

(i) $\Rightarrow$ (ii) Let I be a left ideal of R . It is not hard to see that the set $\mathfrak{I} = \{\varphi_a \mid a \in I\}$ is a left ideal of the ring $S = \text{End}_R(R_R)$. By Proposition 2.4, the right ideal $\mathfrak{I}(R_R) = \sum_{a \in I} aR = IR$ is a direct summand of R_R .

(ii) $\Rightarrow$ (iii) is immediate.

(iii) $\Rightarrow$ (iv) We claim that R is a semiprime ring. Suppose, contrary to our claim, that R contains a nonzero two-sided ideal I with $I^2 = 0$. By assumption, there exists a right ideal K of R such that $I \oplus K = R$. This implies that $I^2 \oplus KI = I$. Hence $KI = I \subseteq K$. Therefore, $I = 0$, a contradiction. Applying [3, Lemma 3.1], we deduce that every two-sided ideal of R is generated by a central idempotent. From [10, 3.5], it follows that R is a finite product of simple rings.

(iv) $\Rightarrow$ (i) Let I be a fully invariant submodule of the R -module R_R . Then, I is a two-sided ideal of R . By [10, 3.5], I is a direct summand of R_R . The result follows from Corollary 2.2. $\square$

Applying Proposition 2.10, we obtain the following result which shows that the quasi-dual Baer property is left-right symmetric for any ring.

Corollary 2.11 *A ring R is right quasi-dual Baer if and only if it is left quasi-dual Baer.*

In the next proposition, we provide a characterization of indecomposable quasi-dual Baer modules over any ring R .

Proposition 2.12 *Let M be an indecomposable R -module. Then, the following are equivalent:*

- (i) M is quasi-dual Baer;
- (ii) For every fully invariant proper submodule N of M , $\text{Hom}_R(M, N) = 0$.

Proof (i) $\Rightarrow$ (ii) This follows from Proposition 2.1.

(ii) $\Rightarrow$ (i) This follows from the fact that $\mathfrak{D}(M) = \text{End}_R(M)$ and $\mathfrak{D}(N) = 0$ for every fully invariant proper submodule N of M . $\square$

Corollary 2.13 *Let R be a local ring with maximal right ideal m . Assume that $M = E(R/m)$ is a quasi-dual Baer module. Then R/m is injective (i.e., R is a division ring) or $\text{Rad}(M) = M$.*

Proof Put $S = R/m$. Assume that S is an injective R -module. Then R is a right V-ring. This yields $m = 0$ and hence R is a division ring. Now assume that $E(S) \neq S$. Suppose that M has a maximal submodule L . As R is a local ring, there exists an R -isomorphism $\lambda : M/L \rightarrow S$. Let $\pi : M \rightarrow M/L$ denote the natural epimorphism. Then, $\lambda\pi \in \text{Hom}_R(M, S)$ and $\lambda\pi \neq 0$. Therefore, $\text{Hom}_R(M, S) \neq 0$. From Proposition 2.12, it follows that S is not fully invariant in M , a contradiction. Consequently, $\text{Rad}(M) = M$. $\square$

A ring R is called right Bass if $\text{Rad}(M) \neq M$ for every nonzero right R -module M . It is well known that every right perfect ring is right Bass (see [2, Remark 28.5(3)]).

Corollary 2.14 *Let R be a local ring with maximal right ideal m and $S = R/m$. Assume that $\text{Rad}(E(S)) \neq E(S)$ (e.g., R is a local right perfect ring). Then, the following are equivalent:*

- (i) $E(S)$ is quasi-dual Baer;
- (ii) R is a division ring.

Proof This follows from Corollary 2.13. $\square$

Next, we deal with some special cases of direct sums of quasi-dual Baer modules.

Proposition 2.15 *Let an R -module $M = M_1 \oplus M_2$ be a direct sum of submodules M_1 and M_2 such that $\text{Rad}(M_1) = M_1$ and M_2 is semisimple. Let $S = \text{End}_R(M)$. Then, the following are equivalent:*

- (i) M is quasi-dual Baer;
- (ii) M_1 is quasi-dual Baer and $\mathfrak{I}(M_2) \cap M_1 \subseteq \mathfrak{I}(M_1)$ for every left ideal $\mathfrak{I}$ of S ;
- (iii) M_1 is quasi-dual Baer and $\mathfrak{I}(M_2) \cap M_1 \subseteq \mathfrak{I}(M_1)$ for every two-sided ideal $\mathfrak{I}$ of S .

- Proof* (i) $\Rightarrow$ (ii) Clearly, M_1 is a quasi-dual Baer module by Corollary 2.5. Now let $\mathcal{J}$ be a left ideal of S . Therefore, $\mathcal{J}(M)$ is a fully invariant submodule of M . Then $\mathcal{J}(M) = (\mathcal{J}(M) \cap M_1) \oplus (\mathcal{J}(M) \cap M_2)$. Note that $\mathcal{J}(M) = K_1 + K_2$, where $K_1 = \mathcal{J}(M_1)$ and $K_2 = \mathcal{J}(M_2)$. Moreover, $M_1 = \text{Rad}(M)$ is a fully invariant submodule of M . Therefore, $K_1 \subseteq M_1$. By modularity, we have $\mathcal{J}(M) \cap M_1 = K_1 + (K_2 \cap M_1)$. This yields $\mathcal{J}(M) = (K_1 + (K_2 \cap M_1)) \oplus (\mathcal{J}(M) \cap M_2)$. Since $K_2 \cap M_1$ is semisimple, there exists a semisimple submodule N of $K_2 \cap M_1$ such that $K_1 + (K_2 \cap M_1) = K_1 \oplus N$. Therefore, $\mathcal{J}(M) = (K_1 \oplus N) \oplus (\mathcal{J}(M) \cap M_2)$. As M is quasi-dual Baer, $\mathcal{J}(M)$ is a direct summand of M by Proposition 2.4. This implies that $K_1 \oplus N$ is a direct summand of M_1 . Thus $\text{Rad}(N) = N$ as $\text{Rad}(M_1) = M_1$. But N is semisimple. Then $N = 0$. It follows that $\mathcal{J}(M) = K_1 \oplus (\mathcal{J}(M) \cap M_2)$. Hence, $K_1 + (K_2 \cap M_1) = K_1$. We thus get $K_2 \cap M_1 \subseteq K_1$.
- (ii) $\Rightarrow$ (iii) This is clear.
- (iii) $\Rightarrow$ (i) Let $\mathcal{J}$ be a nonzero two-sided ideal of $S = \text{End}_R(M)$. Let $\pi_1 : M \rightarrow M_1$ be the projection map and $\mu_1 : M_1 \rightarrow M$ be the inclusion map. Consider the subset $\mathcal{J}_1 = \{\pi_1 f \mu_1 \mid f \in \mathcal{J}\}$ of $S_1 = \text{End}_R(M_1)$. It is easily seen that $\mathcal{J}_1$ is a left ideal of S_1 . Since M_1 is quasi-dual Baer, $N_1 = \mathcal{J}_1(M_1) = \mathcal{J}_1(M_1)$ is a direct summand of M_1 by Proposition 2.4. Since $\mathcal{J}(M)$ is a fully invariant submodule of M , we have $\mathcal{J}(M) = (\mathcal{J}(M) \cap M_1) \oplus (\mathcal{J}(M) \cap M_2)$. As M_2 is semisimple, there exists a semisimple submodule $L \leq \mathcal{J}(M_2)$ such that $\mathcal{J}(M) = N_1 + \mathcal{J}(M_2) = N_1 \oplus L$. By modularity, we have $\mathcal{J}(M) \cap M_1 = N_1 \oplus (L \cap M_1)$. By hypothesis, $L \cap M_1 \subseteq N_1$. Thus $L \cap M_1 = 0$. Therefore, $\mathcal{J}(M) \cap M_1 = N_1$ is a direct summand of M_1 . Since $\mathcal{J}(M) \cap M_2$ is a direct summand of M_2 , $\mathcal{J}(M)$ is a direct summand of M . Using again Proposition 2.4, it follows that M is quasi-dual Baer. $\square$

The next result provides a necessary condition for a direct sum of a module M_1 with $\text{Rad}(M_1) = M_1$ and a semisimple module M_2 to be quasi-dual Baer.

Corollary 2.16 *Let an R -module $M = M_1 \oplus M_2$ be a direct sum of submodules M_1 and M_2 such that $\text{Rad}(M_1) = M_1$ and M_2 is semisimple. Assume that M is quasi-dual Baer. Then, $\text{Im} \varphi$ is not fully invariant in M_1 for all $0 \neq \varphi \in \text{Hom}_R(M_2, M_1)$.*

Proof Suppose to the contrary that there exists a nonzero $\varphi \in \text{Hom}_R(M_2, M_1)$ such that $\text{Im} \varphi$ is fully invariant in M_1 . Consider the endomorphism ψ of M defined by $\psi(x_1) = 0$ for every $x_1 \in M_1$ and $\psi(x_2) = \varphi(x_2)$ for every $x_2 \in M_2$. Let $S = \text{End}_R(M)$ and consider the left ideal $\mathcal{J} = S\psi$ of S . By Proposition 2.15, we have $\mathcal{J}(M_2) \cap M_1 \subseteq \mathcal{J}(M_1)$. Since $\varphi(M_2)$ is fully invariant in M_1 and $M_1 = \text{Rad}(M)$ is fully invariant in M , it follows that $\varphi(M_2) = \psi(M)$ is fully invariant in M . Note that $\mathcal{J}(M_2) = \varphi(M_2)$ and $\mathcal{J}(M_1) = 0$. So $\varphi(M_2) \cap M_1 = 0$; that is $\varphi(M_2) = 0$. This contradicts the choice of φ . $\square$

Example 2.17 Let L be a simple R -module such that $\text{Rad}(E(L)) = E(L)$ (e.g., we can take a simple module L over a Dedekind domain R). Consider the R -module $M = E(L) \oplus L$ and let $\mu : L \rightarrow E(L)$ be the inclusion map. Note that $\text{Im} \mu$ is fully invariant in $E(L)$. By Corollary 2.16, M is not quasi-dual Baer.

The next corollary is a direct consequence of Proposition 2.15.

Corollary 2.18 *Let an R -module $M = M_1 \oplus M_2$ be a direct sum of submodules M_1 and M_2 such that $\text{Rad}(M_1) = M_1$, M_2 is semisimple and $\text{Hom}_R(M_2, M_1) = 0$. Then, the following are equivalent:*

- (i) M is quasi-dual Baer;
- (ii) M_1 is quasi-dual Baer.

We conclude this section by providing conditions under which a direct sum of quasi-dual Baer modules is quasi-dual Baer.

Proposition 2.19 *Let $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$ such that each M_λ ($\lambda \in \Lambda$) is a fully invariant submodule of M which is a quasi-dual Baer R -module. Then M is quasi-dual Baer.*

Proof Let $S = \text{End}_R(M)$ and for each $\lambda \in \Lambda$, let $S_\lambda = \text{End}_R(M_\lambda)$. Let $\mathcal{J}$ be a left ideal of the ring S . Since M_λ is fully invariant in M for all $\lambda \in \Lambda$, we obtain $\mathcal{J}(M) = \bigoplus_{\lambda \in \Lambda} \mathcal{J}(M_\lambda)$. For each $\lambda \in \Lambda$ and $f \in S$, let f_λ denote the endomorphism of M_λ defined by $f_\lambda(x) = f(x)$ for all $x \in M_\lambda$. It is not hard to see that $\mathcal{J}_\lambda = \{f_\lambda \mid f \in \mathcal{J}\}$ is a left ideal of S_λ . Since each M_λ ($\lambda \in \Lambda$) is quasi-dual Baer, it follows that $\mathcal{J}_\lambda(M_\lambda)$ is a direct summand of M_λ for every $\lambda \in \Lambda$ (see Proposition 2.4). Therefore $\mathcal{J}(M)$ is a direct summand of M . Hence, by Proposition 2.4, we conclude that M is a quasi-dual Baer module. $\square$

3 Modules over commutative rings

This section is devoted to the study of some properties of quasi-dual Baer modules over commutative rings. Throughout this section, R will denote a commutative ring, unless otherwise stated.

Proposition 3.1 *Let R be a commutative ring and let M be a cyclic R -module. Then M is quasi-dual Baer if and only if M is semisimple.*

Proof Without loss of generality, we can assume that $M = R/I$ for some ideal I of R . It is easily seen that the module M_R is quasi-dual Baer if and only if the module $M_{R/I}$ is quasi-dual Baer. The result follows from Proposition 2.10. $\square$

A module M is said to have the strong summand sum property (SSSP for short) if the sum of any family of direct summands of M is a direct summand of M . The class of dual Baer modules was characterized in [4, Theorem 2.1] as that of modules M having the SSSP such that for every $\varphi : M \rightarrow M$, $Im\varphi$ is a direct summand of M . To obtain a partial analogue of this characterization for quasi-dual Baer modules, Amouzegar and Talebi [1] introduced the FI-strong summand sum property (FI-SSSP for short). A module M is said to have the FI-SSSP if the sum of any family of fully invariant direct summands of M is a direct summand of M . The next result is taken from [1, Lemmas 2.1 and 2.2].

Lemma 3.2 *Let R be a ring (not necessarily commutative) and let M be a quasi-dual Baer module. Then:*

- (1) M has the FI-SSSP, and
- (2) for every endomorphism φ of M such that $Im\varphi$ is fully invariant in M , $Im\varphi$ is a direct summand of M .

Proposition 3.3 *Let M be a module over a commutative ring R . If M is quasi-dual Baer, then $M\mathfrak{a}$ is a direct summand of M for any ideal $\mathfrak{a}$ of R .*

Proof Let $\mathfrak{a}$ be an ideal of R and let $r \in \mathfrak{a}$. It is clear that $Mr = Imf_r$ where f_r is the R -endomorphism of M defined by $f_r(x) = xr$ for every $x \in M$. Moreover, it is easily seen that Mr is a fully invariant submodule of M . Now using Lemma 3.2, it follows that Imf_r is a direct summand of M for all $r \in \mathfrak{a}$. Note that M has the FI-SSSP (see Lemma 3.2). Therefore, $M\mathfrak{a} = \sum_{r \in \mathfrak{a}} Mr = \sum_{r \in \mathfrak{a}} Imf_r$ is a direct summand of M . $\square$

The converse of Proposition 3.3 is not true, in general, as shown in the following example. This example shows also that the class of quasi-dual Baer modules is not closed under direct products.

Example 3.4 Let $\mathbf{P}$ denote the set of all prime numbers. Consider the $\mathbb{Z}$ -module $M = \prod_{p \in \mathbf{P}} \mathbb{Z}/p\mathbb{Z}$. Then the torsion submodule of M is $T(M) = \oplus_{p \in \mathbf{P}} \mathbb{Z}/p\mathbb{Z}$. For each $p \in \mathbf{P}$, let f_p denote the endomorphism of M defined by $(x_p)_{p \in \mathbf{P}} \mapsto (y_p)_{p \in \mathbf{P}}$ such that $y_q = 0$ for every $q \neq p$ and $y_p = x_p$. Let $\mathfrak{J}$ denote the left ideal of $S = \text{End}_{\mathbb{Z}}(M)$ generated by all the f_p ($p \in \mathbf{P}$). Then $\mathfrak{J}(M) = T(M)$ is not a direct summand of M . Therefore M is not quasi-dual Baer by Proposition 2.4. On the other hand, it is easily seen that $M(n\mathbb{Z}) = Mn$ is a direct summand of M for every integer n .

Corollary 3.5 *Let M be a nonzero projective module over a commutative ring R . If M is a quasi-dual Baer R -module, then $\text{Rad}(M) = 0$.*

Proof Assume that M is quasi-dual Baer. Let $J = \text{Rad}(R)$. Since M is projective, we have $\text{Rad}(M) = MJ$ by [2, Proposition 17.10]. From Proposition 3.3, it follows that $\text{Rad}(M)$ is a direct summand of M . So there exists a submodule N of M such that $M = \text{Rad}(M) \oplus N$. Clearly, we have $\text{Rad}(N) = 0$. Therefore, $\text{Rad}(\text{Rad}(M)) = \text{Rad}(M)$. Thus, $\text{Rad}(M) = 0$ by [2, Proposition 17.14]. $\square$

Remark 3.6 Let R be a commutative ring and let M be an indecomposable quasi-dual Baer R -module. Using Proposition 3.3 and applying similar arguments as in the proof of [9, Proposition 4.4], we conclude that either $\text{Rad}(M) = M$ or M is a simple R -module. But this is not true, in general, if the ring R is not commutative. Indeed, we can consider the example given in [9, Example 4.10 and Remark 4.11].

Corollary 3.7 *Let M be an indecomposable $\mathbb{Z}$ -module. Then, the following are equivalent:*

- (i) M is quasi-dual Baer;
- (ii) M is dual Baer;
- (iii) $M \cong \mathbb{Q}$ or $M \cong \mathbb{Z}(p^\infty)$ or $M \cong \mathbb{Z}/p\mathbb{Z}$, where p is a prime number.

Proof (i) $\Rightarrow$ (iii) Note that for every $\mathbb{Z}$ -module N with $\text{Rad}(N) = N$, N is an injective module. Now, this implication follows easily from Remark 3.6.

(iii) $\Rightarrow$ (ii) This follows from [4, Corollary 3.5].

(ii) $\Rightarrow$ (i) This is immediate. $\square$

Recall that a ring R is called semilocal if $R/\text{Rad}(R)$ is a semisimple ring. The next result characterizes quasi-dual Baer modules over commutative semilocal rings.

Theorem 3.8 *Let M be a nonzero module over a commutative semilocal ring R and let $S = \text{End}_R(M)$. Then, the following conditions are equivalent:*

- (i) M is a quasi-dual Baer R -module;
- (ii) (a) $M = M_1 \oplus M_2$ is a direct sum of submodules M_1 and M_2 such that $\text{Rad}(M_1) = M_1$ is quasi-dual Baer and M_2 is semisimple, and
(b) $\mathfrak{I}(M_2) \cap M_1 \subseteq \mathfrak{I}(M_1)$ for every two-sided ideal $\mathfrak{I}$ of S .

Proof (i) $\Rightarrow$ (ii) Let $J = \text{Rad}(R)$. By [2, Corollary 15.18], we have $\text{Rad}(M) = MJ$. Applying Proposition 3.3, we deduce that $\text{Rad}(M)$ is a direct summand of M . Set $M_1 = \text{Rad}(M)$ and let M_2 be a submodule of M such that $M = M_1 \oplus M_2$. Clearly, $\text{Rad}(M_2) = M_2J = 0$ (see [2, Corollary 15.18]). Therefore M_2 is an R/J -module. Since R/J is a semisimple ring, it follows that M_2 is a semisimple R -module. Note that M_1 is quasi-dual Baer by Corollary 2.5. This proves (a). Note that (b) follows directly from Proposition 2.15.

(ii) $\Rightarrow$ (i) By Proposition 2.15. $\square$

In the following result, we describe the structure of quasi-dual Baer modules over commutative perfect rings.

Corollary 3.9 *Let M be a nonzero module over a commutative perfect ring R . Then the following are equivalent:*

- (i) M is a quasi-dual Baer module;
- (ii) M is a dual Baer module;
- (iii) M is a semisimple module.

Proof (i) $\Rightarrow$ (iii) Since R is a semilocal ring (see [2, Theorem 28.4]), there exists a semisimple submodule N of M such that $M = \text{Rad}(M) \oplus N$ by Theorem 3.8. But $\text{Rad}(M) \ll M$ by [2, Remark 28.5(3)]. Then, $\text{Rad}(M) = 0$. It follows that $M = N$ is a semisimple R -module.

(iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) These implications are obvious. $\square$

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