Arab. J. Math. (2021) 10:331–349
https://doi.org/10.1007/s40065-021-00310-8
Arabian Journal of Mathematics

Moussa Bzeih · **Toufic El Arwadi** · **Mohammad Hindi**

Numerical analysis and simulation for Rayleigh beam equation with dynamical boundary controls

Received: 2 August 2020 / Accepted: 4 January 2021 / Published online: 15 February 2021

Abstract In this paper, the Rayleigh beam system with two dynamical boundary controls is treated. Theoretically, the well-posedness of the weak solution is obtained. Later, we discretize the system by using the Implicit Euler scheme in time and the P^3 Hermite finite element in space. In addition, we show the decay of the discrete energy and we establish some a priori error estimates. Finally, some numerical simulations are presented.

Mathematics Subject Classification 35R37 · 35K10 · 65M60 · 65M15

1 Introduction

In [1], Wehbe considered the Rayleigh beam equation with two dynamical boundary controls and the energy decay was theoretically established. In [2], the authors consider a clamped Rayleigh beam equation subject to only one dynamical boundary feedback. First, they considered the Rayleigh beam equation subject to only one dynamical boundary control moment, and later, they considered the Rayleigh beam equation subject to only one dynamical boundary control force and established in both cases a theoretical energy decay.

In [3], Rao considered the exact controllability of a hybrid system consisting of an elastic beam, clamped at one end and attached at the other end to a rigid antenna. Such a system is governed by one partial differential equation and two ordinary differential equations. Using the HUM method, they proved that the hybrid system is exactly controllable in an arbitrarily short time in the usual energy space.

In [4], the authors established, using a multiplier method, the polynomial energy decay rate for the smooth solutions of Kirchhoff plates equations. Consequently, they obtained the strong stability in the absence of compactness of the infinitesimal operator.

In [5], Rincon and Copetti studied numerically a locally damped wave equation. Error estimates for the semi-discrete and fully discrete schemes in the energy norm were provided.

In [6], a dynamic contact problem between a viscoelastic beam and a deformable obstacle is considered. The classical Timoshenko beam model is used and the contact is modeled using the well-known normal compliance contact condition. Fully discrete approximations were introduced, A priori error estimates were proved to obtain the linear convergence of the algorithm under an additional regularity condition. A numerical analysis for Bresse system is done in [8] and [7].

M. Bzeih (✉) · T. El Arwadi · M. Hindi
Faculty of sciences, Beirut Arab University, Beirut, Lebanon
E-mail: moussabzeih81@gmail.com

T. El Arwadi
E-mail: t.elarwadi@bau.edu.lb

M. Hindi
E-mail: myh223@students.bau.edu.lb

The aim of this paper is to study numerically the equation of Rayleigh beam, which is clamped at one end and subjected to two dynamical boundary controls at the other end. At an instant time t, the position $y(x,t)$ of a given point x of the beam is governed by the following system of equations:

$$\begin{cases} y_{tt} - \gamma y_{xxtt} + y_{xxxx} &= 0, \\ y(0,t) = y_x(0,t) &= 0, \\ y_{xx}(1,t) + \eta(t) &= 0, \\ y_{xxx}(1,t) - \gamma y_{xtt}(1,t) &= \xi(t), \end{cases} \tag{1}$$

where $\gamma > 0$ is a physical constant, and ξ and η designate, respectively, the dynamical boundary force and moment controls applied at the free end of the beam. The dynamical controls ξ and η are given by the following integral system:

$$\begin{cases} \eta_t(t) - y_{xt}(1,t) + \eta(t) = 0, \\ \xi_t(t) - y_t(1,t) + \xi(t) \quad = 0, \end{cases} \tag{2}$$

where $0 < x < 1,\ t > 0$ and the following initial conditions:

$$y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x), \quad 0 < x < 1, \quad \eta(0) = \eta_0, \xi(0) = \xi_0 \in \mathbb{R}.$$

Let y be a smooth solution of the system (1)–(2). The associated energy $E(t)$ is defined by:

$$E(t) = \frac{1}{2}\Big(\int_0^1 \left(|y_t|^2 + \gamma |y_{xt}|^2 + |y_{xx}|^2\right) \mathrm{d}x + |\eta(t)|^2 + |\xi(t)|^2 \Big). \tag{3}$$

A straightforward calculations yields:

$$E'(t) = -\,|\eta(t)|^2 - |\xi(t)|^2 \leq 0. \tag{4}$$

Hence, system (1)–(2) is dissipative in the sense that the energy $E(t)$ is a decreasing function of the time.

The notion of a dynamical control has been studied for the first time by the automaticians in the finite-dimensional case (see Francis [9]). While, in the infinite-dimensional case, the notion of dynamical controls was studied by Russell [10].

Our purpose is to use the finite-element method and to obtain error estimates for the approximation of (1). Moreover, a fully discrete implicit scheme is proposed and analyzed. An outline of the contents of this paper is as follows. In Sect. 2, we show the well-posedness using the Faedo–Galerkin method (see[11]), as well as some regularity results. In Sect. 3, a semi-discrete Galerkin approximation to the solution of (1.1) is analyzed, and in Sect. 4, a fully discrete scheme is considered. We use the P^3 Hermite functions in space and the backward Euler method in time. We show that the fully discrete energy decays and derive some stability and error estimates. Finally, the results of numerical experiments illustrating the theoretical results are presented in the last section.

2 Existence and uniqueness

Denote by $|.|$ and $(.,.)$ the norm and scalar product in $L^2(0,1)$, respectively, and introduce the energy space:

$$V = H_E^2(0,1) = \{\chi \in H^2(0,1)/\chi(0) = \chi_x(0) = 0\}.$$

Denote also by

$$H_E^1(0,1) = \{\chi \in H^1(0,1)/\chi(0) = 0\}.$$

Multiply $(1)_1$ by a test function $\upsilon \in H_E^2(0,1)$, and then use Green's formula with the initial and boundary conditions to obtain the weak form of (1).

Find $y(.,t) \in H_E^2(0,1)$, $\xi(t) \in \mathbb{R}$ and $\eta(t) \in \mathbb{R}$, such that, for all $\upsilon \in H_E^2(0,1)$:

$$\begin{cases} (y_{tt}, \upsilon) + \gamma\,(y_{xtt}, \upsilon_x) + (y_{xx}, \upsilon_{xx}) + \xi \upsilon(1) + \eta \upsilon_x(1) = 0, \\ \eta_t(t) - y_{xt}(1,t) + \eta(t) = 0, \\ \xi_t(t) - y_t(1,t) + \xi(t) = 0. \end{cases} \tag{5}$$

The next theorem provides us with the existence and uniqueness of the solution of (1)–(2) using the Faedo–Galerkin method.

Theorem 2.1 *Assume that $y_0, y_1 \in H_E^2(0,1)$. Then, for each $T > 0$, there exist a unique solution to (1)–(2) under the following regularity:*

(1) $y \in L^\infty(0,T,H_E^2(0,1)\cap H^3(0,1))$,
(2) $y_t \in L^\infty(0,T,H_E^2(0,1))$,
(3) $y_{tt} \in L^\infty(0,T,H_E^1(0,1))$,
(4) $\xi, \xi_t, \eta, \eta_t \in L^\infty(0,T)\cap L^2(0,T)$,
(5) $(\gamma y_{tt} - y_{xx})_{xx} = y_{tt} \in L^\infty(0,T,H_E^1(0,1))$,
(6) $y_{tt} - \gamma y_{xxtt} + y_{xxxx} = 0\, a.e\, in\ (0,1)\times(0,T)$,
(7) $y_{xx}(1,t) + \eta(t) = 0$,
(8) $y_{xxx}(1,t) - \gamma y_{xtt}(1,t) = \xi(t)$,
(9) $y(0) = y_0,\ y_t(0) = y_1,\ \xi(0) = \xi_0,\ and\ \eta(0) = \eta_0$.

Proof Let V_m be the subspace spanned by $\{\phi_1, \phi_2, ..., \phi_m\}$, where $\{\phi_i\}_{i=1}^\infty$ is a smooth basis for V.

There exist two sequences (see [11] p. 513) (y_0^m) and (y_1^m), such that:

$$\begin{cases} \forall m,\ y_0^m \in V_m,\ \ y_0^m \to y_0 \text{ in } V, \\ \forall m,\ y_1^m \in V_m,\ \ y_1^m \to y_1 \text{ in } V. \end{cases} \tag{6}$$

Hence, we have to find $y^m(x,t) = \sum_{i=1}^m d_{\text{im}}(t)\phi_i(x)$, ξ^m and $\eta^m \in \mathbb{R}$, satisfying $\forall \upsilon \in V_m$:

$$\begin{cases} \left(y_{tt}^m, \upsilon\right) + \gamma\left(y_{xtt}^m, \upsilon_x\right) + \left(y_{xx}^m, \upsilon_{xx}\right) + \xi^m\upsilon(1) + \eta^m\upsilon_x(1) = 0, \\ \eta_t^m(t) - y_{xt}^m(1,t) + \eta^m(t) = 0, \\ \xi_t^m(t) - y_t^m(1,t) + \xi^m(t) = 0, \end{cases} \tag{7}$$

with the initial conditions $y^m(x,0) = y_0^m$, $y_t^m(x,0) = y_1^m$, $\xi^m(0) = \xi_0$, $\eta^m(0) = \eta_0$ and $\upsilon \in V_m$. This initial value problem has a local solution on $[0,T_m]\times[0,T_m]\times[0,T_m]$ by the standard ODE theory. The a priori estimates established below allow us to extend the solution to $[0,T]\times[0,T]\times[0,T]$ for any given T.

Estimate I:

Take $\upsilon = 2y_t^m$ in $(7)_1$ to get:

$$2\left(y_{tt}^m, y_t^m\right) + 2\gamma\left(y_{xtt}^m, y_{xt}^m\right) + 2\left(y_{xx}^m, y_{xxt}^m\right) + 2\xi^m y_t^m(1,t) + 2\eta^m y_{xt}^m(1,t) = 0.$$

Using $(7)_2$ and $(7)_3$, we obtain:

$$\frac{d}{dt}\left(|y_t^m|^2 + \gamma|y_{xt}^m|^2 + |y_{xx}^m|^2 + |\xi^m|^2 + |\eta^m|^2\right) + 2|\xi^m|^2 + 2|\eta^m|^2 = 0.$$

Integrating from 0 to t yields:

$$|y_t^m|^2 + \gamma|y_{xt}^m|^2 + |y_{xx}^m|^2 + |\xi^m|^2 + |\eta^m|^2 + 2\int_0^t \left(|\xi^m|^2 + |\eta^m|^2\right) ds =$$

$$|y_1^m|^2 + \gamma|y_{1x}^m|^2 + |y_{0xx}^m|^2 + |\xi_0^m|^2 + |\eta_0^m|^2 \le$$

$$C\left(|y_1|^2 + \gamma|y_{1x}|^2 + |y_{0xx}|^2 + |\xi_0|^2 + |\eta_0|^2\right).$$

The assumption on the initial data gives the following:
y^m is bounded in $L^\infty(0,T,H_E^2(0,1))$,
y_t^m is bounded in $L^\infty(0,T,H_E^1(0,1))$,
ξ^m and η^m are bounded in $L^\infty(0,T)\cap L^2(0,T)$,
and the solution can be extended to $[0,T]\times[0,T]\times[0,T]$. Also, there exist three sub-sequences still denoted by (y^m), (ξ^m), (η^m), such that

$y^m \longrightarrow y$ weak-star in $L^\infty(0,T,H_E^2(0,1))$,
$y_t^m \longrightarrow y_t$ weak-star in $L^\infty(0,T,H_E^1(0,1))$,
$\xi^m \longrightarrow \xi$ weak-star in $L^\infty(0,T)$,
$\xi^m \longrightarrow \xi$ weak in $L^2(0,T)$,
$\eta^m \longrightarrow \eta$ weak-star in $L^\infty(0,T)$, and
$\eta^m \longrightarrow \eta$ weak in $L^2(0,T)$.

Estimate II:
Differentiating (7) with respect to t and let $\hat{y}^m = y_t^m$, we get:

$$\begin{cases} \left(\hat{y}_{tt}^m, \upsilon\right) + \gamma\left(\hat{y}_{xtt}^m, \upsilon_x\right) + \left(\hat{y}_{xx}^m, \upsilon_{xx}\right) + \xi_t^m \upsilon(1) + \eta_t^m \upsilon_x(1) = 0, \\ \eta_{tt}^m(t) - \hat{y}_{xt}^m(1,t) + \eta_t^m(t) = 0, \\ \xi_{tt}^m(t) - \hat{y}_t^m(1,t) + \xi_t^m(t) = 0. \end{cases} \tag{8}$$

Take $\upsilon = 2\hat{y}_t^m$ in $(8)_1$:

$$2\left(\hat{y}_{tt}^m, \hat{y}_t^m\right) + 2\gamma\left(\hat{y}_{xtt}^m, \hat{y}_{xt}^m\right) + 2\left(\hat{y}_{xx}^m, \hat{y}_{xxt}^m\right) + 2\xi_t^m \hat{y}_t^m(1,t) + 2\eta_t^m \hat{y}_{xt}^m(1,t) = 0;$$

use $(8)_2$ and $(8)_3$ to obtain:

$$\frac{d}{\mathrm{d}t}\left(|\hat{y}_t^m|^2 + \gamma|\hat{y}_{xt}^m|^2 + |\hat{y}_{xx}^m|^2 + |\xi_t^m|^2 + |\eta_t^m|^2\right) + 2|\xi_t^m|^2 + 2|\eta_t^m|^2 = 0.$$

Integrating from 0 to t, we get:

$$\begin{aligned} &|\hat{y}_t^m|^2 + \gamma|\hat{y}_{xt}^m|^2 + |\hat{y}_{xx}^m|^2 + |\xi_t^m|^2 + |\eta_t^m|^2 + 2\int_0^t \left(|\xi_t^m|^2 + |\eta_t^m|^2\right) ds = \\ &|y_{tt}^m(0)|^2 + \gamma|y_{xtt}^m(0)|^2 + |y_{xx}^m(0)|^2 + |\xi_t^m(0)|^2 + |\eta_t^m(0)|^2. \end{aligned} \tag{9}$$

We have to bound $|y_{tt}^m(0)|$, $|y_{xtt}^m(0)|$, $|\xi_t^m(0)|$, and $|\eta_t^m(0)|$.

Use after integrating by parts $(7)_1$) with $\upsilon = y_{tt}^m(0)$ and $t = 0$, we get:

$$|y_{tt}^m(0)|^2 + \gamma|y_{xtt}^m(0)|^2 = -(y_{xx}^m(0), y_{tt}^m(0)) \leq \frac{1}{2}|y_{xx}^m(0)|^2 + \frac{1}{2}|y_{tt}^m(0)|^2;$$

hence:

$$|y_{tt}^m(0)|^2 \leq C \text{ and } |y_{xtt}^m(0)|^2 \leq C.$$

Now, use $(7)_2$ for $t = 0$ to obtain:
$\eta_t^m(0) = y_{xt}^m(1,0) - \eta^m(0) = y_{1x}^m(1) - \eta_0$.
Note that

$$y_{1x}^m(1) = \int_0^1 y_{1xx}^m(x)\mathrm{d}x \leq C|y_{1xx}^m| \leq C|y_{1xx}|,$$

and the regularity on the initial data gives:

$$|\eta_t^m(0)| \leq C|y_{1xx}| + |\eta_0| \leq C.$$

In the same way, we get:

$$|\xi_t^m(0) \leq C|y_{1x}| + |\xi_0| \leq C.$$

Combine these results with (9) to get:

$$\begin{aligned} &|\hat{y}_t^m|^2 + \gamma|\hat{y}_{xt}^m|^2 + |\hat{y}_{xx}^m|^2 + |\xi_t^m|^2 + |\eta_t^m|^2 + 2\int_0^t \left(|\xi_t^m|^2 + |\eta_t^m|^2\right) ds = \\ &|y_{tt}^m(0)|^2 + \gamma|y_{xtt}^m(0)|^2 + |y_{xx}^m(0)|^2 + |\xi_t^m(0)|^2 + |\eta_t^m(0)|^2; \end{aligned}$$

therefore:
$\hat{y}^m$ is bounded in $L^\infty(0, T, H_E^2(0,1))$,
$\hat{y}_t^m$ is bounded in $L^\infty(0, T, H_E^1(0,1))$, and
ξ_t^m and η_t^m are bounded in $L^\infty(0,T) \cap L^2(0,T)$.
We conclude that
$y_t^m \longrightarrow y_t$ weak star in $L^\infty(0, T, H_E^2(0,1))$,
$y_{tt}^m \longrightarrow y_{tt}$ weak star in $L^\infty(0, T, H_E^1(0,1))$,

$\xi_t^m \longrightarrow \xi_t$ weak star in $L^\infty(0,T)$,
$\xi_t^m \longrightarrow \xi_t$ weak in $L^2(0,T)$,
$\eta_t^m \longrightarrow \eta_t$ weak star in $L^\infty(0,T)$, and
$\eta_t^m \longrightarrow \eta_t$ weak in $L^2(0,T)$.
Using the convergence results, we can pass to the limit $m \to +\infty$ in (7) to obtain:

$$\begin{cases} (y_{tt},\upsilon)+\gamma\,(y_{xtt},\upsilon_x)+(y_{xx},\upsilon_{xx})+\xi\upsilon(1)+\eta\upsilon_x(1)=0,\\ \eta_t(t)-y_{xt}(1,t)+\eta(t)=0,\\ \xi_t(t)-y_t(1,t)+\xi(t)=0. \end{cases} \tag{10}$$

Using $(10)_1$ for $\upsilon\in D(0,1)$, we get:

$$(\gamma y_{tt}-y_{xx},\upsilon_{xx})=(y_{tt},\upsilon),$$

and since, $y_{tt}\in L^\infty(0,T,H_E^1(0,1))$, we get:
$\gamma y_{tt}-y_{xx}\in H^3(0,1)$, a.e $t\in[0,T]$, and:

$$y_{tt}-\gamma y_{xxtt}+y_{xxxx}=0 \quad a.e \quad in \quad (0,1)\times(0,T).$$

However, $y_{tt}(.,t)\in H^1(0,1)$, so
$y_{xx}(.,t)\in H^1(0,1)$; that is, $y(.,t)\in H^3(0,1)$.
Using Greens formula to Eq. $(10)_1$, we get $\forall\upsilon\in V$:

$$(y_{tt},\upsilon)-\gamma\,(y_{xxtt},\upsilon)+(y_{xxxx},\upsilon)+$$

$$(\gamma y_{xtt}(1,t)-y_{xxx}(1,t)+\xi(t))\,\upsilon(1)+(y_{xx}(1,t)+\eta)\,\upsilon_x(1)=0;$$

hence:

$$\big[\gamma y_{xtt}(1,t)-y_{xxx}(1,t)+\xi(t)\big]\upsilon(1)+\big[y_{xx}(1,t)+\eta\big]\upsilon_x(1)=0, \tag{11}$$

use (11) with $\upsilon\in V$ such that $\upsilon(1)=0,\ \upsilon_x(1)\neq 0$, we get:

$$y_{xx}(1,t)+\eta=0,$$

use (11) again with $\upsilon\in V$, such that $\upsilon(1)\neq 0,\ \upsilon_x(1)=0$, we get:

$$y_{xxx}(1,t)-\gamma y_{xtt}(1,t)=\xi(t),$$

and so, (y,ξ,η) verify the boundary conditions.

Uniqueness:
let (y_1,ξ_1,η_1) and (y_2,ξ_2,η_2) be two solutions and take
$w=y_1-y_2,\ \sigma=\xi_1-\xi_2$ and $\lambda=\eta_1-\eta_2$, and then, we have:

$$\begin{cases} (w_{tt},\upsilon)+\gamma\,(w_{xtt},\upsilon_x)+(w_{xx},\upsilon_{xx})+\sigma\upsilon(1)+\lambda\upsilon_x(1)=0,\\ \lambda_t(t)-w_{xt}(1,t)+\lambda(t)=0,\\ \sigma_t(t)-w_t(1,t)+\sigma(t)=0. \end{cases}$$

Take $\upsilon=2w_t$, and then integrate from 0 to t to obtain:

$$|w_t|^2+\gamma|w_{xt}|^2+|w_{xx}|^2+|\sigma|^2+|\lambda|^2+2\int_0^t(|\sigma|^2+|\lambda|^2)ds=$$

$$|w_t(0)|^2+\gamma|w_{xt}(0)|^2+|w_{xx}(0)|^2+|\sigma(0)|^2+|\lambda(0)|^2=0.$$

Therefore, $w_t=w_{xt}=w_{xx}=0$; therefore, $w=\sigma=\lambda=0$. □

3 Semi-discrete approximation

Let $0 = x_0 < x_1 < \cdots < x_{s+1} = 1$ be a uniform partition of the interval $I = (0, 1)$ into sub-intervals $I_j = (x_{j-1}, x_j)$, $j = 1, ..., s+1$ of length $h = \frac{1}{s+1}$, and we denote by $V_E^h \subset H_E^2(I)$ the space:

$$V_E^h = \{\chi \in H_E^2(I) \cap C^1([0,1]) \text{ such that } \chi_{|[x_j, x_{j+1}]} \in \mathbb{P}_3 \text{ for all } 0 \le j \le s+1\},$$

and by π_E^h, the elliptic projection $\pi_E^h : H_E^2(I) \longrightarrow V_E^h$, defined by:

$$(\pi_E^h \psi)(x_i) = \psi(x_i) \text{ and } (\pi_E^h \psi)_x(x_i) = \psi_x(x_i) \ \forall \psi \in H_E^2(I),$$

and satisfying the following estimate (see [12]):

$$|\psi - \pi_E^h \psi|_{H^1(I)} \le Ch|\psi_{xx}|, \quad \forall \psi \in H_E^2(I). \tag{12}$$

Moreover, for all $\psi \in H_E^2(I) \cap H^3(I)$, we have:

$$|\psi_{xx} - (\pi_E^h \psi)_{xx}| \le Ch|\psi_{xxx}|. \tag{13}$$

We admit the following lemma:

Lemma 3.1 *For all $\psi \in H_E^2(I)$ and $\chi \in V_E^h$, we have:*

$$((\pi_E^h \psi)_{xx}, \chi_{xx}) = (\psi_{xx}, \chi_{xx}).$$

The semi-discrete finite element to (5) is to find for all $0 \le t \le T$, $y^h(t) \in V_E^h$ and $\xi^h(t), \eta^h(t) \in \mathbb{R}$, such that

$$\begin{cases} \left(y_{tt}^h, W\right) + \gamma\left(y_{xtt}^h, W_x\right) + \left(y_{xx}^h, W_{xx}\right) + \xi^h W(1) + \eta^h W_x(1) = 0, \\ \eta_t^h(t) - y_{xt}^h(1,t) + \eta^h(t) = 0, \\ \xi_t^h(t) - y_t^h(1,t) + \xi^h(t) = 0. \end{cases} \tag{14}$$

$\forall W \in V_E^h$, with $y^h(0) = \pi_E^h y_0$, $y_t^h(0) = \pi_E^h y_1$, $\xi^h(0) = \xi_0$, and $\eta^h(0) = \eta_0$.

Lemma 3.2 $|(\pi_E^h y_i)_{xx}|^2 = |y_{ixx}|^2$, $|(\pi_E^h y_i)_x|^2 \le C|y_{ixx}|^2$, *and* $|\pi_E^h y_i|^2 \le C|y_{ixx}|^2$, *for* $i = 0, 1$.

Proof $\forall\ \chi \in V_E^h$, we have by lemma (3.1):

$$((\pi_E^h y_i)_{xx}, \chi_{xx}) = (y_{ixx}, \chi_{xx}),$$

choose $\chi = \pi_E^h y_i \in V_E^h$, and use lemma (3.1) again, we get:

$$|(\pi_E^h y_i)_{xx}|^2 = (y_{ixx}, (\pi_E^h y_i)_{xx}) = (y_{ixx}, y_{ixx}) = |y_{ixx}|^2.$$

Now:
$|(\pi_E^h y_i)_x|^2 = |y_{ix} - (\pi_E^h y_i)_x + y_{ix}|^2 \le 2|y_{ix} - (\pi_E^h y_i)_x|^2 + 2|y_{ix}|^2.$
Using (12) and the Poincare inequality, we get:
$|(\pi_E^h y_i)_x|^2 \le C|y_{ixx}|^2,$
Poincare inequality again yields:
$|\pi_E^h y_i|^2 \le C|(\pi_E^h y_i)_x|^2 \le C|y_{ixx}|^2.$ □

Define the semi-discrete energy by:

$$E^h(t) = \frac{1}{2}\left(\int_0^1 (|y_t^h|^2 + \gamma|y_{xt}^h|^2 + |y_{xx}^h|^2)\mathrm{d}x + |\xi^h|^2 + |\eta^h|^2\right).$$

To show that the semi-discrete energy decays, we choose $W = 2y_t^h$, as a test function in (14), and hence:

$$2\left(y_{tt}^h, y_t^h\right) + 2\gamma\left(y_{xtt}^h, y_{xt}^h\right) + 2\left(y_{xx}^h, y_{xxt}^h\right) + 2\xi^h y_t^h(1,t) + 2\eta^h y_{xt}^h(1,t) = 0.$$

Using $(14)_2$ and $(14)_3$ to obtain:

$$\frac{d}{dt}\left(|y_t^h|^2+\gamma|y_{xt}^h|^2+|y_{xx}^h|^2+|\xi^h|^2+|\eta^h|^2\right)+2|\xi^h|^2+2|\eta^h|^2=0.$$

Therefore, we have:

$$\frac{d}{dt}E^h(t)\leq 0.$$

Finally, integrating from 0 to t to get stability:

$$|y_t^h|^2+\gamma|y_{xt}^h|^2+|y_{xx}^h|^2+|\xi^h|^2+|\eta^h|^2+2\int_0^t\left(|\xi^h|^2+|\eta^h|^2\right)ds=$$

$$|y_t^h(0)|^2+\gamma|y_{xt}^h(0)|^2+|y_{xx}^h(0)|^2+|\xi^h(0)|^2+|\eta^h(0)|\leq$$

$$C\left(|\pi_E^h y_1|^2+\gamma|(\pi_E^h y_1)_x|^2+|(\pi_E^h y_0)_{xx}|^2+|\xi_0|^2+|\eta_0|\right)\leq C.$$

Where we used lemma (3.2) and the regularity on the initial data, and hence:

$$E^h(t)\leq C.$$

The next theorem provides us with an error bounds for the piecewise linear approximation (14).

Theorem 3.3 *Under the assumption of Theorem* (2.1)*, the estimate*

$$|y_t-y_t^h|^2+\gamma|y_{xt}-y_{xt}^h|^2+|y_{xx}-y_{xx}^h|^2+|\xi-\xi^h|^2+|\eta-\eta^h|^2\leq Ch^2,$$

holds for all $0\leq t\leq T$.

Proof Introducing $\hat{y}=y_t$ and $\hat{y}^h=y_t^h$, hence the continuous (5) and the semi-discrete (14) problems can be written as:

$$\begin{cases}(\hat{y}_t,\upsilon)+\gamma(\hat{y}_{xt},\upsilon_x)+(y_{xx},\upsilon_{xx})+\xi\upsilon(1)+\eta\upsilon_x(1)=0,\\ \eta_t(t)-y_{xt}(1,t)+\eta(t)=0,\\ \xi_t(t)-y_t(1,t)+\xi(t)=0,\end{cases}$$

and

$$\begin{cases}(\hat{y}_t^h,W)+\gamma(\hat{y}_{xt}^h,W_x)+(y_{xx}^h,W_{xx})+\xi^h W(1)+\eta^h W_x(1)=0,\\ \eta_t^h(t)-y_{xt}^h(1,t)+\eta^h(t)=0,\\ \xi_t^h-(t)y_t^h(1,t)+\xi^h(t)=0,\end{cases}$$

for all $W\in V_E^h$ and $\upsilon\in H_E^2(I)$.
In particular, the first equality is true $\forall W\in V_E^h$.
Take $\upsilon=W\in V_E^h$ and subtract the two equalities to get:

$$(\hat{y}_t-\hat{y}_t^h,W)+\gamma(\hat{y}_{xt}-\hat{y}_{xt}^h,W_x)+(y_{xx}-y_{xx}^h,W_{xx})+(\xi-\xi^h)W(1)+(\eta-\eta^h)W_x(1)=0;$$

that is:

$$\begin{cases}(\hat{y}_t-\pi_E^h\hat{y}_t+\pi_E^h\hat{y}_t-\hat{y}_t^h,W)+\gamma(\hat{y}_{xt}-\pi_E^h\hat{y}_{xt}+\pi_E^h\hat{y}_{xt}-\hat{y}_{xt}^h,W_x)+\\ (y_{xx}-\pi_E^h y_{xx}+\pi_E^h y_{xx}-y_{xx}^h,W_{xx})+(\xi-\xi^h)W(1)+(\eta-\eta^h)W_x(1)=0.\end{cases}$$

Let $\hat{e}^h=\hat{y}^h-\pi_E^h\hat{y}$ and $e^h=y^h-\pi_E^h y$, and we get:

$$\begin{cases}(\hat{e}_t^h,W)+\gamma(\hat{e}_{xt}^h,W_x)+(e_{xx}^h,W_{xx})-(\xi-\xi^h)W(1)-(\eta-\eta^h)W_x(1)=\\ (\hat{y}_t-\pi_E^h\hat{y}_t,W)+\gamma(\hat{y}_{xt}-\pi_E^h\hat{y}_{xt},W_x)+(y_{xx}-\pi_E^h y_{xx},W_{xx}).\end{cases}$$

Take $W = \hat{e}^h \in V_E^h$ and use (13), to obtain:
$(\pi_E^h \hat{y})(1) = (\pi_E^h \hat{y})(x_s) = y_t(x_s) = y_t(1) = \xi_t + \xi$,
$(\pi_E^h \hat{y})_x(1) = (\pi_E^h \hat{y}_x)(x_s) = y_{xt}(x_s) = y_{xt}(1) = \eta_t + \eta$, and
hence $W(1) = \hat{e}^h(1) = \hat{y}^h(1) - (\pi_E^h \hat{y})(1) = (\xi_t^h - \xi_t) + (\xi^h - \xi)$,
also, $W_x(1) = (\eta_t^h - \eta_t) + (\eta^h - \eta)$;
therefore:

$$\begin{cases} \dfrac{1}{2}\dfrac{d}{\mathrm{d}t}\left[|\hat{e}^h|^2 + \gamma|\hat{e}_x^h|^2 + |e_{xx}^h|^2 + |\xi - \xi^h|^2 + |\eta - \eta^h|^2\right] + |\xi - \xi^h|^2 + |\eta - \eta^h|^2 = \\ (\hat{y}_t - \pi_E^h \hat{y}_t, \hat{e}^h) + \gamma(\hat{y}_{xt} - \pi_E^h \hat{y}_{xt}, \hat{e}_x^h) + (y_{xx} - \pi_E^h y_{xx}, \hat{e}_{xx}^h). \end{cases} \tag{15}$$

Using (12), we have:
$(\hat{y}_t - \pi_E^h \hat{y}_t, \hat{e}^h) \le 2|\hat{y}_t - \pi_E^h \hat{y}_t|^2 + 2|\hat{e}^h|^2 \le 2Ch^2|\hat{y}_{tx}|^2 + 2|\hat{e}^h|^2$,
and $(\hat{y}_{xt} - \pi_E^h \hat{y}_{xt}, \hat{e}_x^h) \le 2Ch^2|\hat{y}_{xxt}|^2 + 2|\hat{e}_x^h|^2$.
Lemma (3.1) yields:
$(y_{xx} - \pi_E^h y_{xx}, \hat{e}_{xx}^h) = 0$.
Combining these results with (15) to obtain:

$$\begin{cases} \dfrac{1}{2}\dfrac{d}{\mathrm{d}t}\left[|\hat{e}^h|^2 + \gamma|\hat{e}_x^h|^2 + |e_{xx}^h|^2 + |\xi - \xi^h|^2 + |\eta - \eta^h|^2\right] + \\ |\xi - \xi^h|^2 + |\eta - \eta^h|^2 \le 2Ch^2|\hat{y}_{tx}|^2 + 2C\gamma h^2|\hat{y}_{xxt}|^2 + 2\gamma|\hat{e}_x^h|^2 + 2|\hat{e}^h|^2. \end{cases}$$

Integrating from 0 to t, we find:

$$\begin{cases} |\hat{e}^h|^2 + \gamma|\hat{e}_x^h|^2 + |e_{xx}^h|^2 + |\xi - \xi^h|^2 + |\eta - \eta^h|^2 + \displaystyle\int_0^t |\xi - \xi^h|^2 + |\eta - \eta^h|^2)ds \le \\ 2Ch^2 \displaystyle\int_0^t \left(|\hat{y}_{tx}|^2 + \gamma|\hat{y}_{xxt}|^2\right)ds + 2\int_0^t \left(|\hat{e}^h|^2 + \gamma|\hat{e}_x^h|^2\right)ds + \\ |\hat{e}^h(0)|^2 + \gamma|\hat{e}_x^h(0)|^2 + |e_{xx}^h(0)|^2 + |\xi(0) - \xi^h(0)|^2 + (|\eta(0) - \eta^h(0)|^2. \end{cases}$$

Use (12) to obtain:
$\left|\hat{e}^h(0)\right| = |\hat{y}^h(0) - \pi_E^h \hat{y}(0)| = |\pi_E^h y_1 - \hat{y}(0)| = |\pi_E^h y_1 - y_1| \le Ch|y_{1xx}|$,
and
$|\hat{e}_x^h(0)| \le Ch|y_{1xx}|$.
By (13), we have:
$|e_{xx}^h(0)| = |y_{xx}^h(0) - \pi_E^h y_{xx}(0)| = |y_{0xx} - \pi_E^h y_{0xx}| \le Ch|y_{0xxx}|$.
Use the above estimates with the fact that:

$$\xi(0) - \xi^h(0) = \eta(0) - \eta^h(0) = 0,$$

we get:

$$\begin{cases} |\hat{e}^h|^2 + \gamma|\hat{e}_x^h|^2 + |e_{xx}^h|^2 + |\xi - \xi^h|^2 + |\eta - \eta^h|^2 + \\ \displaystyle\int_0^t |\xi - \xi^h|^2 + |\eta - \eta^h|^2)ds \le \\ C\gamma h^2|y_{1xx}|^2 + Ch^2|y_{0xxx}|^2 + 2Ch^2 \displaystyle\int_0^t \left(|\hat{y}_{tx}|^2 + \gamma|\hat{y}_{xxt}|^2\right)ds \\ +2\displaystyle\int_0^t \left(|\hat{e}^h|^2 + \gamma|\hat{e}_x^h|^2\right)ds. \end{cases}$$

The regularity on y_0, y_1, and y, gives:

$$
\begin{aligned}
&|\hat{e}^h|^2+\gamma|\hat{e}^h_x|^2+|e^h_{xx}|^2+|\xi-\xi^h|^2+|\eta-\eta^h|^2\le\\
&\quad Ch^2+2\int_0^t\Big(|\hat{e}^h|^2+\gamma|\hat{e}^h_x|^2\Big)ds.
\end{aligned}
$$

Applying Gronwall inequality, to obtain:

$$
|\hat{e}^h|^2+\gamma|\hat{e}^h_x|^2\le Ch^2+Ch^2\int_0^t e^{C(t-s)}ds\le Ch^2.
$$

Therefore:

$$|\hat{e}^h|^2+\gamma|\hat{e}^h_x|^2+|e^h_{xx}|^2+|\xi-\xi^h|^2+|\eta-\eta^h|^2\le Ch^2.$$

Finally:

$$
\begin{cases}
|y_t-y^h_t|^2+\gamma|y_{xt}-y^h_{xt}|^2+|y_{xx}-y^h_{xx}|^2+|\xi-\xi^h|^2+|\eta-\eta^h|^2=\\
|y_t-\pi^h_E y_t+\pi^h_E y_t-y^h_t|^2+|y_{xt}-\pi^h_E y_{xt}+\pi^h_E y_{xt}-y^h_{xt}|^2+\\
|y_{xx}-\pi^h_E y_{xx}+\pi^h_E y_{xx}-y^h_{xx}|^2+|\xi-\xi^h|^2+|\eta-\eta^h|^2\le\\
C(|\hat{e}^h|^2+\gamma|e^h_{xt}|^2+|\xi-\xi^h|^2+|\eta-\eta^h|^2)+\\
C(|y_t-\pi^h_E y_t|^2+|y_{xt}-\pi^h_E y_{xt}|^2+|y_{xx}-\pi^h_E y_{xx}|^2)\le\\
Ch^2(|y_{xxt}|^2+|y_{xxt}|^2+|y_{xxx}|^2)+Ch^2\le Ch^2,
\end{cases}
$$

where we used the regularity on y. □

4 Fully discrete approximation

In this section, we introduce a fully discrete finite-element method to (11).
Given an integer $N>0$, our numerical scheme can be stated as find $y^n, n=2,...,N$ and $\xi^n,\eta^n, n=1,...,N$, such that, $\forall W\in V^h_E$, we have:

$$
\begin{cases}
\dfrac{1}{(\Delta t)^2}(y^{n+1}-2y^n+y^{n-1},W)+\dfrac{\gamma}{(\Delta t)^2}(y^{n+1}_x-2y^n_x+y^{n-1}_x,W_x)+\\
(y^{n+1}_{xx},W_{xx})+\xi^{n+1}W(1)+\eta^{n+1}W_x(1)=0,\\
\frac{1}{\Delta t}(\eta^{n+1}-\eta^n)-\frac{1}{\Delta t}(y^{n+1}_x(1)-y^n_x(1))+\eta^{n+1}=0,\\
\frac{1}{\Delta t}(\xi^{n+1}-\xi^n)-\frac{1}{\Delta t}(y^{n+1}(1)-y^n(1))+\xi^{n+1}=0,
\end{cases}
\tag{16}
$$

where $\Delta t=T/N$ is the time step, $y^0=\pi^h_E y_0$, $y^1=y^0+\Delta t\pi^h_E y_1$, $\xi^0=\xi_0$, and $\eta^0=\eta_0$.

Writing $y^n(x)=\sum_{i=1}^{2s+2}c^n_i\mu_i(x)$, where $\{\mu_i\}_{i=1}^{2s+2}$ is a $\mathbb{P}^3$ Hermite basis for V^h_E, which will be defined in Sect. 5.
We find that the method defined requires the linear system of $(2s+4)$ algebraic equations which can be written in the matrix form as:

$$
MC^{n+1}=KC^n+LC^{n-1},\quad \text{for}\quad n=1,2,\dots N. \tag{17}
$$

Here:

$$
\begin{aligned}
M&=\Big[A+\gamma B+(\Delta t)^2D+(\Delta t)^2E+F+G\Big],\\
K&=\Big[2A+2\gamma B+F'+G'\Big],\\
L&=\Big[-A-\gamma B\Big],
\end{aligned}
$$

with

$$
\left\{
\begin{array}{l}
A_{ij} = (\mu_i, \mu_j), A_{(2s+4)j} = A_{i(2s+4)} = A_{(2s+3)j} = A_{i(2s+3)} = 0, \quad \forall\, 1 \le i, j \le 2s+4, \\
B_{ij} = (\mu_{ix}, \mu_{jx}), B_{i(2s+4)} = B_{j(2s+4)} = B_{i(2s+3)} = B_{j(2s+3)} = 0, \quad \forall\, 1 \le i, j \le 2s+4, \\
D_{ij} = (\mu_{ixx}, \mu_{jxx}), D_{i(2s+4)} = D_{j(2s+4)} = D_{i(2s+3)} = D_{j(2s+3)} = 0, \quad \forall\, 1 \le i, j \le 2s+4, \\
E_{ij} = 0, \quad \forall\, 1 \le i, j \le 2s+4, \text{ except } E_{s+1,2s+4} = E_{2s+2,2s+3} = 1, \\
F_{ij} = 0, \quad \forall\, 1 \le i, j \le 2s+4, \text{ except } F_{2s+3,2s+2} = -1, F_{2s+3,2s+3} = 1 + \Delta t, \\
F'_{ij} = 0, \quad \forall\, 1 \le i, j \le 2s+4, \text{ except } F'_{2s+3,2s+2} = -1\; F'_{2s+3,2s+3} = 1, \\
G_{ij} = 0, \quad \forall\, 1 \le i, j \le 2s+4, \text{ except } G_{2s+4,s+1} = -1,\; G_{2s+4,2s+4} = 1 + \Delta t, \\
G'_{ij} = 0, \quad \forall\, 1 \le i, j \le 2s+4, \text{ except } G'_{2s+4,s+1} = -1,\; G'_{2s+4,2s+4} = 1.
\end{array}
\right.
\tag{18}
$$

And $C^n = (c_1^n, c_2^n, ..., c_{2s+2}^n, \eta^n, \xi^n)^T$, is the vector to be determined at each time step. Since the matrix M is non-singular, the system has a unique solution.

In a similar manner to the continuous case, the decay of the energy associated with the fully discrete problem and stability estimates are proved.
Let

$$
\hat{y}^{n+1} = (y^{n+1} - y^n)/\Delta t, \hat{\xi}^{n+1} = (\xi^{n+1} - \xi^n)/\Delta t \quad \text{and} \quad \hat{\eta}^{n+1} = (\eta^{n+1} - \eta^n)/\Delta t
$$

define the energy of the fully discrete problem by:

$$
E^{n+1} = \frac{1}{2}\big(|\hat{y}^{n+1}|^2 + \gamma|\hat{y}_x^{n+1}|^2 + |y_{xx}^{n+1}|^2 + |\xi^{n+1}|^2 + |\eta^{n+1}|^2\big), \qquad n = 0, 1, \cdots, (N-1). \tag{19}
$$

Taking, $W = \hat{y}^{n+1}$ in $(16)_1$, we get:

$$
\left\{
\begin{array}{l}
\dfrac{1}{\Delta t}(\hat{y}^{n+1} - \hat{y}^n, \hat{y}^{n+1}) + \dfrac{\gamma}{\Delta t}(\hat{y}_x^{n+1} - \hat{y}_x^n, \hat{y}_x^{n+1}) + \\
(y_{xx}^{n+1}, \hat{y}_{xx}^{n+1}) + \xi^{n+1}\hat{y}^{n+1}(1) + \eta^{n+1}\hat{y}_x^{n+1}(1) = 0.
\end{array}
\right.
$$

Using $(16)_2$ and $(16)_3$, we get:
$\hat{y}_x^{n+1}(1) = \hat{\eta}^{n+1} + \eta^{n+1}$, and
$\hat{y}^{n+1}(1) = \hat{\xi}^{n+1} + \xi^{n+1}$:

$$
\left\{
\begin{array}{l}
\dfrac{1}{\Delta t}(\hat{y}^{n+1} - \hat{y}^n, \hat{y}^{n+1}) + \dfrac{\gamma}{\Delta t}(\hat{y}_x^{n+1} - \hat{y}_x^n, \hat{y}_x^{n+1}) + (y_{xx}^{n+1}, \hat{y}_{xx}^{n+1}) + \\
|\xi^{n+1}|^2 + \xi^{n+1}\hat{\xi}^{n+1} + |\eta^{n+1}|^2 + \eta^{n+1}\hat{\eta}^{n+1} = 0.
\end{array}
\right.
$$

Using the elementary equality:

$$
(a, b) = \frac{1}{2}(|a|^2 + |b|^2 - |a-b|^2), \quad \forall a, b \in \mathbb{R}. \tag{20}
$$

We get:

$$
\left\{
\begin{array}{l}
\dfrac{1}{2\Delta t}\left[|\hat{y}^{n+1} - \hat{y}^n|^2 + |\hat{y}^{n+1}|^2 - |\hat{y}^n|^2\right] \\
+\dfrac{\gamma}{2\Delta t}\left[|\hat{y}_x^{n+1} - \hat{y}_x^n|^2 + |\hat{y}_x^{n+1}|^2 - |\hat{y}_x^n|^2\right] \\
+\dfrac{1}{2\Delta t}\left[|y_{xx}^{n+1} - y_{xx}^n|^2 + |y_{xx}^{n+1}|^2 - |y_{xx}^n|^2\right] \\
+\dfrac{1}{2\Delta t}\left[|\eta^{n+1} - \eta^n|^2 + |\eta^{n+1}|^2 - |\eta^n|^2\right] \\
+\dfrac{1}{2\Delta t}\left[|\xi^{n+1} - \xi^n|^2 + |\xi^{n+1}|^2 - |\xi^n|^2\right] \\
+|\xi^{n+1}|^2 + |\eta^{n+1}|^2 = 0.
\end{array}
\right.
\tag{21}
$$

Therefore:

$$
\big(E^{n+1} - E^n\big)/\Delta t \le 0.
$$

For the stability, we multiply (21) by $2\Delta t$ and sum from $i = 1$ to n, to get:

$$\begin{cases} \sum_{i=1}^{n}(|\hat{y}^{i+1} - \hat{y}^{i}|^2) + |\hat{y}^{n+1}|^2 + \gamma \sum_{i=1}^{n}(|\hat{y}_x^{i+1} - \hat{y}_x^{i}|^2) + \gamma|\hat{y}_x^{n+1}|^2 + \\ \sum_{i=1}^{n}(|y_{xx}^{i+1} - y_{xx}^{i}|^2) + |y_{xx}^{n+1}|^2 + \sum_{i=1}^{n}(|\eta^{i+1} - \eta^{i}|^2) + |\eta^{n+1}|^2 + \\ \sum_{i=1}^{n}(|\xi^{i+1} - \xi^{i}|^2) + |\xi^{n+1}|^2 + \sum_{i=1}^{n}(|\eta^{i+1}|^2) + \sum_{i=1}^{n}(|\xi^{i+1}|^2) = \\ |\hat{y}^1|^2 + \gamma|\hat{y}_x^1|^2 + |y_{xx}^1|^2 + |\eta^1|^2 + |\xi^1|^2. \end{cases}$$

Use lemma (3.2) and the definitions of y^1 and $\hat{y}^1$, to obtain:

$$|\hat{y}^1| = |\pi_E^h y_1| \le C|y_{1xx}| \le C,$$

and

$$|\hat{y}_x^1| = |(\pi_E^h y_1)_x| \le C|y_{1xx}| \le C.$$

Using lemma (3.2), we get:

$|y_{xx}^1| = |(\pi_E^h y_0)_{xx} + \Delta t(\pi_E^h y_1)_{xx}| \le |(\pi_E^h y_0)_{xx}| + |\Delta t(\pi_E^h y_1)_{xx}| = |y_{0xx} + \Delta t y_{1xx}| \le C.$

To bound η^1, we use $(16)_2$ for $n = 0$, the second part of (13) with $x_s = 1$, the definition of y^1, and the fact that $y_1(1) \le |y_{1x}|$.

Therefore, we have:

$$\begin{cases} |\eta^1| = \frac{1}{1+\Delta t}|\eta^0 + \Delta t(\pi_E^h y_1)(1)| \le \frac{1}{1+\Delta t}(|\eta^0| + |\Delta t y_1(1)|) \\ \quad \le \frac{1}{1+\Delta t}(|\eta^0| + C|y_{1x}|) \le C. \end{cases}$$

In the same way, we use $(16)_3$ to get $|\xi^1| \le C$.

Therefore:

$$E^{n+1} \le C.$$

4.1 Error Estimate

Theorem 4.1 *Assume that the assumption of theorem 2.1 holds, and then, if* $y \in H^3(0, T, L^2(I)) \cap H^2(0, T, H^2(I))$ *and* $\eta, \xi \in H^2(0, T)$ *we have:*

$$|y_t(t_n) - \hat{y}^n|^2 + \gamma|\hat{y}_x(t_n) - \hat{y}_x^n|^2 + |y_{xx}(t_n) - y_{xx}^n|^2 + |\eta(t_n) - \eta^n|^2 + |\xi(t_n) - \xi^n|^2 \le C\Big[(\Delta t)^2 + h^2\Big],$$

where C is independent of t and h.

Proof We use the standard decompositions:

$$\begin{cases} y^n - y(t_n) = y^n - \pi_E^h y(t_n) + \pi_E^h y(t_n) - y(t_n) = e^n + \rho^n, \\ \hat{y}_x^n - \hat{y}_x(t_n) = y_x^n - \pi_E^h \hat{y}_x(t_n) + \pi_E^h \hat{y}_x(t_n) - \hat{y}_x(t_n) = \hat{e}_x^n + \hat{\rho}_x^n, \end{cases}$$

where:

$$\begin{cases} e^n = y^n - \pi_E^h y(t_n), \quad \rho^n = \pi_E^h y(t_n) - y(t_n), \\ b^n = \xi^n - \xi(t_n), \quad d^n = \eta^n - \eta(t_n), \\ \hat{e}^n = \hat{y}^n - \pi_E^h \hat{y}(t_n), \\ \hat{\rho}^n = \pi_E^h \hat{y}(t_n) - \hat{y}(t_n), \\ \hat{b}^n = \hat{\xi}^n - \hat{\xi}(t_n), \quad \hat{d}^n = \hat{\eta}^n - \hat{\eta}(t_n). \end{cases}$$

Let us rewrite (16)$_1$ as:

$$\begin{cases} \frac{1}{\Delta t}(\hat{y}^{n+1}-\hat{y}^n, W)+\frac{\gamma}{\Delta t}(\hat{y}_x^{n+1}-\hat{y}_x^n, W_x)+(y_{xx}^{n+1}, W_{xx}) \\ +\xi^{n+1}W(1)+\eta^{n+1}W_x(1)=0, \forall W \in V_E^h; \end{cases}$$

therefore:

$$\begin{cases} \frac{1}{\Delta t}(\hat{e}^{n+1}-\hat{e}^n, W)+\frac{\gamma}{\Delta t}(\hat{e}_x^{n+1}-\hat{e}_x^n, W_x)+ \\ (e_{xx}^{n+1}, W_{xx})+\xi^{n+1}W(1)+\eta^{n+1}W_x(1)= \\ -\frac{1}{\Delta t}\left(\pi_E^h\hat{y}(t_{n+1})-\pi_E^h\hat{y}(t_n), W\right) \\ -\frac{\gamma}{\Delta t}\left(\left(\pi_E^h\hat{y}(t_{n+1})\right)_x-\left(\pi_E^h\hat{y}(t_n)\right)_x, W_x\right)-\left(\left(\pi_E^h y(t_{n+1})\right)_{xx}, W_{xx}\right). \end{cases}$$

Using lemma (3.1), we get:

$$\begin{cases} \frac{1}{\Delta t}(\hat{e}^{n+1}-\hat{e}^n, W)+\frac{\gamma}{\Delta t}(\hat{e}_x^{n+1}-\hat{e}_x^n, W_x)+ \\ (e_{xx}^{n+1}, W_{xx})+\xi^{n+1}W(1)+\eta^{n+1}W_x(1)= \\ -\frac{1}{\Delta t}\left(\pi_E^h\hat{y}(t_{n+1})-\pi_E^h\hat{y}(t_n), W\right) \\ -\frac{\gamma}{\Delta t}\left(\left(\pi_E^h\hat{y}(t_{n+1})\right)_x-\left(\pi_E^h\hat{y}(t_n)\right)_x, W_x\right)-(y_{xx}(t_{n+1}), W_{xx}). \end{cases}$$

Using (5) with $t=t_{n+1}$ and $\upsilon=W$, we get:

$$\begin{aligned} &(y_{xx}(t_{n+1}), W_{xx}) \\ &\quad=-\frac{\gamma}{\Delta t}\left(\hat{y}_x(t_{n+1})-\hat{y}(t_n)_x, W_x\right)-(\hat{y}_t(t_{n+1}), W)-\xi(t_{n+1})W(1)-\eta(t_{n+1})W_x(1); \end{aligned}$$

therefore:

$$\begin{cases} \frac{1}{\Delta t}(\hat{e}^{n+1}-\hat{e}^n, W)+\frac{\gamma}{\Delta t}(\hat{e}_x^{n+1}-\hat{e}_x^n, W_x)+(e_{xx}^{n+1}, W_{xx})+\xi^{n+1}W(1)+\eta^{n+1}W_x(1)= \\ -\frac{1}{\Delta t}\left(\pi_E^h\hat{y}(t_{n+1})-\pi_E^h\hat{y}(t_n), W\right)-\frac{\gamma}{\Delta t}\left(\left(\pi_E^h\hat{y}(t_{n+1})\right)_x-\left(\pi_E^h\hat{y}(t_n)\right)_x, W_x\right) \\ +\frac{\gamma}{\Delta t}\left(\hat{y}_x(t_{n+1})-\hat{y}(t_n)_x, W_x\right)+(\hat{y}_t(t_{n+1}), W)+\xi(t_{n+1})W(1)+\eta(t_{n+1})W_x(1); \end{cases}$$

hence:

$$\begin{cases} \frac{1}{\Delta t}(\hat{e}^{n+1}-\hat{e}^n, W)+\frac{\gamma}{\Delta t}(\hat{e}_x^{n+1}-\hat{e}_x^n, W_x)+(e_{xx}^{n+1}, W_{xx})+b^{n+1}W(1)+d^{n+1}W_x(1)= \\ \frac{\gamma}{\Delta t}\left(\hat{\rho}_x^n-\hat{\rho}_x^{n+1}, W_x\right)+\left(\hat{y}_t(t_{n+1})-\frac{\pi_E^h\hat{y}(t_{n+1})-\pi_E^h\hat{y}(t_n)}{\Delta t}, W\right). \end{cases}$$

Take $W=\hat{e}^{n+1}$, and use the fact that:

$$\begin{cases} \hat{e}^{n+1}(1)=\hat{y}^{n+1}(1)-(\pi_E^h\hat{y}(t_{n+1}))(1) \\ =\hat{\xi}^{n+1}+\xi^{n+1}-\hat{y}(1, t_{n+1}) \\ =\hat{\xi}^{n+1}+\xi^{n+1}-\hat{\xi}(t_{n+1})-\xi(t_{n+1}) \\ =\left(\hat{\xi}^{n+1}-\hat{\xi}(t_{n+1})\right)+\left(\xi^{n+1}-\xi(t_{n+1})\right) \\ =b^{n+1}+\hat{b}^{n+1}, \end{cases}$$

and

$$\begin{cases} \hat{e}_x^{n+1}(1)=\hat{y}_x^{n+1}(1)-(\pi_E^h\hat{y}(t_{n+1}))_x(1) \\ =\hat{\eta}^{n+1}+\eta^{n+1}-\hat{y}_x(1, t_{n+1}) \\ =\hat{\eta}^{n+1}+\eta^{n+1}-\hat{\eta}(t_{n+1})-\eta(t_{n+1}) \\ =\left(\hat{\eta}^{n+1}-\hat{\eta}(t_{n+1})\right)+\left(\eta^{n+1}-\eta(t_{n+1})\right) \\ =d^{n+1}+\hat{d}^{n+1}, \end{cases}$$

we get:

$$\begin{cases} \frac{1}{\Delta t}(\hat{e}^{n+1}-\hat{e}^{n},\hat{e}^{n+1})+\frac{\gamma}{\Delta t}(\hat{e}_x^{n+1}-\hat{e}_x^{n},\hat{e}_x^{n+1})+(e_{xx}^{n+1},\hat{e}_{xx}^{n+1})+ \\ |b^{n+1}|^2+|d^{n+1}|^2+b^{n+1}\hat{b}^{n+1}+d^{n+1}\hat{d}^{n+1}= \\ \frac{\gamma}{\Delta t}\Big(\hat{\rho}_x^{n}-\hat{\rho}_x^{n+1},\hat{e}_x^{n+1}\Big)+\Big(\hat{y}_t(t_{n+1})-\frac{\pi_E^h\hat{y}(t_{n+1})-\pi_E^h\hat{y}(t_n)}{\Delta t},\hat{e}^{n+1}\Big). \end{cases} \tag{22}$$

Note that:

$$\xi_t(t_{n+1})=\frac{\xi(t_{n+1})-\xi(t_n)}{\Delta t}+\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}(s-t_n)\xi_{tt}(s)ds;$$

therefore:

$$\begin{cases} \hat{b}^{n+1}=\hat{\xi}^{n+1}-\hat{\xi}(t_{n+1}) \\ \quad=\frac{1}{\Delta t}\Big[(\xi^{n+1}-\xi(t_{n+1}))-(\xi^{n}-\xi(t_n))-\int_{t_n}^{t_{n+1}}(s-t_n)\xi_{tt}(s)ds\Big] \\ \quad=\frac{1}{\Delta t}[b^{n+1}-b^{n}-\theta^{n+1}], \end{cases}$$

where

$$\theta^{n+1}=\int_{t_n}^{t_{n+1}}(s-t_n)\xi_{tt}(s)ds.$$

Therefore, (20) gives:

$$\begin{cases} b^{n+1}\hat{b}^{n+1}=\frac{1}{\Delta t}b^{n+1}\Big(b^{n+1}-b^{n}\Big)-\frac{1}{\Delta t}b^{n+1}\theta^{n+1}= \\ \frac{1}{2\Delta t}\Big[|b^{n+1}-b^{n}|^2+|b^{n+1}|^2-|b^{n}|^2\Big]-\frac{1}{\Delta t}b^{n+1}\theta^{n+1}\geq \\ \frac{1}{2\Delta t}\Big[|b^{n+1}-b^{n}|^2+|b^{n+1}|^2-|b^{n}|^2\Big]-|b^{n+1}|^2-\frac{1}{4\Delta t}|\theta^{n+1}|^2, \end{cases}$$

where we used Young's Inequality $\Big(\pm ab\leq \delta a^2+\frac{1}{4\delta}b^2\Big)$, with $\delta=\Delta t$. In the same way, we show that:

$$d^{n+1}\hat{d}^{n+1}\geq\frac{1}{2\Delta t}\Big[|d^{n+1}-d^{n}|^2+|d^{n+1}|^2-|d^{n}|^2\Big]-|d^{n+1}|^2-\frac{1}{4\Delta t}|\phi^{n+1}|^2,$$

where

$$\phi^{n+1}=\int_{t_n}^{t_{n+1}}(s-t_n)\eta_{tt}(s)ds.$$

Combine the above with (22) to obtain:

$$\begin{cases} \frac{1}{2\Delta t}\left[|\hat{e}^{n+1}-\hat{e}^{n}|^2+|\hat{e}^{n+1}|^2-|\hat{e}^{n}|^2\right] \\ +\frac{\gamma}{2\Delta t}\left[|\hat{e}_x^{n+1}-\hat{e}_x^{n}|^2+|\hat{e}_x^{n+1}|^2-|\hat{e}_x^{n}|^2\right] \\ +\frac{1}{2\Delta t}\left[|e_{xx}^{n+1}-e_{xx}^{n}|^2+|e_{xx}^{n+1}|^2-|e_{xx}^{n}|^2\right] \\ \frac{1}{2\Delta t}\left[|b^{n+1}-b^{n}|^2+|b^{n+1}|^2-|b^{n}|^2\right]+ \\ \frac{1}{2\Delta t}\left[|d^{n+1}-d^{n}|^2+|d^{n+1}|^2-|d^{n}|^2\right]\leq \\ \frac{\gamma}{\Delta t}\Big(\hat{\rho}_x^{n}-\hat{\rho}_x^{n+1},\hat{e}_x^{n+1}\Big)+\Big(\hat{y}_t(t_{n+1})-\frac{\pi_E^h\hat{y}(t_{n+1})-\pi_E^h\hat{y}(t_n)}{\Delta t},\hat{e}^{n+1}\Big)+ \\ \frac{1}{4\Delta t}|\theta^{n+1}|^2+\frac{1}{4\Delta t}|\phi^{n+1}|^2= \\ \frac{\gamma}{\Delta t}H+I+\frac{1}{4\Delta t}J+\frac{1}{4\Delta t}K. \end{cases} \tag{23}$$

Using the definition of $\hat{\rho}$ and (12), we get:

$$|\hat{\rho}_x^{n+1} - \hat{\rho}_x^n|^2 = \int_{t_n}^{t_{n+1}} |\hat{\rho}_{xt}(s)|^2 ds = \int_{t_n}^{t_{n+1}} \left| \left(\pi_E^h \hat{y}(s)\right)_{xt} - \hat{y}_{xt}(s)\right|^2 ds \le Ch^2 \int_{t_n}^{t_{n+1}} |\hat{y}_{xxt}(s)|^2 ds;$$

hence:

$$H = \left(\hat{\rho}_x^n - \hat{\rho}_x^{n+1}, \hat{e}_x^{n+1}\right) \le 2|\hat{\rho}_x^{n+1} - \hat{\rho}_x^n|^2 + 2|\hat{e}_x^{n+1}|^2 \le 2Ch^2 \int_{t_n}^{t_{n+1}} |\hat{y}_{xxt}(s)|^2 ds + 2|\hat{e}_x^{n+1}|^2.$$

Moreover:

$$\begin{aligned}&\left|\hat{y}_t(t_{n+1}) - \frac{\pi_E^h \hat{y}(t_{n+1}) - \pi_E^h \hat{y}(t_n)}{\Delta t}\right| \le \left|\hat{y}_t(t_{n+1}) - \frac{\hat{y}(t_{n+1}) - \hat{y}(t_n)}{\Delta t}\right| \\ &+ \tfrac{1}{\Delta t}\left|(\hat{y}(t_{n+1}) - \hat{y}(t_n)) - (\pi_E^h \hat{y}(t_{n+1}) - \pi_E^h \hat{y}(t_n))\right| = I_1 + I_2.\end{aligned}$$

We have to bound I_1 and I_2:

$$\begin{aligned}I_1 &= \left|\hat{y}_t(t_{n+1}) - \frac{\hat{y}(t_{n+1}) - \hat{y}(t_n)}{\Delta t}\right| = \left|\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (s - t_n)\hat{y}_{tt}(s) ds\right| \\ &\le \frac{1}{\Delta t}\Big(\int_{t_n}^{t_{n+1}} (s - t_n)^2 ds\Big)^{\frac{1}{2}} \Big(\int_{t_n}^{t_{n+1}} |\hat{y}_{tt}(s)|^2 ds\Big)^{\frac{1}{2}} \\ &= \frac{(\Delta t)^{\frac{1}{2}}}{3}\Big(\int_{t_n}^{t_{n+1}} |\hat{y}_{tt}(s)|^2 ds\Big)^{\frac{1}{2}},\end{aligned}$$

and

$$\begin{aligned}&I_2 = \tfrac{1}{\Delta t}\left|(\hat{y}(t_{n+1}) - \hat{y}(t_n)) - (\pi_E^h \hat{y}(t_{n+1}) - \pi_E^h \hat{y}(t_n))\right| = \\ &\tfrac{1}{\Delta t}\left|\int_{t_n}^{t_{n+1}} \hat{\rho}_t(s) ds\right| \le \frac{1}{\Delta t}\Big(\Delta t \int_{t_n}^{t_{n+1}} |\hat{\rho}_t(s)|^2 ds\Big)^{1/2},\end{aligned}$$

again

$$\left|\hat{\rho}_t(s)\right| = \left|\left(\pi_E^h \hat{y}(s)\right)_t - \hat{y}_t(s)\right| \le Ch\left|\hat{y}_{xxt}(s)\right|;$$

hence:

$$I_2 \le Ch(\Delta t)^{\frac{1}{2}}\Big(\int_{t_n}^{t_{n+1}} |y_{xxtt}(s)|^2 ds\Big)^{\frac{1}{2}}.$$

Young's Inequality with $\delta = \Delta t$ gives:

$$\begin{aligned}&I = \left(\hat{y}_t(t_{n+1}) - \frac{\pi_E^h \hat{y}(t_{n+1}) - \pi_E^h \hat{y}(t_n)}{\Delta t}, \hat{e}^{n+1}\right) \le \\ &2\Delta t I_1^2 + 2\Delta t I_2^2 + \frac{2}{\Delta t}|\hat{e}^{n+1}|^2 \le \frac{2(\Delta t)^2}{9}\int_{t_n}^{t_{n+1}} |\hat{y}_{tt}(s)|^2 ds + \\ &Ch^2(\Delta t)^2 \int_{t_n}^{t_{n+1}} |y_{xxtt}(s)|^2 ds + \frac{2}{\Delta t}|\hat{e}^{n+1}|^2.\end{aligned}$$

Using the Cauchy–Schwarz inequality, we get:

$$\theta^{n+1} = \int_{t_n}^{t_{n+1}} (s - t_n)\xi_{tt}(s) ds \le C(\Delta t)^{\frac{3}{2}}\Big(\int_{t_n}^{t_{n+1}} |\xi_{tt}(s)|^2 ds\Big)^{\frac{1}{2}};$$

hence:

$$J = |\theta^{n+1}|^2 \le C(\Delta t)^3 \int_{t_n}^{t_{n+1}} |\xi_{tt}(s)|^2 ds;$$

also:

$$K = |\phi^{n+1}|^2 \le C(\Delta t)^3 \int_{t_n}^{t_{n+1}} |\eta_{tt}(s)|^2 ds.$$

The above estimates together with (23) give:

$$\begin{cases} \frac{1}{2\Delta t}\left[|\hat{e}^{n+1}-\hat{e}^{n}|^2+|\hat{e}^{n+1}|^2-|\hat{e}^{n}|^2\right]+ \\ \frac{\gamma}{2\Delta t}\left[|\hat{e}_x^{n+1}-\hat{e}_x^{n}|^2+|\hat{e}_x^{n+1}|^2-|\hat{e}_x^{n}|^2\right]+ \\ \frac{1}{2\Delta t}\left[|e_{xx}^{n+1}-e_{xx}^{n}|^2+|e_{xx}^{n+1}|^2-|e_{xx}^{n}|^2\right]+ \\ \frac{1}{2\Delta t}\left[|b^{n+1}-b^{n}|^2+|b^{n+1}|^2-|b^{n}|^2\right]+ \\ \frac{1}{2\Delta t}\left[|d^{n+1}-d^{n}|^2+|d^{n+1}|^2-|d^{n}|^2\right]\le \\ \frac{2\gamma Ch^2}{\Delta t}\int_{t_n}^{t_{n+1}}|\hat{y}_{xxt}(s)|^2ds+\frac{2\gamma}{\Delta t}|\hat{e}_x^{n+1}|^2+ \\ \frac{2(\Delta t)^2}{9}\int_{t_n}^{t_{n+1}}|\hat{y}_{tt}(s)|^2ds+C(\Delta t)^2h^2\int_{t_n}^{t_{n+1}}|y_{ttxx}(s)|^2ds+\frac{2}{\Delta t}|\hat{e}^{n+1}|^2+ \\ C(\Delta t)^2\int_{t_n}^{t_{n+1}}|\xi_{tt}(s)|^2ds+C(\Delta t)^2\int_{t_n}^{t_{n+1}}|\eta_{tt}(s)|^2ds. \end{cases}$$

Multiply by $2\Delta t$ and sum from $i=1$ to n:

$$\begin{cases} \sum_{i=1}^{n}|\hat{e}^{i+1}-\hat{e}^{i}|^2+|\hat{e}^{n+1}|^2+\gamma\sum_{i=1}^{n}|\hat{e}_x^{i+1}-\hat{e}_x^{i}|^2+\gamma|\hat{e}_x^{n+1}|^2+ \\ \sum_{i=1}^{n}|e_{xx}^{i+1}-e_{xx}^{i}|^2+|e_{xx}^{n+1}|^2+\sum_{i=1}^{n}|b^{i+1}-b^{i}|^2+|b^{n+1}|^2+ \\ \sum_{i=1}^{n}|b^{i+1}-b^{i}|^2+|d^{n+1}|^2 \\ \le C(\Delta t)^2\Big(\int_{t_1}^{t_{n+1}}|\hat{y}_{tt}(s)|^2ds+h^2\int_{t_1}^{t_{n+1}}|y_{ttxx}(s)|^2ds+\int_{t_1}^{t_{n+1}}|\xi_{tt}(s)|^2ds+ \\ \int_{t_1}^{t_{n+1}}|\eta_{tt}(s)|^2ds\Big)+4\sum_{i=1}^{n}\left[|\hat{e}^{i+1}|^2+\gamma|\hat{e}_x^{i+1}|^2\right]+|\hat{e}^1|^2+|\hat{e}_x^1|^2+|e_{xx}^1|^2+|b^1|^2+|d^1|^2\le C \\ (\Delta t)^2\left[\|y_{ttt}\|^2_{L^2(0,T,L^2(I))}+h^2\|y_{tt}\|^2_{L^2(0,T,H_E^2(I))}+\|\xi_{tt}\|^2_{L^2(0,T)}+\|\eta_{tt}\|^2_{L^2(0,T)}\right]+ \\ 4\sum_{i=1}^{n}\left[|\hat{e}^{i+1}|^2+\gamma|\hat{e}_x^{i+1}|^2\right]+|\hat{e}^1|^2+|\hat{e}_x^1|^2+|e_{xx}^1|^2+|b^1|^2+|d^1|^2 \\ \le C(\Delta t)^2+Ch^2(\Delta t)^2+|\hat{e}^1|^2+|\hat{e}_x^1|^2+|e_{xx}^1|^2+|b^1|^2+|d^1|^2+4\sum_{i=1}^{n}\left[|\hat{e}^{i+1}|^2+\gamma|\hat{e}_x^{i+1}|^2\right]. \end{cases} \tag{24}$$

We have to bound $|e_{xx}^1|$, $|\hat{e}_x^1|$, $|\hat{e}^1|$, $|b^1|$, and $|d^1|$:

$$\begin{cases} (e_{xx}^1,\upsilon_{xx})=(y_{xx}^1-(\pi_E^h y(t_1))_{xx},\upsilon_{xx})=(y_{xx}^0+\Delta t(\pi_E^h y_1)_{xx}-(\pi_E^h y(t_1))_{xx},\upsilon_{xx}) \\ =((\pi_E^h y_0)_{xx}+\Delta t(\pi_E^h y_1)_{xx}-(\pi_E^h y(t_1))_{xx},\upsilon_{xx})=(y_{0xx}+y_{1xx}-y_{xx}(t_1),\upsilon_{xx}) \\ =(y_{xx}(0)+\Delta t\, y_{xxt}(0)-y_{xx}(t_1),\upsilon_{xx}), \end{cases}$$

where we used the definition of π_E^h, y^1 and y^0.

However:

$$y_{xx}(t_1) = y_{xx}(0) + \Delta t y_{xxt}(0) + \frac{(\Delta t)^2}{2} y_{xxtt}(\tau_1);$$

so:

$$(e^1_{xx}, \upsilon_{xx}) = \Big(-\frac{(\Delta t)^2}{2} y_{xxtt}(\tau_1), \upsilon_{xx}\Big).$$

Choosing $\upsilon = e^1$, we get:

$$|e^1_{xx}|^2 \leq \frac{1}{2}(\Delta t)^2 |y_{xxtt}(\tau_1)||e^1_{xx}|, \quad \text{that is,} \quad |e^1_{xx}| \leq C(\Delta t)^2.$$

To bound $|\hat{e}^1|$ and $|\hat{e}^1_x|$, we will bound $|\hat{e}^1_{xx}|$, and then, we use the Poincare inequality.

Knowing that, $\hat{y}^1 = (y^1 - y^0)/\Delta t$ and $y^1 = y^0 + \Delta t \pi_E^h y_1$.

We have:

$$\begin{cases} (\hat{e}^1_{xx}, \upsilon_{xx}) = (\hat{y}^1_{xx} - (\pi_E^h \hat{y}(t_1))_{xx}, \upsilon_{xx}) \\ = \dfrac{1}{\Delta t}(y^1_{xx} - y^0_{xx}, \upsilon_{xx}) - (\pi_E^h \hat{y}(t_1))_{xx}, \upsilon_{xx}) \\ = \Big((\pi_E^h y_1)_{xx}, \upsilon_{xx}\Big) - ((\pi_E^h \hat{y}(t_1))_{xx}, \upsilon_{xx}) \\ = (y_{1xx}, \upsilon_x) - (y_{xxt}(t_1), \upsilon_{xx}) \\ = (y_{xxt}(0) - y_{xxt}(t_1), \upsilon_{xx}), \end{cases}$$

but, $y_{xxt}(t_1) = y_{xxt}(0) + \Delta t y_{xxtt}(\tau_2)$, so, $(\hat{e}^1_x, \upsilon_x) = -\Delta t(y_{ttx}(\tau_2), \upsilon_x)$.

Choose $\upsilon = \hat{e}^1$, we get:

$$|\hat{e}^1_{xx}|^2 \leq C\Delta t |(y_{xxtt}(\tau_2)||\hat{e}^1_{xx}|, \text{ and hence} |\hat{e}^1_{xx}| \leq C\Delta t.$$

Using Poincare inequality, we get $|\hat{e}^1| \leq C|\hat{e}^1_x| \leq C|\hat{e}^1_{xx}| \leq \Delta t$.

To bound $|b^1|$, we use the fact:

$$\eta(t_1) = \eta(t_0) + \Delta t \eta_t(\sigma), with 0 < \sigma < t_1;$$

therefore, we have:

$$|b^1| = |\eta^1 - \eta(t_1)| = |\eta^1 - \eta^0 + \eta(0) - \eta(t_1)| = \Delta t |\hat{\eta}^1 - \eta_t(\sigma)| \leq C\Delta t.$$

Also, $|d^1| \leq C\Delta t$.

Using (24) and the above estimates, we get:

$$\begin{cases} |\hat{e}^{n+1}|^2 + \gamma|\hat{e}^{n+1}_x|^2 + |e^{n+1}_{xx}|^2 + |b^{n+1}|^2 + |d^{n+1}|^2 \\ \leq C(\Delta t)^2 + Ch^4 + 4C \displaystyle\sum_{i=1}^{n} \Big[|\hat{e}^{i+1}|^2 + \gamma|\hat{e}^i_x|^2\Big]. \end{cases}$$

Applying the discrete Gronwall inequality to obtain:

$$|\hat{e}^{n+1}|^2 + \gamma|\hat{e}^{n+1}_x|^2 + |e^{n+1}_{xx}|^2 + |b^{n+1}|^2 + |d^{n+1}|^2 \leq C\Big[(\Delta t)^2 + h^4\Big].$$

Now:

$$\begin{cases} |y_t(t_n) - \hat{y}^n|^2 + \gamma|y_{xt}(t_n) - \hat{y}^n_x|^2 + |y_{xx}(t_n) - y^n_{xx}|^2 + |\xi(t_n) - \xi^n|^2 + |\eta(t_n) - \eta^n|^2 \\ \leq C\Big[|\hat{e}^{n+1}|^2 + \gamma|\hat{e}^{n+1}_x|^2 + |e^{n+1}_{xx}|^2 + |b^{n+1}|^2 + |d^{n+1}|^2 + |\hat{\rho}^n|^2 + \gamma|\hat{\rho}^n_x|^2 + |\rho^n_{xx}|^2\Big] \\ \leq C\Big[(\Delta t)^2 + (\Delta t)^4 + h^4\Big] + C\Big[|\hat{\rho}^n|^2 + \gamma|\hat{\rho}^n_x|^2 + |\rho^n_{xx}|^2\Big]. \end{cases}$$

Using (12), we have:

$$|\hat{\rho}^n| = |\pi_E^h \hat{y}(t_n) - \hat{y}(t_n)| \leq Ch|\hat{y}_{xx}(t_n)| \leq Ch,$$

also $|\hat{\rho}_x^n| \leq Ch$ and $|\rho_{xx}^n| \leq Ch$. Hence:

$$\begin{cases} |y_t(t_n) - \hat{y}^n|^2 + \gamma |y_{xt}(t_n) - \hat{y}_x^n|^2 + |y_{xx}(t_n) - y_{xx}^n|^2 + |\xi(t_n) - \xi^n|^2 + |\eta(t_n) - \eta^n|^2 \leq \\ C(h^2 + h^4 + (\Delta t)^2). \end{cases}$$

Finally:

$$|y_t(t_n) - \hat{y}^n|^2 + \gamma |y_{xt}(t_n) - \hat{y}_x^n|^2 + |y_{xx}(t_n) - y_{xx}^n|^2 + |\xi(t_n) - \xi^n|^2 + |\eta(t_n) - \eta^n|^2 \leq C(h^2 + (\Delta t)^2). \qquad \square$$

5 Finite-element method

In the finite-element method, the basis functions are usually polynomials of any degree, defined in each finite element. In this paper, we are going to use the Hermite finite element (see [12] Page 168).

To define a basis for V_E^h, we introduce the two reference functions:

$$\phi(x) = \begin{cases} (1+x)^2(1-2x) & \text{if } x \in [-1, 0], \\ (1-x)^2(1+2x) & \text{if } x \in [0, 1], \\ 0 & \text{elsewhere}, \end{cases}$$

and

$$\psi(x) = \begin{cases} x(1+x)^2 & \text{if } x \in [-1, 0], \\ x(1-x)^2 & \text{if } x \in [0, 1], \\ 0 & \text{elsewhere}. \end{cases}$$

Define the following bases functions by:

$$\phi_j(x) = \phi\left(\frac{x - x_j}{h}\right) \; \forall\, 1 \leq j \leq s+1$$

and

$$\psi_j(x) = h\psi\left(\frac{x - x_j}{h}\right) \; \forall\, 1 \leq j \leq s+1.$$

Note that, for all $1 \leq i, j \leq s+1$, we have:

$$\phi_j(x_i) = \delta_{ij}, \phi_{jx}(x_i) = 0, \psi_j(x_i) = 0 \text{ and } \psi_{jx}(x_i) = \delta_{ij}.$$

With the above, V_E^h becomes a subspace of $H_E^2(I)$ of dimension $2s+2$, and for all $y^h \in V_E^h$, y^h can be written in the form:

$$y^h(x) = \sum_{j=1}^{s+1} y^h(x_j)\phi_j(x) + \sum_{j=1}^{s+1} y_x^h(x_j)\psi_j(x) \; \forall x \in [0, 1].$$

let us rename the basis as $\{\mu_i\}_{i=1}^{2s+2}$, where:

$$\mu_i(x) = \phi_i(x) = \phi\left(\tfrac{x - x_i}{h}\right) \; \forall\, 1 \leq i \leq s+1,$$
$$\mu_i(x) = \psi_{i-s-1}(x) = h\psi\left(\tfrac{x - x_{i-s-1}}{h}\right) \; \forall\, s+2 \leq j \leq 2s+2.$$

5.1 Numerical simulations

The numerical analysis is performed for $\gamma = 0.1$, $\Delta t = 0.05$, $h = 1/15$, $y_0(x) = x^2(1-x)$, $y_1(x) = -x^2(1-x)$, $\eta_0 = 3/(1+\Delta t)$, $\eta_1 = (4(20+\Delta t))/(5(1+\Delta t))$, $\xi_0 = 1$, and $\xi_1 = 1/(1+\Delta t)$.

The graphs in Figs. 1, 2, 3, and 4 represent, respectively, the time evolution of the beam's position $y^n(x, t)$, moment control η^n, force control ξ^n, and the discrete energy E^n.

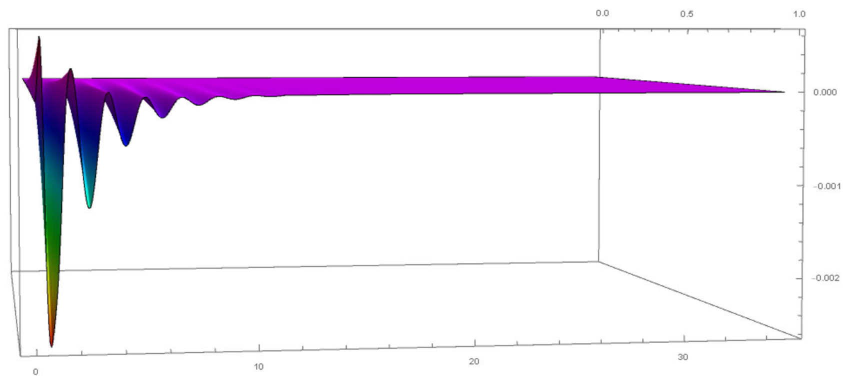

Fig. 1 Position $y^n(x, t)$

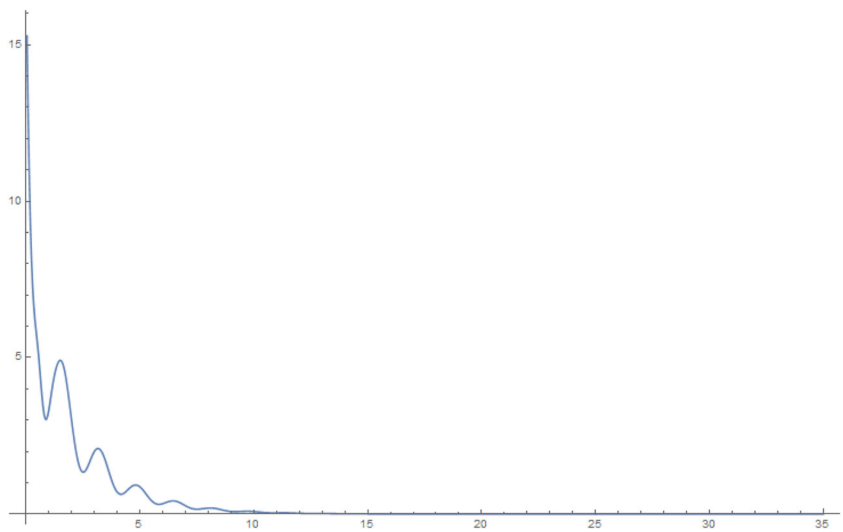

Fig. 2 Moment control η^n

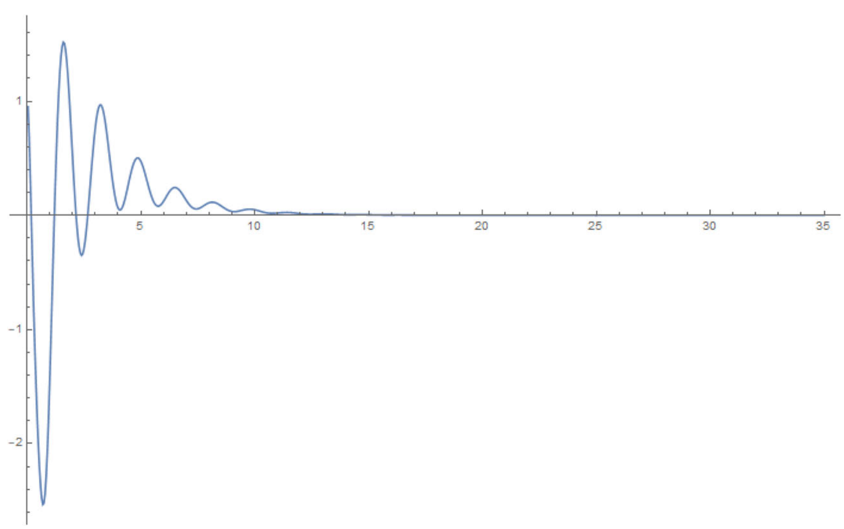

Fig. 3 Force control ξ^n

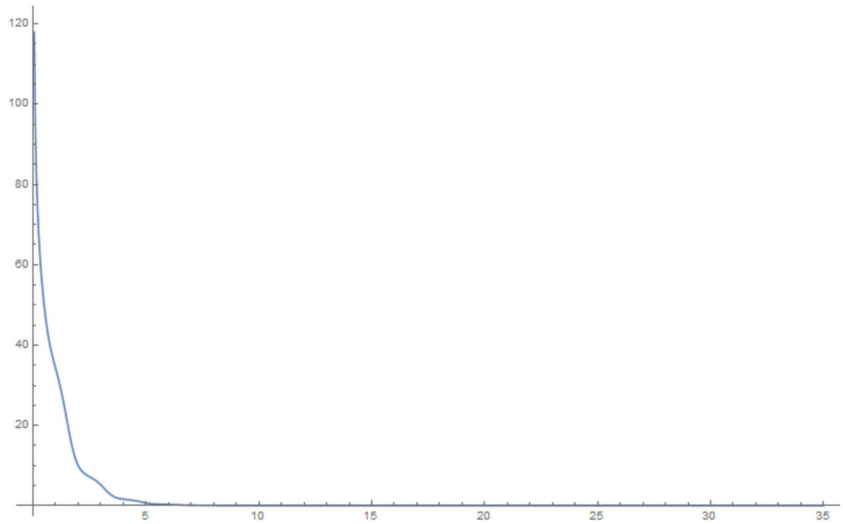

Fig. 4 Full discrete energy E^n.

References

1. Allaire, G.: Numerical analysis and optimization, an introduction to mathematical modelling and numerical simulation. Oxford University Press. First edition, USA (2007)
2. El Arwadi, T., Youssef, W.: On the stabilization of the Bresse beam with Kelvin Voigt damping. Appl. Math. Optim. (2019)
3. Copetti, M.I.M.; Fernandez, J.R.: A dynamic contact problem involving a Timoshenko beam model. Appl. Numer. Math. **63**, 117–128 (2013)
4. Dautray, R., Lions, J.: Mathematical analysis and numerical methods for science and technology, vol. 5 of Evolution Problems I, Springer-Verlag, (2000)
5. El Arwadi, T.; Copetti, M.I.M.; Youssef, W.: On the theoretical and numerical stability of the thermoviscoelastic Bresse system. Z. Angew. Math. Mech. **99**, 120 (2019)
6. Francis, B.A.: H^∞control theory, Lecture Notes in Control and Sciences, (1986)
7. Rao, B.: Exact boundary controllability of a hybrid system of elasticity by the HUM method, ESIAM: COCV, 6 (2001), 183199
8. Rao, B.; Wehbe, A.: Polynomial energy decay rate and strong stability of Kirchhoff plates with non-compact resolvent. J. Evol. Equ. **5**, 137152 (2005)
9. Rincon, M.; Copetti, M.I.M.: Numerical analysis for a locally damped wave equation. J. Appl. Anal. Comput. **3**, 169–182 (2013)
10. Russell, D.L.: A General framework for the study of indirect damping mechanisms in elastic systems. J. Math. Anal. Appl. (1993)
11. Sammoury, M.A.; Nicaise, S.; Mercier, D.; Wehbe, A.: Optimal energy decay rate for rayleigh beam equation with only one dynamic boundary control. Boletim da Sociedade Paranaense de Matematica. **35**, 131–171 (2017). https://doi.org/10.5269/bspm.v35i3.29266
12. Wehbe, A.: Optimal energy decay rate in the Rayleigh beam equation with boundary dynamical controls. Bull. Belg. Math. Soc. **12**, 1–16 (2005)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.