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On measure algebras associated to locally compact groups

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Abstract We shall consider measure algebras associated to locally compact groups, bounded operators between them and properties of the underlying measures. We take into account the second dual of measure algebras provided with the Arens products together with tools of Gélfand theory.

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1 Introduction

Let G denote a locally compact Hausdorff group G and let $E = \mathrm{C}_0(G)$ be the space of continuous complex valued functions which converge to zero at infinity. As usual, $\mathrm{C}_{00}(G)$ is the subspace of E of continuous functions with compact support. Endowed with the natural vector space structure, the uniform norm and the usual conjugation E becomes an abelian B^*-algebra. Its dual space E^* is realized as the space $M(G)$ of complex regular Borel measures on G. This result by Kakutani generalizes a well known landmark by Riesz [8,9].

If $m, n \in M(G)$ and $f \in E$ let $\langle f, m * n\rangle = \int_G \int_G f(ab)\mathrm{d}m(a)\mathrm{d}n(b)$. Then $m * n \in M(G)$ and with this product $M(G)$ becomes an associative complex Banach algebra. Given a Banach space B let us write $\varkappa_B : B \hookrightarrow B^{**}$ to the usual isometric immersion of B into its double dual space B^{**}. In particular, $\varkappa_E(E)$ is a closed submodule of $M(G)^*$. For, let $m \in M(G)$, $f \in E$ and let us see that $m\varkappa_E(f) \in \varkappa_E(E)$. To this end let $\{n_i\}_{i\in I}$ be a net in $M(G)$ so that $w^* - \lim_{i\in I} n_i = 0_{M(G)}$. By Fubini's theorem for each $i \in I$ we can write

$$\langle n_i, m\varkappa_E(f)\rangle = \langle n_i * m, \varkappa_E(f)\rangle = \langle f, n_i * m\rangle = \int_G \int_G f(ab)\mathrm{d}m(b)\mathrm{d}n_i(a).$$

However, the function $a \in G \to \int_G f(ab)\mathrm{d}m(b)$ belongs to E (cf. [7], Lemma 19.5). Thus $\lim_{i\in I}\langle n_i, m\varkappa_E(f)\rangle = 0$, i.e., $m\varkappa_E(f)$ is w^*-continuous and the claim holds. Furthermore, $\varkappa_E(E)^* \approx M(G)$ and $M(G)$ becomes a dual Banach algebra. Indeed, a little modification of the above reasoning shows that the product of $M(G)$ is separately w^*-continuous.

Since E is an abelian B^*-algebra it is Arens regular, i.e., the first $\square$ and second $\lozenge$ Arens products on E^{**} coincide [1,12]. Here, given $\Phi, \Psi \in E^{**}$ by Goldstine's theorem there are bounded nets $\{f_i\}_{i\in I}, \{g_j\}_{j\in J}$ in E so that $\Phi = w^* - \lim_{i\in I} \varkappa_E(f_i)$, $\Psi = w^* - \lim_{j\in J} \varkappa_E(g_j)$ and we are writting

$$\Phi\square\Psi = w^* - \lim_{i\in I}\lim_{j\in J} \varkappa_E(f_i g_j),\ \Phi\lozenge\Psi = w^* - \lim_{j\in J}\lim_{i\in I} \varkappa_E(f_i g_j).$$

Thus, $(E^{**}, \square)$ becomes an abelian B^*-algebra as E is abelian and regular (cf. [2], Th. 7.1). It is unitary since E has a bounded approximate identity (cf. [11], Lemma 1.1). Let $\breve{G}$ be the spectrum or structure space

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of $(E^{**}, \Box)$ and let $\gamma_{E^{**}} : E^{**} \to C(\widetilde{G})$ be the Gélfand transform of $(E^{**}, \Box)$. Then $\widetilde{G}$ becomes a w^*-compact subset of the unit sphere of $E^{***} = M(G)^{**}$ and $\gamma_{E^{**}}$ is an isometric isomorphism of B^*-algebras. So $\gamma^*_{E^{**}} : M(\widetilde{G}) \to M(G)^{**}$ establishes an isometric isomorphism of Banach spaces. So there is induced a Banach algebra structure in $M(\widetilde{G})$ with respect to which $\gamma^*_{E^{**}}$ becomes a Banach algebra homomorphism: given $\widetilde{\mu}, \widetilde{\nu} \in M(\widetilde{G})$ it suffices to write

$$\widetilde{\mu} * \widetilde{\nu} = (\gamma^*_{E^{**}})^{-1}(\gamma^*_{E^{**}}(\widetilde{\mu})\Box\gamma^*_{E^{**}}(\widetilde{\nu})).$$

Our task in this article is to explore connections between $M(G)$ and $M(\widetilde{G})$. In Sect. 2 we shall introduce in Theorem 2.1 bilateral Banach algebra homomorphisms between them. We shall enumerate various properties of these mappings, their connections and their behavior on measures (preservation of order or incidence on the absolute value of measures). In Theorem 2.2 we shall relate $M(G)^*$, by means of an isometric isomorphism of unital abelian B^*-algebras, with a subalgebra of operators on $M(G)$ in the sense of [13–15]. In Sect. 3 we shall consider the support problem, that is to say in what extent we can relate the support of a given measure and the support of its image under the previous operators.

Notation 1.1 *Let $\delta : G \to M(G)$, $\delta(a)(f) = f(a)$ and $\widetilde{\delta} : \widetilde{G} \to M(\widetilde{G})$, $\widetilde{\delta}(M)(\widetilde{f}) = \widetilde{f}(M)$, where $a \in G$, $M \in \widetilde{G}$, $f \in E$ and $\widetilde{f} \in C(\widetilde{G})$. Plainly δ and $\widetilde{\delta}$ are well defined (τ_G, w^*) and $(\tau_{\widetilde{G}}, w^*)$ continuous functions respectively. The function $\mathfrak{h} : G \to \widetilde{G}$ so that $\mathfrak{h} = \varkappa_{M(G)} \circ \delta$ is well defined although it is not continuous. Let $C_b(G)$ be the Banach space of complex bounded continuous functions on G endowed with the uniform norm.*

We shall denote $\mathfrak{B}(G)$, $K(G)$ and $\mathfrak{B}_b(G)$ to the classes of Borel, of compact subsets and of bounded Borel functions of G respectively. Besides $\mathfrak{B}_{loc}(G)$ will be the class of locally Borel subsets of G, i.e., given $S \subseteq G$ we have $S \in \mathfrak{B}_{loc}(G)$ if and only if $S \cap K \in \mathfrak{B}(G)$ for all $K \in K(G)$. Given a positive measure $m \in M(G)^+$ and $S \in \mathfrak{B}_{loc}(G)$ we say that S is an m-zero set if for any $K \in K(G)$ we have $m(S \cap K) = 0$. If $m \in M(G)$ and $B \in \mathfrak{B}(G)$ then

$$| m |= \sup \left\{ \sum_{n=1}^{m} | m(B_n) |: m \in \mathbb{N} \text{ and } \{B_1, \dots, B_m\} \in \mathcal{P}_{fdb}(B) \right\},$$

where $\mathcal{P}_{fdb}(B)$ is the class of finite disjoint borelian partitions of B. As it is known, $\| m \| =| m | (G)$.

Given $m, n \in M(G)$ we shall write $m << n$ or $m \perp n$ according as m is n-continuous or m and n are mutually singular respectively. For other notation or terminology the reader can see [6].

2 Some relations between $M(G)$ and $M(\widetilde{G})$

Theorem 2.1 *The following assertions hold:*

(1) *The mapping $A : M(\widetilde{G}) \to M(G)$, $A(\widetilde{\mu})(f) = \int_{\widetilde{G}} M(\varkappa_E(f))d\widetilde{\mu}(M)$ is a well defined contractive homomorphism of Banach algebras, where $\widetilde{\mu} \in M(\widetilde{G})$ and $f \in E$.*
(2) *$A : M_{\mathbb{R}}(\widetilde{G})^+ \to M_{\mathbb{R}}(G)^+$ and $| A(\widetilde{\mu}) | \leq A(| \widetilde{\mu} |)$ for all $\widetilde{\mu} \in M_{\mathbb{R}}(\widetilde{G})$.*
(3) *Further, $| A(\widetilde{\mu}) | \leq A(| \widetilde{\mu} |)$ for all $\widetilde{\mu} \in M(\widetilde{G})$.*
(4) *The mapping $B : M(G) \to M(\widetilde{G})$, $B(m)(\widetilde{f}) = \langle m, \gamma^{-1}_{E^{**}}(\widetilde{f})\rangle$, with $m \in M(G)$ and $\widetilde{f} \in C(\widetilde{G})$, is a well defined isometric homomorphism of Banach algebras.*
(5) *$A \circ B = Id_{M(G)}$.*
(6) *The operator $\Pi = \varkappa_{E^*} \circ \varkappa^*_E \in B(M(G)^{**})$ is a *-algebra homomorphism with respect to both Arens products.*
(7) *$\mathfrak{R}(\Pi) = \varkappa_{M(G)}(M(G))$ and $M(G)^{**}$ is the direct product of the closed subalgebra $\varkappa_{M(G)}(M(G))$ and the ideal $\varkappa_E(E)^{\perp}$, or*

$$M(G)^{**} = \varkappa_{M(G)}(M(G)) \ltimes \varkappa_E(E)^{\perp}.$$

Consequently,

$$\begin{aligned} M(\widetilde{G}) &= \mathfrak{R}(B) \ltimes (\gamma^*_{E^{**}})^{-1}[\varkappa_E(E)^{\perp}] \\ &= \mathfrak{R}(B) \ltimes \ker(A). \end{aligned}$$

(8) $B \circ A = (\gamma^*_{E^{**}})^{-1} \circ \Pi \circ \gamma^*_{E^{**}}$

Proof (1) As $A = (\gamma_{E^{**}} \circ \varkappa_E)^*$ then A is a weak-* to weak-* continuous linear operator. Let $\widetilde{\mu}, \widetilde{\nu} \in M(\widetilde{G})$. Since $\gamma^*_{E^{**}}$ is a weak-* to weak-* isomorphism between $M(\widetilde{G})$ and $M(G)^{**}$ by Goldstine's theorem there are bounded nets $\{m_i\}_{i\in I}, \{n_j\}_{j\in J}$ in $M(G)$ so that

$$\widetilde{\mu} = w^* - \lim_{i\in I}(\gamma^*_{E^{**}})^{-1}(\varkappa_{M(G)}(m_i)),$$
$$\widetilde{\nu} = w^* - \lim_{j\in J}(\gamma^*_{E^{**}})^{-1}(\varkappa_{M(G)})(n_j)).$$

We observe that $A \circ (\gamma^*_{E^{**}})^{-1} = \varkappa^*_E$ and $\varkappa^*_E \circ \varkappa_{M(G)} = Id_{M(G)}$. So, given $f \in E$ we can write

$$\begin{aligned}
\langle f, A(\widetilde{\mu}) * A(\widetilde{\nu})\rangle &= \lim_{i\in I}\langle f, \varkappa^*_E(\varkappa_{M(G)}(m_i)) * A(\widetilde{\nu})\rangle \\
&= \lim_{i\in I}\lim_{j\in J}\langle f, m_i * \varkappa^*_E(\varkappa_{M(G)}(n_j))\rangle \\
&= \lim_{i\in I}\lim_{j\in J}\langle f, m_i * n_j\rangle \\
&= \langle \varkappa_E(f), \gamma^*_{E^{**}}(\widetilde{\mu})\Box\gamma^*_{E^{**}}(\widetilde{\nu})\rangle \\
&= \langle f, (A \circ (\gamma^*_{E^{**}})^{-1})(\gamma^*_{E^{**}}(\widetilde{\mu})\Box\gamma^*_{E^{**}}(\widetilde{\nu})) \\
&= \langle f, A(\widetilde{\mu} * \widetilde{\nu})\rangle.
\end{aligned}$$

(2) Let $\widetilde{\mu} \in M_{\mathbb{R}}(\widetilde{G})^+$ and $f \in E$. Then

$$\sigma(\varkappa_E(f), E^{**}) \subseteq \sigma(f, E) \tag{2.1}$$

and $A(\widetilde{\mu})(f) = \int_{\widetilde{G}} \gamma_{E^{**}}(\varkappa_E(f))d\widetilde{\mu}$. If $f \in E^+$ its spectrum is contained in $[0, +\infty)$. Besides, the Gélfand transform of $\varkappa_E(f)$ maps onto its spectrum and so $A(\widetilde{\mu})(f) \geq 0$.
More generally, given $\widetilde{\mu} \in M_{\mathbb{R}}(\widetilde{G})$ by be the Jordan decomposition of $\widetilde{\mu}$ we have $\widetilde{\mu} = \widetilde{\mu}^+ - \widetilde{\mu}^-$. Consequently $A(\widetilde{\mu}) = A(\widetilde{\mu}^+) - A(\widetilde{\mu}^-)$ and so $A(\widetilde{\mu})^\pm \leq A(\widetilde{\mu}^\pm)$ (cf. [10], p. 135). Thus

$$| A(\widetilde{\mu}) |= A(\widetilde{\mu})^+ + A(\widetilde{\mu})^- \leq A(\widetilde{\mu}^+) + A(\widetilde{\mu}^-) = A(| \widetilde{\mu} |).$$

(3) Let $\widetilde{\mu} \in M(\widetilde{G})$ and $f \in E^+$. Then

$$| A(\widetilde{\mu}) | (f) = \sup\{| A(\widetilde{\mu})(g) |: g \in E \text{ such that } | g |\leq f\} \tag{2.2}$$

(cf. [7], Th. 14.5). So, given $g \in E$ so that $| g |\leq f$ we see that

$$| A(\widetilde{\mu})(g) |=| \int_{\widetilde{G}} \gamma_{E^{**}}(\varkappa_E(g))\mathrm{d}\widetilde{\mu} |\leq \int_{\widetilde{G}} | \gamma_{E^{**}}(\varkappa_E(g)) | \mathrm{d} | \widetilde{\mu} | . \tag{2.3}$$

Besides

$$| \gamma_{E^{**}}(\varkappa_E(g)) |\leq \gamma_{E^{**}}(\varkappa_E(| g |)) \text{ in } C(\widetilde{G}). \tag{2.4}$$

For, given $M \in \widetilde{G}$ we obtain that

$$\begin{aligned}
| \gamma_{E^{**}}(\varkappa_E(g)) |^2 (M) &= \gamma_{E^{**}}(\varkappa_E(g))(M)\gamma_{E^{**}}(\varkappa_E(g))(M)^- \\
&= M(\varkappa_E(g))M(\varkappa_E(g))^- \\
&= M(\varkappa_E(g))M(\varkappa_E(g)^*) \\
&= M(\varkappa_E(g)\Box\varkappa_E(\overline{g})) \\
&= M(\varkappa_E(| g |^2)) \\
&= M(\varkappa_E(| g |))^2
\end{aligned}$$

and (2.4) follows. Moreover, by (2.1) and (2.3) we get

$$\begin{aligned}| A(\widetilde{\mu})(g) | &\le \int_{\widetilde{G}} \gamma_{E^{**}}(\varkappa_E(| g |))\mathrm{d} | \widetilde{\mu} | \\ &\le \int_{\widetilde{G}} \gamma_{E^{**}}(\varkappa_E(f))\mathrm{d} | \widetilde{\mu} | \\ &= A(| \widetilde{\mu} |)(f)\end{aligned}$$

and the assertion follows by (2.2).

(4) It is immediate because $B = (\gamma_{E^{**}}^{-1})^* \circ \varkappa_{E^*}$, i.e., B is realized as composition of isometries.

(5) It is straightforward.

(6) The linear operator $\varkappa_E^* : M(G)^{**} \to M(G)$ is weak-* to weak-* continuous. Besides the convolution product of $M(G)$ is separately w^*-continuous and $\varkappa_E^*$ becomes a Banach algebra homomorphism with respect to both Arens products on $M(G)^{**}$. The assertion fo-llows because $\varkappa_{E^*}$ is an algebra homomorphism between $M(G)$ and $M(G)^{**}$.

(7) It is easy to see that Π is a projection, $\ker(\Pi) = \varkappa_E(E)^{\perp}$, $\mathfrak{R}(\Pi)$ is closed and it consists of w^*-continuous linear functionals. Besides $\Pi \,|_{\varkappa_{M(G)(M(G))}} = \mathrm{Id}_{\varkappa_{M(G)}(M(G))}$ and the first assertion holds.
If $M \in M(G)^{**}$ then

$$M = (M - \Pi(M)) + \Pi(M)$$

and $M - \Pi(M) \in \varkappa_E(E)^{\perp}$, i.e., $M(G)^{**} = \varkappa_E(E)^{\perp} + \mathfrak{R}(\Pi)$.
Let $N \in M(G)^{**}$ so that $\Pi(N) \in \varkappa_E(E)^{\perp}$. For all $f \in E$ we have

$$0 = \langle \varkappa_E(f), \Pi(N)\rangle = \langle \varkappa_E(f), \varkappa_{E^*}(\varkappa_E^*(N))\rangle = \langle \varkappa_E(f), N\rangle,$$

i.e., $N \in \varkappa_E(E)^{\perp}$. Consequently $\Pi(N) = 0_{M(G)^{**}}$ and we can write $M(G)^{**} = \varkappa_E(E)^{\perp} \bigoplus \mathfrak{R}(\Pi)$. Since Π is an algebra homomorphism the claim follows.

(8) It is immediate.

$\square$

Theorem 2.2 *Let $\mathfrak{R}_G$ be the commutator in $\mathcal{B}[M(G)]$ of the subalgebra generated by the set $\{P_B : B \in \mathfrak{B}(G)\}$, where $P_B(m) = I_B\mathrm{d}m$ for each $m \in M(G)$ and $B \in \mathfrak{B}(G)$.*

(1) *$\mathfrak{R}_G$ is an abelian B^*-subalgebra of $\mathcal{B}[M(G)]$.*

(2) *Considering $M(G)^*$ as the second dual space of E endowed with the first or second Arens product the following mapping establishes an isometric isomorphism of B^*-algebras where $\Phi \in M(G)^*$ and $B \in \mathfrak{B}(G)$.*

Proof (1) Given $T \in \mathcal{B}[M(G)]$ let $T^*(m) = T(m^*)^*$ for all $m \in M(G)$. It is easily seen that $T^* \in \mathcal{B}[M(G)]$ and $\mathcal{B}[M(G)]$ becomes a Banach *-algebra. Indeed, $\mathfrak{R}_G$ is a Banach *-subalgebra of $\mathcal{B}[M(G)]$. For, let $B \in \mathfrak{B}(G)$, $m \in M(G)$, $f \in E$. Then

$$\langle f, P_B^*(m)\rangle = \langle f^*, I_B m^*\rangle^- = \left(\int_B f^*\mathrm{d}m^*\right)^- = \int_B f\mathrm{d}m = \langle f, P_B(m)\rangle$$

i.e., $P_B^* = P_B$. Consequently, given $T \in \mathfrak{R}_G$ we see that

$$T^* \circ P_B = (T \circ P_B^*)^* = (T \circ P_B)^* = (P_B \circ T)^* = P_B^* \circ T^* = P_B \circ T^*.$$

By linearity we conclude that $T^* \in \mathfrak{R}_G$. The abelianity of $\mathfrak{R}_G$ is consequence of [13], Th. 1.5.

(2) It is straightforward to see that $\beth \in \mathcal{B}[M(G)^*, \mathfrak{R}_G]$. If $\Phi \in M(G)$ we have

$$\begin{aligned}\| \beth(\Phi) \| &= \sup_{\|m\|=1} \| \beth(\Phi)(m) \| \\ &= \sup_{\|m\|=1} \sup\left\{\sum_{B\in\pi} |\langle I_B\mathrm{d}m, \Phi\rangle| : \pi \in \mathcal{P}\right\}\end{aligned}$$

where $\mathcal{P}$ denote the class of finite disjoint borelian partitions of G. By linearity, given $\pi \in \mathcal{P}$ and $m \in [M(G)]_1$ there is $h_\pi \in [\mathfrak{B}_b(G)]_1$ so that

$$\sum_{B\in\pi} |\langle I_B \mathrm{d}m, \Phi\rangle| = \langle h_\pi \mathrm{d}m, \Phi\rangle.$$

As $\| h_\pi \mathrm{d}m \| \leq 1$ we see that $\| \beth(\Phi) \| \leq \| \Phi \|$. Now, if $\epsilon > 0$ there exists $n \in [M(G)]_1$ so that

$$\| \Phi \| - \epsilon < |\langle n, \Phi\rangle| = |\beth(\Phi)(n)(G)| \leq \| \beth(\Phi)(n) \|$$

and we showed that $\beth$ is isometric.
Now we prove that $\beth$ is onto. For, given $T \in \mathfrak{R}_G$ let $\Psi : M(G) \to \mathbb{C}$ so that $\langle p, \Psi\rangle = T(p)(G)$ if $p \in M(G)$. Clearly, $\Psi \in M(G)^*$ and given $B \in \mathfrak{B}(G)$ and $p \in M(G)$ we have

$$\beth(\Psi)(p)(B) = \langle I_B dp, \Psi\rangle = T(P_B(p))(G) = P_B(T(p))(G) = T(p)(B)$$

and so $\beth(\Psi) = T$.

Now, given $\Xi \in M(G)^*$, $n \in M(G)$ then $\beth(\Xi)(n) = n\Xi$ in $M(G)$. For, let $\Xi = w^* - \lim_{s\in S} \varkappa_E(g_s)$ for some bounded net $\{g_s\}_{s\in S}$ in E. Let us observe that given a simple Borel function $\mathfrak{f} = \sum_{j=1}^n z_j I_{B_j}$ we can write

$$\begin{aligned}\langle \mathfrak{f}, \beth(\Xi)(n)\rangle &= \sum_{j=1}^n z_j \langle I_{B_j}, \beth(\Xi)(n)\rangle = \sum_{j=1}^n z_j \beth(\Xi)(n)(B_j)\\ &= \sum_{j=1}^n z_j\, \Xi(I_{B_j} n) = \lim_{s\in S} \sum_{j=1}^n z_j \int_{B_j} g_s \mathrm{d}n = \lim_{s\in S} \int_G \mathfrak{f} g_s \mathrm{d}n.\end{aligned}$$

Consequently, let $f \in E$. It can be represented as the uniform limit of a bounded sequence of simple Borel functions $\{\mathfrak{f}_k\}_{k\in\mathbb{N}}$ (cf. [6], §20. Th. B). By Lebesgue dominated convergence theorem and the Arens regularity of E, we have

$$\begin{aligned}\langle f, \beth(\Xi)(n)\rangle &= \int_G f \mathrm{d}\beth(\Xi)(n)\\ &= \lim_{k\to\infty} \int_G \mathfrak{f}_k \mathrm{d}\beth(\Xi)(n)\\ &= \lim_{k\to\infty} \lim_{s\in S} \int_G \mathfrak{f}_k g_s \mathrm{d}n\\ &= \langle n, \varkappa_E(f)\Box\Xi\rangle\\ &= \langle n, \varkappa_E(f)\Diamond\Xi\rangle\\ &= \lim_{s\in S} \lim_{k\to\infty} \int_G \mathfrak{f}_k g_s \mathrm{d}n\\ &= \lim_{s\in S} \langle g_s, fn\rangle\\ &= \langle fn, \Xi\rangle\\ &= \langle f, n\Xi\rangle\end{aligned}$$

and the claim holds.

Let $\Phi_1, \Phi_2 \in M(G)^*$, $p \in M(G)$, $g \in E$. We have

$$\begin{aligned}\langle g, \beth(\Phi_1\Box\Phi_2)(p)\rangle &= \langle g, p(\Phi_1\Box\Phi_2)\rangle\\ &= \langle gp, \Phi_1\Box\Phi_2\rangle\\ &= \langle gp, \Phi_1\Diamond\Phi_2\rangle\\ &= \langle (gp)\Phi_1, \Phi_2\rangle\\ &= \langle g(p\Phi_1), \Phi_2\rangle\\ &= \langle g, (p\Phi_1)\Phi_2\rangle\\ &= \langle g, \beth(\Phi_2)[\beth(\phi_1)(p)]\rangle\\ &= \langle g, (\beth(\Phi_2)\circ\beth(\phi_1))(p)\rangle,\end{aligned}$$

i.e., $\daleth(\Phi_1\Box\Phi_2) = \daleth(\Phi_2)\circ\daleth(\phi_1)$. Since E^{**} is abelian we conclude that $\daleth$ is a homomorphism.

With the above notation, let $\Xi = w^* - \lim_{s\in S}\varkappa_E(g_s)$ in $M(G)^*$. Then $\Xi^* = w^* - \lim_{s\in S}\varkappa_E(g_s^*)$ and given $m \in M(G)$ and $f \in E$ we have

$$\begin{aligned}\langle f, \daleth(\Xi^*)(m)\rangle &= \lim_{s\in S}\int_G f g_s^* \mathrm{d}m\\ &= \lim_{s\in S}\langle f g_s^*, m\rangle\\ &= \lim_{s\in S}\langle f^* g_s, m^*\rangle^-\\ &= \lim_{s\in S}\left(\int_G f^* g_s \mathrm{d}m^*\right)^-\\ &= \langle f^*, \daleth(\Xi)(m^*)\rangle^-\\ &= \langle f, \daleth(\Xi)(m^*)^*\rangle\\ &= \langle f, \daleth(\Xi)^*(m)\rangle\end{aligned}$$

and $\daleth$ is a *-homomorphism. □

Definition 2.3 A closed subspace $\mathfrak{L}$ of $M(G)$ is called an L-space if whenever μ is in $\mathfrak{L}$ and ν is μ-continuous then $\nu \in \mathfrak{L}$.

Definition 2.4 Let X_1, X_2 be compact spaces. A bounded linear operator Θ between $M(X_1)$ and $M(X_2)$ is called L-homomorphism if the following three conditions hold:

(a) If $m \in M(X_1)$ then $\Theta(m)(X_2) = m(X_1)$.
(b) $\Theta[M(X_1)^+] \subseteq M(X_2)^+$.
(c) If $m_1 \in M(X_1)^+$, $n \in M(X_2)$ and $0 \le n \le \Theta(m_1)$ then there exists $m_2 \in M(X_1)$ so that $0 \le m_2 \le m_1$ and $\Theta(m_2) = n$.

Theorem 2.5 *With the above notation the following assertions hold:*

(1) *Let X_G be the structure space of $\mathfrak{R}_G$. There are an isometric isomorphism of Banach -$*$-algebras $\Lambda : M(G) \to M(X_G)$ and an isometric surjective isomorphism of C^*-algebras $\Sigma : C(\widetilde{G}) \to C(X_G)$ so that $B = \Sigma^* \circ \Lambda$ and $\gamma_{E^{**}} = \Sigma^{-1}\circ\gamma_{\mathfrak{R}_G}\circ\daleth$, where $\gamma_{\mathfrak{R}_G}$ denotes the Gélfand transform of $\mathfrak{R}_G$.*
(2) *(cf. [13], Th. 1.8) Given $F \in C(X_G)$ and $m \in M(G)$ then*

$$\daleth^{-1}[\gamma_{\mathfrak{R}_G}^{-1}(F)](m) = \int_{X_G} F\mathrm{d}\Lambda(m) = \gamma_{\mathfrak{R}_G}^{-1}(F)(m)(G).$$

(3) *Σ^* is an L-homomorphism.*

Proof (1) Let $\Sigma : C(\widetilde{G}) \to C(X_G)$ so that $\Sigma = \gamma_{\mathfrak{R}_G}\circ\daleth\circ\gamma_{E^{**}}^{-1}$. Besides, let $\Lambda : M(G) \to M(X_G)$ so that

$$\langle F, \Lambda(m)\rangle = \langle m, \daleth^{-1}(\gamma_{\mathfrak{R}_G}^{-1})(F)\rangle \text{ if } m \in M(G) \text{ and } F \in C(X_G).$$

Plainly Σ is an isometric surjective isomorphism of C^*-algebras and Λ is a linear isometry. Given $\widetilde{f} \in C(\widetilde{G})$ and $m \in M(G)$ we obtain

$$\begin{aligned}\langle \widetilde{f}, (\Sigma^*\circ\Lambda)(m)\rangle &= \langle \Sigma(\widetilde{f}), \Lambda(m)\rangle\\ &= \langle m, (\daleth^{-1}\circ\gamma_{\mathfrak{R}_G}^{-1}\circ\Sigma)(\widetilde{f})\rangle\\ &= \langle m, \gamma_{E^{**}}^{-1}(\widetilde{f})\rangle\\ &= \langle \widetilde{f}, B(m)\rangle\end{aligned}$$

and so $B = \Sigma^*\circ\Lambda$.
(2) It is immediate.
(3) Notice that Σ is a *-homomorphism between $C(\widetilde{G})$ and $C(X_G)$ and $\Sigma(1_{C(\widetilde{G})}) = 1_{C(X_G)}$. Consequently Σ^* becomes an L-homomorphism (cf. [13], Th. 1.14). □

Remark 2.6 The image of the isometric -*-isomorphism Λ between $M(G)$ and $M(X_G)$ is an L-subspace of $M(X_G)$. Moreover, it preserves the absolute value of measures and the properties of continuity and singularity. In other words, let $m, n \in M(G)$. Then $|\ \Lambda(m)\ | = \Lambda(|\ m\ |)$, if $m << n$ then $\Lambda(m) << \Lambda(n)$ and if $m \perp n$ then $\Lambda(m) \perp \Lambda(n)$ (cf. [13], Th. 1.10). As we saw before B factors as the composition of Λ and an L-homomorphism Σ^* between $M(X_G)$ and $M(\widetilde{G})$. We cannot go any further in connection with similar properties of B, even when the compact spaces X_G and $\widetilde{G}$ are homeo-morphic and when Σ^* defines an L-homomorphism.

3 The support problem

Let us recall that the support supp(m) of a measure m on G is the complement of the union of all m-zero open subsets of G. It is clear that the support of m has empty intersection with an open set O if and only if $m(O) = 0$ (cf. [7], Th. 11.25). If $m \in M(G)$ we write supp(m) =supp$(|m|)$. In the general case, an element $a \in G$ belongs to the support of a measure m if and only if any of the following equivalent conditions hold: (i) each open neighborhood of a has $|m|$-positive measure. (ii) $\int_G |f|d|m| > 0$ if $f \in C_{00}(G)$ and $f(a) \neq 0$. (iii) For each open neighborhood U of a there is $g \in E$ with compact support contained in U so that $\int_G g\mathrm{d}m \neq 0$ (cf. [4], Ch. 13, §19).

Provided with the uniform norm $\mathfrak{B}_b(G)$ is a B*-algebra. Let us write $\urcorner : \mathfrak{B}_b(G) \to M(G)^*$ so that $\urcorner(F)(m) = \int_G F\mathrm{d}m$ if $F \in \mathfrak{B}_b(G)$ and $m \in M(G)$. Hence $\urcorner$ defines an isometric immersion of B*-algebras. Given $a, b \in G$ we see that

$$\langle \urcorner(I_{\{b\}}), \mathfrak{h}(a)\rangle = \urcorner(I_{\{b\}})(\delta(a)) = \delta_{a,b}.$$

Thus $\{M \in \widetilde{G} :|\ \langle \urcorner(I_{\{a\}}), M - \mathfrak{h}(a)\ | < 1\} \cap \mathfrak{h}(G) = \{\mathfrak{h}(a)\}$, i.e., $\mathfrak{h}(G)$ is a discrete subset of $\mathfrak{h}(G)$. Moreover, $\mathfrak{h}(G)$ consists of the whole set of isolated points of $\widetilde{G}$ (cf. [3], Corollary 4.2).

Proposition 3.1 *Let $m \in M(G)$, $M \in \widetilde{G}$. Then $M \notin$ supp(B(m)) if and only if there exists $\Phi \in M(G)^*$ so that*

$$\langle \Phi, M\rangle = 1 \text{ and } supp(B(m)) \cap supp(\gamma_{M(G)^*}(\Phi)) = \emptyset. \tag{3.1}$$

Proof Let $M \in \widetilde{G} -$ supp$(B(m))$. Since $\widetilde{G}$ is compact and separate there is an open neighborhood $\widetilde{U}$ of M in $\widetilde{G}$ so that $\widetilde{U}^{-w^*} \subseteq \widetilde{G} -$ supp(B(m)) (cf. [10], Th. 2.7). Further, there exists a continuous function $\widetilde{f} : \widetilde{G} \to [0, 1]$ so that $\widetilde{f}(M) = 1$ and supp$(\widetilde{f}) \subseteq \widetilde{U}$. Then, $\gamma^{-1}_{E^{**}}(\widetilde{f}) \in M(G)^*$ and

$$\langle \gamma^{-1}_{E^{**}}(\widetilde{f}), M\rangle = \widetilde{f}(M) = 1.$$

On the other hand, let $\Phi \in M(G)^*$ that satisfies (3.1). The complex function $\widetilde{g} = \gamma_{E^{**}}(\Phi)$ is continuous on $\widetilde{G}$ and

$$\widetilde{g}(M) = \langle \Phi, M\rangle = 1.$$

Thus $M \in \{\widetilde{g} \neq 0\}$ and $\{\widetilde{g} \neq 0\}$ is an open neighborhood of M whose w^*- closure does not intersect the support of $B(m)$. Moreover, given $\widetilde{h} \in C(\widetilde{G})$ so that supp$(\widetilde{h}) \subseteq \{\widetilde{g} \neq 0\}$ we have

$$B(m)(\widetilde{h}) = \int_{\widetilde{G}} \widetilde{h}\mathrm{d}B(m) = 0,$$

i.e., $M \notin$ supp$(B(m))$. □

Theorem 3.2 *The following assertions hold:*

(1) *Let $\widetilde{\mu} \in M(\widetilde{G})$. There are unique $m \in M(G)$ and $P \in \varkappa_E(E)^\perp$ so that*

$$\widetilde{\mu} = (\gamma^*_{E^{**}})^{-1}(\varkappa_{M(G)}(m)) + (\gamma^*_{E^{**}})^{-1}(P)$$

and supp$(A(\widetilde{\mu}))$ = supp(m).

(2) *Let $m \in M(G) - \{0\}$. Then $\mathfrak{h}^{-1}$(supp$(B(m)) \subseteq$ supp(m).*

(3) *Furthermore, $\mathfrak{h}^{-1}$(supp$(B(m)) = \{b \in G : m(\{b\}) \neq 0\}$.*

Proof (1) From Theorem 2.1(2.1) we see that

$$M(\widetilde{G}) = (\gamma^*_{E^{**}})^{-1}(\varkappa_{M(G)}(M(G)) \oplus (\gamma^*_{E^{**}})^{-1}(\varkappa_E(E)^{\perp})$$

and the first assertion follows. Given $P \in \varkappa_E(E)^{\perp}$ and $f \in E$ we can write

$$\begin{aligned} A((\gamma^*_{E^{**}})^{-1}(P))(f) &= (\gamma_{E^{**}} \circ \varkappa_E)^*((\gamma^*_{E^{**}})^{-1}(P))(f) \\ &= \kappa^*_E(P)(f) \\ &= \langle \kappa_E(f), P\rangle \\ &= 0, \end{aligned}$$

i.e., $(\gamma^*_{E^{**}})^{-1}(\varkappa_E(E)^{\perp}) \subseteq \ker(A)$. Besides, if $m \in (G)$ we have that

$$\begin{aligned} A((\gamma^*_{E^{**}})^{-1}(\varkappa_{M(G)}(m))(f) &= (\gamma_{E^{**}} \circ \varkappa_E)^*((\gamma^*_{E^{**}})^{-1}(\varkappa_{M(G)}(m))(f) \\ &= \varkappa^*_E(\varkappa_{M(G)}(m))(f) \\ &= \langle \varkappa_E(f), \varkappa_{M(G)}(m)\rangle \\ &= \langle f, m\rangle, \end{aligned}$$

i.e., $A((\gamma^*_{E^{**}})^{-1}(\varkappa_{M(G)}(m)) = m$ and the claim holds.

(2) Given $m \in M(G)$ and $a \in \mathfrak{h}^{-1}(\mathrm{supp}(B(m))$ let $\Phi_a = \gamma^{-1}_{E^{**}}(I_{\{\mathfrak{h}(a)\}})$.
By Goldstine's theorem there is a bounded net $\{f_i\}_{i\in I}$ in E so that $\Phi_a = w^* - \lim_{i\in I} \varkappa_E(f_i)$. We see that

$$1 = I_{\{\mathfrak{h}(a)\}}(\mathfrak{h}(a)) = \gamma_{E^{**}}(\Phi_a)(\mathfrak{h}(a)) = \mathfrak{h}(a)(\Phi_a) = \lim_{i\in I} f_i(a).$$

Besides, there exists $\epsilon > 0$ so that

$$\lim_{i\in I} \left| \int_G f_i \mathrm{d}m \right| = |\langle m, \Phi_a\rangle| = |B(m)(I_{\{\mathfrak{h}(a)\}})| > 2\epsilon$$

because $\mathfrak{h}(a) \in \mathrm{supp}(B(m))$. Let $i_0 \in I$ so that $|\int_G f_i \mathrm{d}m| > \epsilon$ if $i \geq i_0$. Let U be a relatively compact neighborhood of a in G and let us consider a fixed $i \in I$, $i \geq i_0$. Let V be an open set so that $a \in V$ and $V^- \subseteq U$ (cf. [10], Th. 2.7). By Urysohn's Lemma there exists a continuous function $g_U : G \to [0, 1]$ such that $g_U(V^-) = \{1\}$ and $g_U(G - U) = \{0\}$. Both sets $\mathrm{supp}(g_U)$ and $K_i = \{b \in V^- : |f_i(b)| \geq \epsilon/(2 \parallel m \parallel)\}$ become compact. We can write

$$\begin{aligned} \left|\int_G f_i \mathrm{d}m\right| - \left|\int_G f_i g_U \mathrm{d}m\right| &\leq \left|\int_G (f_i - f_i g_U)\mathrm{d}m\right| \\ &\leq \left|\int_{G-K_i} (f_i - f_i g_U)\mathrm{d}m\right| + \left|\int_{K_i} (f_i - f_i g_U)\mathrm{d}m\right| \\ &\leq \int_{G-K_i} |f_i|(1 - g_U)\mathrm{d}|m| \\ &\leq \frac{\epsilon}{2 \parallel m \parallel} \int_{G-K_i} (1 - g_U)\mathrm{d}|m| \\ &\leq \epsilon/2 \end{aligned}$$

Thus

$$\int_G |f_i g_U| \mathrm{d}m \geq \left| \int_G f_i g_U \mathrm{d}m \right| \geq \left| \int_G f_i \mathrm{d}m \right| - \epsilon/2 > \epsilon/2$$

and $|f_i g_U|$ is a non-negative continuous function with compact support contained in U. Since the class of open relatively compact sets is a basis of the topology of G we conclude that $a \in \mathrm{supp}(m)$.

(3) Given $b \in \operatorname{supp}(m)$ then $\gamma_{E^{**}}(\urcorner(I_{\{b\}})) = I_{\{\mathfrak{h}(b)\}}$. For,

$$\gamma_{E^{**}}(\urcorner(I_{\{b\}}))(\mathfrak{h}(b)) = \langle \urcorner(I_{\{b\}}), \mathfrak{h}(b)\rangle = 1.$$

As $\urcorner(I_{\{b\}})$ is idempotent in $(E^{**}, \Box)$, $M(\urcorner(I_{\{b\}}) \in \{0, 1\}$ if $M \in \widetilde{G}$. Let us suppose that there is some $M \in \widetilde{G}$ so that $M(\urcorner(\{I_{\{b\}}\})) = 1$ but $M \neq \mathfrak{h}(b)$. Let us choose $\Phi \in M(G)^*$ so that $M(\Phi) \neq \mathfrak{h}(b)(\Phi)$. Given $n \in M(G)$ we have that $\urcorner(I_{\{b\}})n = \langle n, \urcorner(I_{\{b\}})\rangle\delta(b)$ and so

$$\langle \urcorner n, \Phi\Box\urcorner(I_{\{b\}})\rangle = \langle\langle n, \urcorner(I_{\{b\}})\rangle\delta(b), \Phi\rangle = \langle\delta(b), \Phi\rangle\langle n, \urcorner(I_{\{b\}})\rangle,$$

i.e., $\Phi\Box\urcorner(I_{\{b\}}) = \mathfrak{h}(b)(\Phi)\urcorner(I_{\{b\}})$. But M is a multiplicative and so

$$M(\Phi) = M[\Phi\Box\urcorner(I_{\{b\}})] = \mathfrak{h}(b)(\Phi)$$

and we get a contradiction. So, if $M \neq \mathfrak{h}(b)$ then $M(\urcorner(\{I_{\{b\}}\})) = 0$ and the claim follows.
Therefore, the conclusion follows from the identity

$$B(m)(I_{\{\mathfrak{h}(b)\}}) = \langle m, \urcorner(I_{\{b\}})\rangle = \langle\{b\}, m\rangle.$$

□

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