Arab. J. Math. (2021) 10:91–101
https://doi.org/10.1007/s40065-021-00314-4

Arabian Journal of Mathematics

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H-convergence of a class of quasilinear equations in perforated domains beyond periodic setting

Received: 9 September 2020 / Accepted: 18 January 2021 / Published online: 6 February 2021

Abstract In this paper, we aim to study the asymptotic behavior (when $\varepsilon \to 0$) of the solution of a quasilinear problem of the form $-\text{div}\ (A^\varepsilon(\cdot, u^\varepsilon)\nabla u^\varepsilon) = f$ given in a perforated domain $\Omega\backslash T_\varepsilon$ with a Neumann boundary condition on the holes T_ε and a Dirichlet boundary condition on $\partial\Omega$. We show that, if the holes are admissible in certain sense (without any periodicity condition) and if the family of matrices $(x, d) \mapsto A^\varepsilon(x, d)$ is uniformly coercive, uniformly bounded and uniformly equicontinuous in the real variable d, the homogenization of the problem considered can be done in two steps. First, we fix the variable d and we homogenize the linear problem associated to $A^\varepsilon(\cdot, d)$ in the perforated domain. Once the H^0-limit $A^0(\cdot, d)$ of the pair $(A^\varepsilon, T^\varepsilon)$ is determined, in the second step, we deduce that the solution u^ε converges in some sense to the unique solution u^0 in $H_0^1(\Omega)$ of the quasilinear equation $-\text{div}\ (A^0(\cdot, u^0)\nabla u) = \chi^0 f$ (where χ^0 is L^∞ weak $\star$ limit of the characteristic function of the perforated domain). We complete our study by giving two applications, one to the classical periodic case and the second one to a non-periodic one.

Mathematics Subject Classification 35B40 · 35B27 · 35J62

1 Introduction

The main goal of this work is to give, in the framework of the H^0-convergence notion (the generalization of the H-convergence to perforated domains), a general homogenization result of a type of quasilinear equations with a mixed Neumann-Dirichlet boundary conditions, beyond the periodic setting. More precisely, we study the asymptotic behaviour of the solution of the following problem:

$$\begin{cases} -\text{div}\ (A^\varepsilon(\cdot, u^\varepsilon)\nabla u^\varepsilon) = f \quad \text{in } \Omega\backslash T_\varepsilon, \\ A^\varepsilon(\cdot, u^\varepsilon)\nabla u^\varepsilon \cdot \nu = 0 \quad \text{on } \partial T_\varepsilon, \\ u^\varepsilon = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^n$, $\{T_\varepsilon\}$ is sequence of compact subsets of Ω, not necessarily periodically distributed, and where $f \in L^2(\Omega)$, $A^\varepsilon : (x, d) \in (\Omega, \mathbb{R}) \longmapsto A^\varepsilon(x, t) \in \mathbb{R}^{n\times n}$ is a sequence of Caratheodory functions uniformly coercive, uniformly bounded and uniformly equicontinuous matrix fields in the variable d. We show that, under a suitable conditions on the equicontinuity modulus and L^p-estimate assumption, there exists a subsequence of ε (still denoted by ε), a positive function $\chi^0 \in L^\infty(\Omega)$ and a matrix field $A^0(\cdot, \cdot)$ which satisfies the same properties as $A^\epsilon(\cdot, \cdot)$ such that $\chi^\varepsilon \rightharpoonup \chi^0$ weakly $\star$ in $L^\infty(\Omega)$,

$$(A^\varepsilon(\cdot, d), T_\varepsilon) \overset{H^0}{\rightharpoonup} A^0(\cdot, d) \quad \text{in } \Omega,\ \forall d \in \mathbb{R}^n,$$

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and, if we denote by $\widetilde{\cdot}$ the extension by 0 from Ω_ε to Ω, we have

$$\begin{cases} \widetilde{u^\varepsilon} \rightharpoonup \chi^0 u^0 \text{ weakly in } L^2(\Omega), \\ \widetilde{A^\varepsilon(u^\varepsilon)\nabla u^\varepsilon} \rightharpoonup A^0(u^0)\nabla u^0 \text{ weakly in } L^2(\Omega)^n, \end{cases}$$

where u^0 is the solution of

$$\begin{cases} -\text{div } (A^0(\cdot, u^0)\nabla u^0) = \chi^0 f \quad \text{in } \Omega, \\ u^0 = 0 \quad \text{on } \partial\Omega. \end{cases}$$

We complete our study by giving two applications of the established compactness results. The first application is for the classical periodic case, where the obtained result coincides (in our framework) with a result given in [7]. While, the second one which concerns a non-periodic case introduced in [5] is an original result.

Our work generalizes that of Murat–Bocardo given in [4] which treated in the general framework of H-convergence the same type of quasilinear equations in fixed domains without holes. The periodic case with Lipschitz continuous coefficients was subsequently processed by Artola–Duvaut in [1]. On the other hand, for periodically perforated domains, the same type of quasilinear equations was firstly studied in Bendib [2] and Bendib–Tcheugoué Teboué [3], with Lipschitz continuous coefficients and linear Robin conditions. After this Cabarrubias–Donato have studied in [7] this equation with a nonlinear Robin condition boundary of the holes and the module of equicontinuity satisfies a suitable assumption introduced by Chipot in [9], but not assumed to be Lipschitz continuous. For the homogenization of other type of Neumann quasilinear equations in perforated domains with data satisfying a general assumptions of abstract homogenization, see for example [8,13] among others.

This article is organized as follows: Sect. 2 is devoted to some preliminary results on the H^0-convergence as introduced by [5]. This notion generalizes that of H-convergence in fixed domains due to Murat–Tartar (see [12,14]). We give at the end of this section, a new result about a pointwise estimate of the dierence of two H^0-limits. In Sect. 3, we present our main compactness results for a class of quasilinear equations in perforated domains in the general framework of H^0-convergence. Section 4 is devoted to the proofs of our results. Finally, in Sect. 5, we give two applications of the obtained compactness results, namely the classical periodic case and a certain non-periodic case.

2 Notations and preliminary results

2.1 Notations

- $\{\varepsilon\}$ denotes a strictly decreasing sequence converging to zero,
- if $\zeta = (\zeta_i)_{1\le i\le n}$ and $\xi = (\xi_i)_{1\le i\le n}$ are two vectors, we set

$$\zeta \cdot \xi = \sum_{i=1}^{n} \zeta_i \xi_i \quad \text{and} \quad |\xi| = \left(\sum_{i=1}^{n} \xi_i^2\right)^{\frac{1}{2}},$$

- for matrix A in $R^{n\times n}$, we set

$$|A| = \sup\{|A\xi| \ \text{ s.t. } \ |\xi| = 1 \ \text{ and } \ \xi \in \mathbb{R}^n\},$$

- $\chi_{\mathcal{O}}$ denotes the characteristic function of a subset $\mathcal{O}$ of $\mathbb{R}^n$,

- for two real numbers α and β such that $0 < \alpha < \beta$, $M(\alpha, \beta; \Omega)$ is the set of the matrix fields $A = (A_{ij})_{1\le i,j\le n}$ defined on Ω such that almost everywhere in Ω, we have

$$\begin{cases} \text{(i) } A_{ij} \in L^\infty(\Omega), \ \text{ for } i, j = 1, \ldots, n, \\ \text{(ii) } \alpha\, |\xi|^2 \le A\xi \cdot \xi, \ \text{ for } \xi \in \mathbb{R}^n, \\ \text{(iii) } A^{-1}\xi \cdot \xi \ge \beta^{-1}|\xi|^2, \ \text{ for } \xi \in \mathbb{R}^n. \end{cases}$$

2.2 Preliminary results on the H-convergence for perforated domains

Since we work in the framework of the H^0-convergence, we recall in this subsection some preliminary results about this notion and we give at the end a useful new result on the pointwise estimate of the dierence of two H^0-limits.

We introduce the perforated domain by

$$\Omega_\varepsilon = \Omega \setminus T_\varepsilon,$$

where $\{T_\varepsilon\}$ is a sequence of compact subsets of Ω and set

$$V_\varepsilon = \left\{ v \in H^1(\Omega_\varepsilon) \; s.t. \; v = 0 \;\; on \; \partial\Omega \right\}.$$

We denote by $\tilde{\cdot}$ the extension by 0 from Ω_ε to Ω and set $\chi^\varepsilon = \chi_{\Omega_\varepsilon}$. In the following ν denotes the outward normal unit vector to the boundary of Ω_ε.

Definition 2.1 ([5]) The sequence $\{T_\varepsilon\}$ is said to be admissible (in Ω) if
i) every L^∞ weak $\star$ limit point of $\{\chi^\varepsilon\}$ is positive almost everywhere in Ω,
ii) there exists a positive real C, independent of ε, and a sequence $\{P_\varepsilon\}$ of linear extension operators such that for each ε

$$\begin{cases} P_\varepsilon \in \mathcal{L}(V_\varepsilon, H_0^1(\Omega)), \\ (P_\varepsilon v)_{|\Omega_\varepsilon} = v, \quad \forall v \in V_\varepsilon, \\ \|\nabla P_\varepsilon v\|_{L^2(\Omega)^n} \leq C \, \|\nabla v\|_{L^2(\Omega_\varepsilon)^n} \, , \quad \forall v \in V_\varepsilon. \end{cases}$$

We denote by $P_\varepsilon^\star$ the adjoint operator of P_ε, which is defined from $H^{-1}(\Omega)$ to V_ε' (dual of V_ε) with $P_\varepsilon^\star$ given for every $g \in H^{-1}(\Omega)$ by

$$\forall v \in V_\varepsilon, \; \langle P_\varepsilon^\star g, v\rangle_{V_\varepsilon', V_\varepsilon} = \langle g, P_\varepsilon v\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}.$$

Definition 2.2 ([5]) Let $A^\varepsilon \in M(\alpha, \beta; \Omega)$ and T_ε be admissible in Ω. We say that the pair $(A^\varepsilon, T_\varepsilon)$ H^0-converges to the matrix $A^0 \in M\left(\alpha', \beta'; \Omega\right)$ and we write $(A^\varepsilon, T_\varepsilon) \overset{H^0}{\rightharpoonup} A^0$ in Ω if and only if for every function g of $L^2(\Omega)$, and every subsequence of ε (still denoted by ε) such that $\chi^\varepsilon \rightharpoonup \chi^0$ weakly $\star$ in $L^\infty(\Omega)$ (χ^0depending upon the subsequence), the solution v^ε of

$$\begin{cases} -\text{div}\,(A^\varepsilon \nabla v^\varepsilon) = g \quad \text{in } \Omega_\varepsilon, \\ (A^\varepsilon \nabla v^\varepsilon)\cdot \nu = 0 \quad \text{on } \partial T_\varepsilon, \\ v^\varepsilon = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

satisfies the weak convergence

$$\begin{cases} i) \; P_\varepsilon(v^\varepsilon) \rightharpoonup v^0 \text{ weakly in } H_0^1(\Omega), \\ ii) \; A^\varepsilon \widetilde{\nabla v^\varepsilon} \rightharpoonup A^0 \nabla v^0 \text{ weakly in } L^2(\Omega)^n, \end{cases} \tag{2.2}$$

where v^0 is the unique solution of the problem

$$\begin{cases} -\text{div}\,(A^0 \nabla v^0) = \chi^0 g \quad \text{in } \Omega, \\ v^0 = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Remark 2.3 (1) In [5] the definition of H^0-convergence is given for $f \in H^{-1}(\Omega)$. This latter and Definition 2.2 are equivalent in view of [5, Theorem 1.5].

(2) In the case of $T_\varepsilon = \emptyset$, this definition reduces to the definition of H-convergence.

The main properties of the H^0-convergence are given by the results below.

Theorem 2.4 (Compactness [5]) *Let $A^\varepsilon \in M(\alpha, \beta; \Omega)$ and T_ε be admissible in Ω. Then, there exists a subsequence of $\{\varepsilon\}$ (still denoted by $\{\varepsilon\}$) and a matrix $A^0 \in M\left(\frac{\alpha}{C^2}, \beta; \Omega\right)$ such that $\{(A^\varepsilon, T_\varepsilon)\}$ H^0-converges to A^0.*

Proposition 2.5 [5] *The pair $(A^\varepsilon, T_\varepsilon)$ H^0-converges to A^0 if and only if $({}^tA^\varepsilon, T_\varepsilon)$ H^0-converges to ${}^tA^0$.*

Finally, we complete the preliminary results by giving a pointwise estimate of the dierence of two H^0-limits. This result needs the following lemma (which is a directly consequence of [5, Proposition 1.14]):

Lemma 2.6 *Assume that* $(A^\varepsilon, T_\varepsilon) \overset{H^0}{\rightharpoonup} A^0$ *in* Ω *and suppose that for every* $\lambda \in \mathbb{R}^{n\times n}$*, there exists a sequence* $\{v_\lambda^\varepsilon\}$ *bounded in* $H^1(\Omega)$ *such that*

$$\begin{cases} (i) \begin{cases} -\mathrm{div}\,\left(\chi^\varepsilon A^\varepsilon \nabla\left(v_\lambda^\varepsilon\right)\right) = P_\varepsilon^\star g_\Lambda^\varepsilon \text{ in } \Omega_\varepsilon, \\ \text{with } g_\lambda^\varepsilon \text{ is in a compact subset of } H^{-1}(\Omega), \end{cases} \\ (ii)\; v_\lambda^\varepsilon \rightharpoonup \lambda x \text{ weakly in } H^1(\Omega). \end{cases} \tag{2.4}$$

Then, if we set

$$N^\varepsilon \lambda = \nabla v_\lambda^\varepsilon, \;\; \forall \lambda \in \mathbb{R}^n, \tag{2.5}$$

we will have $\chi^\varepsilon A^\varepsilon N^\varepsilon \rightharpoonup A^0\lambda$ *weakly in* $L^2(\Omega)^n$ *and* N^ε *is a corrector for the pair* $(A^\varepsilon, T_\varepsilon)$ *in the sense that*

$$\lim_{\varepsilon\to 0} \|\nabla v^\varepsilon - N^\varepsilon v^0\|_{L^1(\Omega_\varepsilon)^n} = 0,$$

where v^ε *and* v^0 *are solutions of* (2.1) *and* (2.3) *respectively.*

We are now able to give a pointwise estimate of the dierence of two H^0-limits.

Theorem 2.7 *Let* T_ε^1 *and* T_ε^2 *be admissible in* Ω*,* $A_1^\varepsilon \in \mathcal{M}(\alpha, \beta; \Omega)$ *and* $A_2^\varepsilon \in \mathcal{M}(\alpha', \beta'; \Omega)$ *such that*

$$\begin{cases} (A_1^\varepsilon, T_\varepsilon^1) \overset{H^0}{\rightharpoonup} A_1^0 \quad \text{in } \Omega, \\ (A_2^\varepsilon, T_\varepsilon^2) \overset{H^0}{\rightharpoonup} A_2^0 \quad \text{in } \Omega, \\ \chi_2^\varepsilon |A_1^\varepsilon(x) - A_2^\varepsilon(x)| \le h^\varepsilon(x) \;\text{ a.e. in } \Omega, \\ \text{with } h^\varepsilon \longrightarrow h^0 \quad \text{strongly in } L^1(\Omega). \end{cases}$$

Assume that

(i) $\chi_1^\varepsilon - \chi_2^\varepsilon \to 0$ *strongly in* $L^1(\Omega)$*,*
(ii) $\left(A_1^\varepsilon, T_\varepsilon^1\right)$ *admits a corrector satisfying* (2.4)–(2.5)*,*
(iii) $\left(A_2^\varepsilon, T_\varepsilon^2\right)$ *admits a corrector* N^ε *satisfying* (2.4)–(2.5) *and*

$$\begin{cases} \exists p > 2, \text{ such that } \|N^\varepsilon\|_{L^p(\Omega)^{n\times n}} \le \rho, \\ \text{with } \rho > 0 \text{ is independent of } \varepsilon. \end{cases}$$

Then,

$$|A_1^0 - A_2^0| \le \sqrt{\frac{\beta\beta'}{\alpha\alpha'}}\, h^0 \quad a.\,e.\text{ in } \Omega. \tag{2.6}$$

Proof The proof is obtained by using Lemma 2.6 and Proposition 2.5, and by following the same techniques used to prove a similar result given for the elasticity case in [11, Theorem 28]. □

Remark 2.8 Assumptions

(i)–(iii) of Theorem 2.7 are reasonable. Indeed,
-(i) is obviously satisfied when $T_\varepsilon^1 = T_\varepsilon^2$ for every ε,
-(ii) is satisfied when there exists a bounded domain O in $\mathbb{R}^n$ in which Ω is relatively compact and for which T_ε is admissible (see the proof of [5, Proposition 1.15]),
-(iii) is satisfied for the classical periodic case and also for the non-periodic case considered in [5].

3 Statement of compactness results

In this section, we give our compactness results for the H^0-convergence of a class of elliptic and uniformly equicontinuous operators in perforated domains. Firstly, we introduce the set $\mathcal{M}_{Equi}(\alpha, \beta, \omega; \Omega)$ in the following definition :

Definition 3.1 For two real numbers $\alpha,\ \beta$ such that $0 < \alpha < \beta$ and ω a function defined from $\mathbb{R}^+$ to $\mathbb{R}^+$ nondecreasing and continuous at 0, $\mathcal{M}_{Equi}(\alpha, \beta, \omega; \Omega)$ denotes the set of all Caratheodory functions

$$A : (x, d) \in (\Omega, \mathbb{R}) \longmapsto A(x, d) \in \mathbb{R}^{n\times n}$$

satisfying the following assumptions:

(i) for every $d \in \mathbb{R}, \quad A(d) \doteq A(\cdot, d) \in \mathcal{M}(\alpha, \beta; \Omega)$,

(ii) for almost every x in Ω and for every $d, d' \in \mathbb{R}$, one has

$$|A(x, d) - A(x, d')| \leq \omega(|d - d'|).$$

Our first main result is the following:

Theorem 3.2 *Let $\{T_\varepsilon\}$ be a sequence admissible in Ω and $\{A^\varepsilon\}$ be a sequence of elements of $\mathcal{M}_{Equi}(\alpha, \beta, \omega; \Omega)$. Assume that $\omega(0) = 0$ and*

$$\begin{cases} \forall d \in \mathbb{R},\ \exists p > 2 \text{ s.t. } (A^\varepsilon(d), T_\varepsilon) \text{ admits a corrector which} \\ \text{satisfies (2.4)-(2.5) and is bounded in } L^p(\Omega) \text{ independently of } \varepsilon. \end{cases} \tag{3.1}$$

Then, there exists a subsequence of $\{\varepsilon\}$ (still denoted by $\{\varepsilon\}$), and an element $A^0 \in \mathcal{M}_{Equi}(\frac{\alpha}{C^2}, \beta, \frac{\beta}{\alpha}\omega; \Omega)$ such that

$$(A^\varepsilon(d), T_\varepsilon) \overset{H^0}{\rightharpoonup} A^0(d) \quad in\ \Omega, \quad \forall d \in \mathbb{R}. \tag{3.2}$$

Moreover, if we suppose that there exists a bounded domain O in $\mathbb{R}^n$ in which Ω is relatively compact and for which T_ε is also admissible, we have

$$(A^\varepsilon(v), T_\varepsilon) \overset{H^0}{\rightharpoonup} A^0(v) \quad in\ \Omega,\ \forall v \in L^1(\Omega). \tag{3.3}$$

Remark 3.3 (i) A similar property to (3.2) is given in [14] in the case of fixed domain when the mapping $d \to A^\varepsilon(\cdot, d)$ is of class C^k (or real analytic) from an open set D of $\mathbb{R}^p$ into $L^\infty(\Omega; L(\mathbb{R}^n; \mathbb{R}^n))$ for every $p \in \mathbb{N}^*$.

(ii) Theorem 3.2 still holds if $d \in \mathbb{R}^p$ and $v \in L^1(\Omega)^p$ for every $p \in \mathbb{N}^*$.

As a consequence of Theorem 3.2, we obtained a general homogenization result for some quasilinear equations in perforated domain beyond periodic setting.

Theorem 3.4 *Let $\{T_\varepsilon\}$ be a sequence admissible in Ω and suppose that there exists a bounded domain O in $\mathbb{R}^n$ in which Ω is relatively compact and for which T_ε is also admissible. Let $\{A^\varepsilon\}$ be a sequence in $\mathcal{M}_{Equi}(\alpha, \beta, \omega; \Omega)$ which satisfies (3.1). Assume that ω is continuous with $\omega(d) > 0\ \forall d > 0$ and*

$$for\ any\ r > 0, \quad \lim_{s\to 0} \int_s^r \frac{\mathrm{d}t}{\omega(t)} = +\infty. \tag{3.4}$$

Then, there exists subsequence of $\{\varepsilon\}$ (still denoted by $\{\varepsilon\}$) with χ^ε converges to a some χ^0 weakly $\star$ in $L^\infty(\Omega)$, such that for every function f of $L^2(\Omega)$, the (unique) solution u^ε of the problem:

$$\begin{cases} -\text{div}\ (A^\varepsilon(u^\varepsilon)\nabla u^\varepsilon) = f \quad in\ \Omega_\varepsilon, \\ A^\varepsilon(u^\varepsilon)\nabla u^\varepsilon \cdot \nu = 0 \quad on\ \partial T_\varepsilon, \\ u^\varepsilon = 0 \quad on\ \partial\Omega, \end{cases} \tag{3.5}$$

satisfies

$$\begin{cases} (i)\ P_\varepsilon(u^\varepsilon) \rightharpoonup u^0 \text{ weakly in } H_0^1(\Omega), \\ (ii)\ \widetilde{u^\varepsilon} \rightharpoonup \chi^0 u^0 \text{ weakly in } L^2(\Omega), \\ (iii)\ \widetilde{A^\varepsilon(u^\varepsilon)\nabla u^\varepsilon} \rightharpoonup A^0(u^0)\nabla u^0 \text{ weakly in } L^2(\Omega)^n, \end{cases} \tag{3.6}$$

where u^0 is the (unique) solution of

$$\begin{cases} -\text{div } (A^0(u^0)\nabla u^0) = \chi^0 f \quad \text{in } \Omega, \\ u^0 = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

with A^0 the family of matrices given by Theorem 3.2.

Remark 3.5 Assumption (3.4) introduced initially in [9] implies that $\lim_{d\to 0} \omega(d) = 0$. If this assumption is replaced by just the fact that $\lim_{d\to 0} \omega(d) = 0$, the uniqueness will no longer be guaranteed for the solutions of (3.5) and (3.7).

4 Proofs of compactness results

We give in this section the proofs of our main results. The proofs are an adaptation of the similar ones given in [4] for fixed domains.

Proof of Theorem 3.2 We give the proof in two steps.

Step 1. Let us prove that there exists $A^0 \in \mathcal{M}_{Equi}(\frac{\alpha}{C^2}, \beta, \frac{\beta}{\alpha}\omega; \Omega)$ which satisfies convergence (3.2) up to subsequence. Using Theorem 2.4 and the diagonal subsequence procedure, we extract a subsequence of $\{\varepsilon\}$ (still denoted by $\{\varepsilon\}$) such that, for every $d \in \mathbb{Q}$, we will have

$$(A^\varepsilon(d), T_\varepsilon)\ H^0\text{-converges to a limit } A^0(d) \in \mathcal{M}\left(\frac{\alpha}{C^2}, \beta; \Omega\right). \tag{4.1}$$

Hence, by the fact that $A^\varepsilon \in \mathcal{M}_{Equi}(\alpha, \beta, \omega; \Omega)$, Assumption (3.1) and Theorem 2.7, we obtain

$$|A^0(x,d) - A^0(x,d')| \le \frac{\beta}{\alpha}\,\omega(|d-d'|) \quad \text{a.e. } x \in \Omega,\ \forall d,\ d' \in \mathbb{Q}.$$

Thus, the mapping

$$\begin{aligned} A^0 :\ \mathbb{Q} &\longrightarrow \mathbb{L}^\infty(\Omega)^{n\times n}, \\ d &\longmapsto A^0(d) \end{aligned}$$

is uniformly continuous. Hence, it is extensible to a mapping (denoted again by A^0) defined and uniformly continuous on all $\mathbb{R}$ (since $\mathbb{Q}$ is dense in $\mathbb{R}$), namely

$$|A^0(x,d) - A^0(x,d')| \le \frac{\beta}{\alpha}\,\omega(|d-d'|), \quad \text{a.e. } x \in \Omega,\ \forall d,\ d' \in \mathbb{R}. \tag{4.2}$$

On the other hand, let $d \in \mathbb{R}$ and $\{d_m\}$ be a sequence in $\mathbb{Q}$ which converges to d as $m \to \infty$. Thanks to Theorem 2.4, there exists a subsequence of $\{\varepsilon\}$ (still denoted by $\{\varepsilon\}$) such that

$$(A^\varepsilon(d), T_\varepsilon)\ H^0\text{-converges to some } A \in \mathcal{M}\left(\frac{\alpha}{C^2}, \beta; \Omega\right). \tag{4.3}$$

Since, for every $\varepsilon > 0$, we have

$$|A^\varepsilon(x,d) - A^\varepsilon(x,d_m)| \le \omega(|d-d_m|), \quad \text{a.e. } x \in \Omega,$$

then from this, (4.1), (4.3), Assumption (3.1) and Theorem 2.7, it comes

$$|A(x) - A^0(x,d_m)| \le \frac{\beta}{\alpha}\omega(|d-d_m|), \quad \text{a.e. } x \in \Omega.$$

This, with (4.2) and by the triangle inequality, we deduce that for almost every x in Ω

$$|A(x) - A^0(x,d)| \le |A(x) - A^0(x,d_m)| + |A^0(x,d) - A^0(x,d_m)| \\ \le 2\frac{\beta}{\alpha}\,\omega(|d-d_m|).$$

Using the continuity of ω at 0, passing to the limit in this inequality as $m \to \infty$, we find

$$A(x) = A^0(x,d), \quad \text{a.e. } x \in \Omega.$$

Step 2. We now show property (3.3). Let $v \in L^1(\Omega)$. Then, $A^\varepsilon(v(\cdot)) \doteq A^\varepsilon(\cdot, v(\cdot))$ belongs to $\mathcal{M}(\alpha,\beta;\Omega)$. Hence, taking into account Theorem 2.4, there exists $B^0 \in \mathcal{M}(\frac{\alpha}{C^2},\beta;\Omega)$ such that to up a subsequence, we have

$$(A^\varepsilon(v), T_\varepsilon) \overset{H^0}{\rightharpoonup} B^0. \tag{4.4}$$

On the other hand, since $v \in L^1(\Omega)$, there exists a sequence of step functions $\{v^m\}$ such that $v^m \to v$ strongly in $L^1(\Omega)$, and v^m is of the form

$$v^m = \sum_{i=1}^{i=k} l_i^m \chi_{Y_i}, \quad \text{a.e. in } \Omega, \tag{4.5}$$

where $\{Y_i\}_{1\le i\le k}$ is a family of disjoint rectangles of $\mathbb{R}^n$ included in Ω and l_i^m real constants. Set

$$\begin{cases} Y_0 = \Omega \backslash \underset{1\le i\le k}{\cup} \overline{Y_i}, \\ \chi_i = \chi_{Y_i} \\ \chi_0 = \chi_{Y_0}. \end{cases}$$

We have

$$\begin{cases} \forall i \in \{1,\ldots,k\},\ |A^\varepsilon(x,v(x)) - A^\varepsilon(x,l_i^m)| \le \omega(|v(x)-l_i^m|) \quad \text{a.e. in } \Omega, \\ |A^\varepsilon(x,v(x)) - A^\varepsilon(x,0)| \le \omega(|v(x)-0|) \quad \text{a.e. in } \Omega, \end{cases} \tag{4.6}$$

and (3.2) gives

$$\begin{cases} A^\varepsilon(l_i^m, T_\varepsilon) \overset{H^0}{\rightharpoonup} A^0(l_i^m), \\ A^\varepsilon((0), T_\varepsilon) \overset{H^0}{\rightharpoonup} A^0(0). \end{cases} \tag{4.7}$$

Hence, using (4.4), (4.6), (4.7), Assumption (3.1), point (ii) of Remark 2.8 and by Theorem 2.7, we obtain

$$\begin{cases} \forall i \in \{1,\ldots,k\},\ |B^0(x) - A^0(x,l_i^m)| \le \frac{\beta}{\alpha}\,\omega(|v(x)-l_i^m|) \quad \text{a.e. in } \Omega, \\ |B^0(x) - A^0(x,0)| \le \frac{\beta}{\alpha}\,\omega(|v(x)-0|) \quad \text{a.e. } in\ \Omega, \end{cases}$$

which implies that for almost every x in Ω

$$|B^0(x) - A^0(x,v^m)| = |B^0(x) - \sum_{i=1}^{i=k} A^0(x,l_i^m)\chi_i(x) + A^0(x,0)\chi_0(x)| \\ \le \sum_{i=1}^{i=k}\chi_i(x)\frac{\beta}{\alpha}\,\omega(|v(x)-l_i^m|) + \chi_0(x)\frac{\beta}{\alpha}\,\omega(|v(x)-0|) \\ = \frac{\beta}{\alpha}\,\omega(|v(x)-v^m(x)|).$$

Moreover, thanks to (4.2), we have

$$|A^0(x,v(x)) - A^0(x,v^m(x))| \le \frac{\beta}{\alpha}\,\omega(|v(x)-v^m(x)|) \quad \text{a.e. in } \Omega.$$

Hence, from this two latter inequalities, it follows from triangle inequality that

$$|B^0(x) - A^0(x, v(x))| \leq |B^0(x) - A^0(x, v^m(x))| + |A^0(x, v^m(x)) - A^0(x, v(x))| \\ \leq 2\frac{\beta}{\alpha}\, \omega(|v(x) - v^m(x)|) \quad \text{a.e. in } \Omega.$$

Since ω is continuous at 0, passing to the limit in this inequality when $m \to \infty$, one obtains

$$B^0(x) = A^0(x, v(x)) \;\; \text{a.e. in } \Omega,$$

which, with (4.4), gives (3.3). □

Proof of Theorem 3.4 First, note that problem (3.5) (respect. (3.7) has a unique solution in $H_0^1(\Omega_\varepsilon)$ (respect. $H_0^1(\Omega)$) thanks to [6].

Second, taking u^ε as a test function in the variational formulation of (3.5), we obtain

$$\|P_\varepsilon u^\varepsilon\|_{H_0^1(\Omega)} \leq C\|u^\varepsilon\|_{H_0^1(\Omega_\varepsilon)} \leq \frac{C}{\alpha}\|f\|_{L^2(\Omega\varepsilon)}.$$

Hence, we can extract a subsequence of $\{\varepsilon\}$ (still denoted by $\{\varepsilon\}$), such that

$$P_\varepsilon u^\varepsilon \rightharpoonup u^0 \quad \text{weakly in } H_0^1(\Omega),$$

hence

$$P_\varepsilon u^\varepsilon \to u^0 \quad \text{strongly in } L^2(\Omega).$$

This implies, for every m, that

$$P_\varepsilon u^\varepsilon - v^m \to u^0 - v^m \quad \text{strongly in } L^2(\Omega),$$

where $\{v^m\}$ is a sequence of functions introduced in (4.5) such that $v^m \to u^0$ strongly in $L^1(\mathbb{R})$. So, thanks to continuity of ω, we get

$$\omega(|P_\varepsilon u^\varepsilon - v^m|) \to \omega(|u^0 - v^m|) \quad \text{strongly in } L^1(\Omega). \tag{4.8}$$

On the other hand, since $A^\varepsilon(P_\varepsilon u^\varepsilon) \doteq A^\varepsilon(\cdot, P_\varepsilon u^\varepsilon(\cdot)) \in \mathcal{M}(\alpha, \beta; \Omega)$, there exists a subsequence of $\{\varepsilon\}$ (still denoted by $\{\varepsilon\}$) and $C^0 \in \mathcal{M}(\frac{\alpha}{C^2}, \beta; \Omega)$, such that

$$(A^\varepsilon(P_\varepsilon u^\varepsilon), T_\varepsilon) \overset{H^0}{\rightharpoonup} C^0, \tag{4.9}$$

but

$$\forall \varepsilon > 0,\; |A^\varepsilon(x, P_\varepsilon u^\varepsilon(x)) - A^\varepsilon(x, v^m(x))| \leq \omega(|P_\varepsilon u^\varepsilon(x) - v^m(x)|), \text{ a.e. in } \Omega,$$

hence by this last inequality, (4.8), (4.9), Theorem 3.2, Assumption (3.1), point (ii) of Remark 2.8 and Theorem 2.7, it comes

$$|C^0(x) - A^0(x, v^m(x))| \leq \frac{\beta}{\alpha}\, \omega(|u^0(x) - v^m(x)|), \text{ a.e. in } \Omega.$$

This gives

$$|C^0(x) - A^0(x, u^0(x))| \leq |C^0(x) - A^0(x, v^m(x))| + |A^0(x, v^m(x)) - A^0(x, u^0(x))| \\ \leq 2\frac{\beta}{\alpha}\, \omega(|u^0(x) - v^m(x)|) \quad \text{a.e. in } \Omega.$$

Since $\lim\limits_{d\to 0} \omega(d) = 0$, passing to the limit in this inequality when $m \to \infty$, we obtain

$$C^0(x) = A^0(x, u^0(x)) \quad \text{a.e. in } \Omega.$$

Then, from this and (4.9), we find

$$(A^\varepsilon(P_\varepsilon u^\varepsilon), T_\varepsilon) \overset{H^0}{\rightharpoonup} A^0(\cdot, u^0(\cdot)),$$

which implies by the definition of the H^0-convergence and the uniqueness of the solutions of (3.5) and (3.7) that

$$\begin{cases} P_\varepsilon u^\varepsilon \rightharpoonup u^0 \text{ weakly in } H_0^1(\Omega), \\ A^\varepsilon \widetilde{(u^\varepsilon)} \nabla u^\varepsilon \rightharpoonup A^0(u^0)\nabla u^0 \text{ weakly in } L^2(\Omega)^n, \end{cases}$$

where u^0 is the solution of (3.7), which completes proof of (i) and (iii) of (3.6).

Finally, we deduce (3.6)ii) from the fact that

$$\chi^\varepsilon \rightharpoonup \chi^0 \quad \text{weakly} \star \text{ in } L^\infty(\Omega)$$

and

$$P_\varepsilon u^\varepsilon \to u^0 \text{ strongly in } L^2(\Omega).$$

□

5 Applications

As an application of our results, we consider the classical periodic case and a non-periodic case.

Let θ a diffeomorphism of class C^2 from $\mathbb{R}^n$ onto $\mathbb{R}^n$ and introduce the holes T_ε defined by

$$\begin{cases} T_\varepsilon = \underset{k\in\mathbb{Z}^n}{\cup} \left\{ S_\varepsilon^k \;\; s.t. \;\; S_\varepsilon^k \subset \Omega, \; \text{dist}(\theta(k\varepsilon), \partial\Omega) > 2\varepsilon \right\}, \\ \text{with } S_\varepsilon^k = \{x \in \mathbb{R}^n \;\; s.t. \;\; |x - \theta(k\varepsilon)| \le \delta\varepsilon\}, \; k \in \mathbb{Z}^n, \end{cases}$$

where $\delta \in]0, 1]$. Let $Y = [-\frac{1}{2}, \frac{1}{2}]^n$ and set

$$A^\varepsilon(x, d)) = A\left(\frac{\theta^{-1}(x)}{\varepsilon}\right),$$

with $A \in \mathcal{M}_{Equi}(\alpha, \beta, \omega; Y)$. Assume that ω is continous with $\omega(d) > 0 \; \forall d > 0$ and

$$\text{for any } r > 0, \quad \lim_{s\to 0} \int_s^r \frac{\mathrm{d}t}{\omega(t)} = +\infty.$$

In what follows, the spherical geometry of the holes can be generalized to the case where a regular boundary hole with a finite number of connected components replace a ball.

5.1 Classical periodic case

Take here $\theta = Id_{\mathbb{R}^n}$ and $\delta = \frac{1}{3}$. Then, the pair $(A^\varepsilon(\cdot, \cdot), T_\varepsilon)$ satisfies all assumptions of Theorem 3.4 and it is well-known that in this case (see [10])

$$\begin{cases} \forall d \in \mathbb{R}, \; (A^\varepsilon(\cdot, d), T_\varepsilon) \overset{H^0}{\rightharpoonup} A^0(d), \\ \chi^\varepsilon \rightharpoonup \frac{|Y^*|}{|Y|} \text{ weakly} \star \text{ in } L^\infty(\Omega), \end{cases}$$

with $A^0(d)$ is independent of x and given by

$$\forall \lambda \in \mathbb{R}^n, \quad A^0(d)\lambda = \frac{1}{|Y|} \int_{Y^\star} A(y, d) \nabla_y v_\lambda(y, d) \mathrm{d}y, \tag{5.1}$$

where

$$Y^* = Y \backslash T, \quad T = \left\{ x \in \mathbb{R}^n \;\; s.t. \;\; |x| \le \frac{1}{3} \right\},$$

and for all $\lambda \in \mathbb{R}^n$, $y \longmapsto v_\lambda(y,d)$ be the solution of

$$\begin{cases} -\text{div } (A(y,d)\nabla v_\lambda(y,d))) = 0 & \text{in } Y^\star, \\ (A(y,d)\nabla v_\lambda(y,d)\nabla u^\varepsilon \cdot \nu = 0 & \text{on } \partial T, \\ v_\lambda(y,d) - \lambda \cdot y \ \ is\ Y - \text{periodic with mean value } 0. \end{cases}$$

In this framework, we have the following result about the convergence of problem (3.5):

Proposition 5.1 *For every $f \in L^2(\Omega)$, the solution u^ε of problem (3.5) satisfies*

$$\begin{cases} (i)\ P_\varepsilon(u^\varepsilon) \rightharpoonup u^0 \quad \textit{weakly in } H_0^1(\Omega), \\ (ii)\ \widetilde{u^\varepsilon} \rightharpoonup \chi^0 u^0 \quad \textit{weakly in } L^2(\Omega), \\ (iii)\ A^\varepsilon \widetilde{(u^\varepsilon)\nabla u^\varepsilon} \rightharpoonup A^0(u^0)\nabla u^0 \quad \textit{weakly in } L^2(\Omega)^n, \end{cases}$$

where u^0 is the solution of

$$\begin{cases} -\text{div } (A^0(u^0)\nabla u^0) = \chi^0 f & \textit{in } \Omega, \\ u^0 = 0 & \textit{on } \partial\Omega. \end{cases}$$

and where A^0 defined by (5.1) belongs to $\mathcal{M}_{Equi}(\frac{\alpha}{C^2}, \beta, \frac{\beta}{\alpha}\omega; \Omega)$.

Remark 5.2 In the geometric framework of this example, Proposition 5.1 coincides with a result given in [7] by using the periodic unfolding, when the nonlinear Robin boundary condition on the holes reduces to the homogeneous Neumann condition.

5.2 Non-periodic case

Consider here the non-periodic perforated domain introduced in [5, Section 3] when studying the corresponding linear case. We suppose that θ^{-1} has a Lipschitz constant κ^{-1} with $\kappa > 2$ and take $\delta = 1$. In this case, from [5, Sections 3-4], we deduce easily that for every $d \in \mathbb{R}$, the pair $(A^\varepsilon, T_\varepsilon)$ satisfies all assumptions of Theorem 3.4 and

$$\begin{cases} \forall d \in \mathbb{R},\ (A^\varepsilon(\cdot,d), T_\varepsilon)) \overset{H^0}{\rightharpoonup} A^0(\cdot,d)\ in\ \Omega, \\ \chi^\varepsilon(\cdot) \rightharpoonup \frac{|Y^*(\cdot)|}{|Y(\cdot)|} \text{ weakly } \star \text{ in } L^\infty(\Omega), \end{cases}$$

with

$$A^0(x,d) = B_x^0(d),$$

where $B_x^0(d)$ is defined by

$$\forall \lambda \in \mathbb{R}^n, \quad B_x^0(d)\lambda = \frac{1}{|Y(x)|}\int_{Y(x)^\star} B(x,y,d)\nabla_y v_\lambda(x,y,d)\mathrm{d}y, \tag{5.2}$$

and where we have

$$\begin{cases} B(x,y,d) = A\left(\left[\nabla\theta(\theta^{-1}(x))\right]^{-1} y, d\right), \\ Y(x) = \{\nabla\theta(\theta^{-1}(x))z \ \ s.t.\ \ z \in Y\}, \\ T_1 = \{z \in \mathbb{R}^n \ \ s.t.\ \ |z| \le 1\}, \\ Y(x)^\star = Y(x) \backslash T_1 \end{cases}$$

and for all $\lambda \in \mathbb{R}^n$, $y \longmapsto v_\lambda(x,y,d)$ be the solution of

$$\begin{cases} -\text{div } (B(x,y,d)\nabla v_\lambda(x,y,d))) = 0 & \text{in } Y(x)^\star, \\ (B(x,y,d)\nabla v_\lambda(x,y,d) \cdot \nu = 0 & \text{on } \partial T_1, \\ v_\lambda(x,y,d) - \lambda \cdot y \ \ is\ Y(x) - \text{periodic with mean value } 0. \end{cases}$$

In this framework, we have the following result about the convergence of problem (3.5):

Proposition 5.3 *For every* $f \in L^2(\Omega)$*, the solution* u^ε *of problem* (3.5) *satisfies*

$$\begin{cases} (i)\ P_\varepsilon(u^\varepsilon) \rightharpoonup u^0 \quad \text{weakly in } H_0^1(\Omega), \\ (ii)\ \widetilde{u^\varepsilon} \rightharpoonup \chi^0 u^0 \quad \text{weakly in } L^2(\Omega), \\ (iii)\ A^\varepsilon \widetilde{(u^\varepsilon)\nabla u^\varepsilon} \rightharpoonup B_x^0(u^0)\nabla u^0 \quad \text{weakly in } L^2(\Omega)^n, \end{cases}$$

where u^0 *is the solution of*

$$\begin{cases} -\text{div}\,(B_x^0(u^0(x))\nabla u^0(x)) = \frac{|Y^*(x)|}{|Y(x)|} f(x) \quad \text{in } \Omega, \\ u^0(x) = 0 \quad \text{on } \partial\Omega \end{cases}$$

and where $(x, d) \mapsto B_x^0(d)$ *defined by* (5.2) *belongs to* $\mathcal{M}_{Equi}(\frac{\alpha}{C^2}, \beta, \frac{\beta}{\alpha}\omega; \Omega)$.

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