RESEARCH ARTICLE

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On a class of Kirchhoff equations involving an anisotropic operator and potential

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Abstract In this work, we are concerned with a class of fractional equations of Kirchhoff type with potential. Using variational methods and a variant of quantitative deformation lemma, we prove the existence of a least energy sign-changing solution. Moreover, the existence of infinitely many solution is established.

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1 Introduction and main result

Consider the following fractional Kirchhoff equation

$$(1 + b[u]_{\alpha}^2) ((-\Delta_x)^{\alpha} u - \Delta_y u) + V(x, y)u = f(u), \quad (x, y) \in \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m, \quad (1.1)$$

where $[u]_{\alpha} = \left(\int_{\mathbb{R}^N} \left(|(-\Delta_x)^{\frac{\alpha}{2}} u|^2 + |\nabla_y u|^2 \right) dx dy \right)^{\frac{1}{2}}$, $\alpha \in (0, 1)$, $n, m \geq 1$, $(-\Delta_x)^{\alpha}$ denotes the fractional Laplacian in x which is defined via the Fourier transform by $\widehat{(-\Delta_x)^{\alpha} u}(X, Y) = |X|^{2\alpha} \widehat{u}(X, Y)$, where $\widehat{u}$ is the Fourier transform of u ; if u is sufficiently smooth, it can be expressed by

$$(-\Delta_x)^{\alpha} u(x, y) = C_{N,\alpha} \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x, y) - u(z, y)}{|x - z|^{n+2\alpha}} dz, \quad (x, y) \in \mathbb{R}^N,$$

with $C_{N,\alpha}$ is a normalization constant and P.V. stands for the Cauchy principal value; and V, f are functions satisfying some conditions which will be specified later.

The presence of the nonlocal term $[u]_{\alpha}^2$ in (1.1) causes some mathematical difficulties and so the study of such a class of equations is of much interest. Moreover, Eq. (1.1) is a fractional version related to the following hyperbolic equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \quad (1.2)$$

which was proposed by Kirchhoff [22] as an extension of the classical D'Alembert's wave equation by considering the changes in the length of the strings produced by transverse vibrations.

On the other hand, when $b = 0$, $V \equiv 1$ and $f(u) = u^p$, equation (1.1) appears in the study of solitary waves of the generalized Benjamin-Ono-Zakharov-Kuznetsov equation

$$u_t + \partial_{x_1} ((-\Delta_x)^{\alpha} u - \Delta_y u + u^p) = 0 \quad (x, y) \in \mathbb{R}^n \times \mathbb{R}^m, \quad (1.3)$$

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see [28] for some local and global well-posedness results on this equation with $m = n = 1$. The anisotropic operator $(-\Delta_x)^\alpha u - \Delta_y u$ is observed in the study of toy models parabolic equations for which local diffusions occur only in certain directions and nonlocal diffusions. It models diffusion sensible to the direction in the Brownian and Lévy-Itô processes. For some regularity and rigidity properties of this operator, the readers can refer to [6, 17].

Recently, Esfahani [15] considered Eq. (1.1) with $b = 0$. Under suitable assumptions on $f \in C^1(\mathbb{R}, \mathbb{R})$, by adapting some arguments developed in [7–9] and using a variant of deformation lemma, the author shows that the equation admits a least energy sign-changing solution.

It is interesting to note that the fractional Laplacian (isotropic operator) problems have attracted a great attention by various study on the existence and multiplicity of solutions; see, e.g., [19, 20, 25, 33]. This interest comes from its significant applications in several areas such as physics, biology, chemistry and finance; see, e.g., [5, 11, 12, 18, 23]. We also refer to [10] for more applications specifically to an audience of mathematicians. We point out that the variational methods leading to existence theories in the fractional setting were introduced in [30].

In [18], Fiscella and Valdinoci first introduced a stationary Kirchhoff variational equation, which models the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string. After that, the fractional Kirchhoff problems have been investigated by many researchers, we cite here [2, 3, 13, 14, 21, 26, 27, 31]. However, to the best of our knowledge, there are no result about existence of solutions for a Kirchhoff-type equation with an operator containing local and nonlocal diffusions except [24]. Inspired by the results mentioned above, especially [1, 4, 15, 21], in the present paper, we will prove the existence of sign-changing solutions and infinitely many solutions for Eq. (1.1). To this aim, we need a compact embedding which is appropriate to our equation and some technical lemmas related to the minimization method on the nodal Nehari manifold.

We assume that $V : \mathbb{R}^N \rightarrow \mathbb{R}$ is continuous and satisfies:

(V₀) $V(x, y) > 0$ for all $(x, y) \in \mathbb{R}^N$ and $V^{-1} \in L^\infty(\mathbb{R}^N)$.

For the nonlinearity f , we assume that $f \in C^1(\mathbb{R}, \mathbb{R})$ and

- (f₁) $\lim_{t \rightarrow 0} \frac{f(t)}{t} = 0$;
- (f₂) $\lim_{|t| \rightarrow \infty} \frac{f(t)}{|t|^{p^*-1}} = 0$, where $p^* = \frac{2(n+m\alpha)}{n+(m-2)\alpha}$;
- (f₃) $\lim_{|t| \rightarrow \infty} \frac{F(t)}{t^4} = +\infty$, where $F(t) = \int_0^t f(\tau) d\tau \geq 0$ for all $t \in \mathbb{R}$;
- (f₄) the function $t \mapsto \frac{f(t)}{|t|^3}$ is increasing on $\mathbb{R} \setminus \{0\}$.

Before stating what we think our main result, we recall our working Sobolev space. Let $\mathcal{H}^\alpha(\mathbb{R}^N)$ be the fractional Sobolev–Liouville space

$$\mathcal{H}^\alpha(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} \left(|(-\Delta_x)^{\frac{\alpha}{2}} u|^2 + |\nabla_y u|^2 + u^2 \right) dx dy < \infty \right\}$$

with the norm

$$\|u\|_{\mathcal{H}^\alpha} = \left(\int_{\mathbb{R}^N} \left(|(-\Delta_x)^{\frac{\alpha}{2}} u|^2 + |\nabla_y u|^2 + u^2 \right) dx dy \right)^{\frac{1}{2}}.$$

The homogeneous fractional Sobolev–Liouville space $\mathcal{D}^\alpha(\mathbb{R}^N)$ is defined by

$$\mathcal{D}^\alpha(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : [u]_\alpha < \infty\},$$

where

$$[u]_\alpha = \left(\int_{\mathbb{R}^N} \left(|(-\Delta_x)^{\frac{\alpha}{2}} u|^2 + |\nabla_y u|^2 \right) dx dy \right)^{\frac{1}{2}}.$$

We search the solutions in the following subspace

$$E_\alpha = \left\{ u \in \mathcal{D}^\alpha(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x, y) dx dy < \infty \right\},$$

endowed with the inner product

$$(u, v)_{E_\alpha} = \int_{\mathbb{R}^N} \left((-\Delta_x)^{\frac{\alpha}{2}} u (-\Delta_x)^{\frac{\alpha}{2}} v + \nabla_y u \nabla_y v \right) dx dy + \int_{\mathbb{R}^N} V(x, y) u v dx dy$$

and its related norm

$$\|u\|_{E_\alpha} = (u, u)_{E_\alpha}^{\frac{1}{2}}.$$

A function $u \in E_\alpha$ is called weak solution of (1.1) if

$$\begin{aligned} (1 + b[u]_\alpha^2) \int_{\mathbb{R}^N} \left((-\Delta_x)^{\frac{\alpha}{2}} u (-\Delta_x)^{\frac{\alpha}{2}} v + \nabla_y u \nabla_y v \right) dx dy + \int_{\mathbb{R}^N} V(x, y) u v dx dy \\ = \int_{\mathbb{R}^N} f(u) v dx dy, \text{ for all } v \in E_\alpha. \end{aligned}$$

A sign-changing solution of (1.1) is a weak solution $u \in E_\alpha$ satisfying $u^\pm \neq 0$, where

$$u^+ = \max(u, 0) \text{ and } u^- = \min(u, 0).$$

The main result can be described by the following theorem.

Theorem 1.1 Assume that (V_0) and (f_1) – (f_4) hold. Then, Eq. (1.1) admits a least energy sign-changing solution. Furthermore, if f is odd, then (1.1) has infinitely many solutions.

Remark 1.2 This result is known for the isotropic case with $N = 3$ and the nonlinear term $f(x, u) = K(x)f(u)$, please refer to [21].

2 Auxiliary results and proof of main theorem

In this section, we give some preliminary results for the proof of Theorem 1.1.

Lemma 2.1 (Cf.[16]) The fractional Sobolev–Liouville space $\mathcal{H}^\alpha(\mathbb{R}^N)$ is continuously embedded into $L^q(\mathbb{R}^N)$ for $q \in [2, p^*]$; and is compactly embedded into $L^q(\mathbb{R}^N)$ for $q \in (2, p^*)$.

According to Lemma 2.1, we have the next result (see [15, Lemma 2.1]).

Lemma 2.2 (Cf.[15]) Assume that (V_0) holds. Then, E_α is continuously embedded into $L^q(\mathbb{R}^N)$ for $q \in [2, p^*]$; and compactly embedded into $L^q(\mathbb{R}^N)$ for $q \in (2, p^*)$;

Obviously, the energy functional associated to (1.1) defined on E_α by

$$I(u) = \frac{1}{2} \|u\|_{E_\alpha}^2 + \frac{b}{4} [u]_\alpha^4 - \int_{\mathbb{R}^N} F(u) dx dy,$$

is of class C^1 and its critical points are solutions of (1.1).

To discuss the existence of sign-changing solutions, we follow some arguments developed in [7–9, 32]. More precisely, we will minimize the functional I on the nodal set

$$\mathcal{M} = \{u \in \mathcal{N} : u^\pm \neq 0, \langle I'(u), u^+ \rangle = \langle I'(u), u^- \rangle = 0\},$$

where

$$\mathcal{N} = \{u \in E_\alpha \setminus \{0\} : \langle I'(u), u \rangle = 0\}.$$

Denote $S_1 = \{u \in E_\alpha : \|u\|_{E_\alpha} = 1\}$. For all fixed $u \in E_\alpha \setminus \{0\}$, we set

$$g_u(t) = I(tu) \text{ for all } t \geq 0.$$

Lemma 2.3 Assume that (V_0) and (f_1) – (f_4) hold. Then

(i) *there exists an unique $t_u > 0$ such that*

$$g'_u(t_u) = 0, \quad g'_u > 0 \text{ on } (0, t_u) \quad \text{and} \quad g'_u < 0 \text{ on } (t_u, \infty);$$

(ii) $\rho := \inf_{u \in S_1} g_u(t_u) > 0$. Furthermore, for each compact $\Lambda \subset S_1$, there exists $C_\Lambda > 0$ such that

$$t_u \leq C_\Lambda \text{ for all } u \in \Lambda;$$

(iii) let $\bar{\Phi} : E_\alpha \setminus \{0\} \rightarrow \mathcal{N}$ and $\Phi : S_1 \rightarrow \mathcal{N}$ be the mappings defined by

$$\bar{\Phi}(u) = t_u u \quad \text{and} \quad \Phi = \bar{\Phi}|_{S_1}.$$

Then, $\bar{\Phi}$ is continuous and Φ is a homeomorphism. Moreover, the inverse of Φ is given by $\Phi^{-1}(u) = \frac{u}{\|u\|_{E_\alpha}}$.

Proof (i) By (f₁)–(f₂), for given $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$F(t) \leq \varepsilon t^2 + C_\varepsilon |t|^{p^*} \text{ for all } t \in \mathbb{R}.$$

Therefore, by Lemma 2.2, for all $t > 0$,

$$\begin{aligned} I(tu) &\geq \frac{t^2}{2} \|u\|_{E_\alpha}^2 - \int_{\mathbb{R}^N} F(tu) dx dy \\ &\geq \frac{t^2}{2} \|u\|_{E_\alpha}^2 - \varepsilon t^2 \int_{\mathbb{R}^N} u^2 dx dy - C_\varepsilon t^{p^*} \int_{\mathbb{R}^N} |u|^{p^*} dx dy \\ &\geq \left(\frac{1}{2} - \varepsilon \|V^{-1}\|_\infty \right) t^2 \|u\|_{E_\alpha}^2 - C C_\varepsilon t^{p^*} \|u\|_{E_\alpha}^{p^*}. \end{aligned} \quad (2.1)$$

Choosing $\varepsilon < \frac{1}{2\|V^{-1}\|_\infty}$, for t_* small enough,

$$I(tu) > 0 \text{ for all } 0 < t < t_*. \quad (2.2)$$

Let $\Omega_0 = \{(x, y) \in \mathbb{R}^N : u(x, y) \neq 0\}$, we have

$$I(tu) \leq \frac{t^2}{2} \|u\|_{E_\alpha}^2 + \frac{bt^4}{4} \|u\|_{E_\alpha}^4 - \int_{\Omega_0} F(tu) dx dy.$$

It follows from Fatou's lemma and (f₃),

$$\limsup_{t \rightarrow \infty} \frac{I(tu)}{\|tu\|_{E_\alpha}^4} \leq \frac{b}{4} - \int_{\Omega_0} \liminf_{t \rightarrow \infty} \frac{F(tu)}{(tu)^4} \frac{u^4}{\|u\|_{E_\alpha}^4} dx dy = -\infty. \quad (2.3)$$

Hence, for some $T > 0$ large enough,

$$I(tu) < 0 \text{ for all } t \geq T. \quad (2.4)$$

By the continuity of g_u , (2.2) and (2.4), there is $t_u > 0$ such that $g_u(t_u) = \max_{t \geq 0} g_u(t)$. Then $t_u u \in \mathcal{N}$. Moreover, using (f₄), it easy to see that t_u is the unique number satisfying $g'_u(t_u) = 0$, and hence (i) follows.

(ii) Let $\rho = \inf_{u \in S_1} g_u(t_u)$. Then $\rho > 0$. Indeed, by (2.1), we can find $\underline{t} > 0$ such that

$$g_u(t_u) \geq \left(\frac{1}{2} - \varepsilon \|V^{-1}\|_\infty \right) \underline{t}^2 - C C_\varepsilon \underline{t}^{p^*} > 0, \text{ for all } u \in S_1.$$

Thus $\rho > 0$.

Now, let $\Lambda \subset S_1$ be a compact set. Suppose by contradiction that there exists $\{u_n\} \subset \Lambda$ such that $t_{u_n} \rightarrow +\infty$. Then, up to subsequence, $u_n \rightarrow u$ for some $u \in \Lambda$. By (2.3), we have

$$\lim_{n \rightarrow \infty} I(t_{u_n} u_n) = -\infty. \quad (2.5)$$

Exploiting (f₄), we can infer that the function $t \mapsto \frac{1}{4}tf(t) - F(t)$ is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$. Therefore

$$\frac{1}{4}tf(t) - F(t) \geq 0 \text{ for all } t \in \mathbb{R}.$$

Since $\langle I'(t_{u_n}u_n), t_{u_n}u_n \rangle = 0$ for all $n \in \mathbb{N}$,

$$\begin{aligned} I(t_{u_n}u_n) &= I(t_{u_n}u_n) - \frac{1}{4}\langle I'(t_{u_n}u_n), t_{u_n}u_n \rangle \\ &= \frac{1}{4}\|t_{u_n}u_n\|_{E_\alpha}^2 + \int_{\mathbb{R}^N} \left(\frac{1}{4}t_{u_n}u_n f(t_{u_n}u_n) - F(t_{u_n}u_n) \right) dx dy \geq 0. \end{aligned}$$

This contradicts with (2.5).

(iii) By (i), $\bar{\Phi}$ is well defined. Let $v \in \mathcal{N}$, then equation $t_u u = v$ with $\|u\|_{E_\alpha} = 1$ has a unique solution $u = \frac{v}{\|v\|_{E_\alpha}}$ with $t_u = \|v\|_{E_\alpha}$. Therefore Φ is invertible and $\Phi^{-1}(v) = \frac{v}{\|v\|_{E_\alpha}}$. We notice that Φ^{-1} is continuous. Now, we prove that $\bar{\Phi}$ is continuous. Let $u_n \rightarrow u$ in $E_\alpha \setminus \{0\}$. By definition of t_{u_n} , we have $t\left(\frac{u_n}{\|u_n\|_{E_\alpha}}\right) = t_{u_n}\|u_n\|_{E_\alpha}$. Since $\left\{\frac{u_n}{\|u_n\|_{E_\alpha}}\right\} \subset S_1$, it follows from (ii) that $t_{u_n}\|u_n\|_{E_\alpha} \rightarrow l_0 > 0$. Thereby $t_{u_n} \rightarrow \frac{l_0}{\|u\|_{E_\alpha}}$. Recall that

$$t_{u_n}^2[u_n]_\alpha^2 + b t_{u_n}^4[u_n]_\alpha^4 + t_{u_n}^2 \int_{\mathbb{R}^N} V(x, y) u_n^2 dx dy = \int_{\mathbb{R}^N} f(t_{u_n}u_n) t_{u_n}u_n dx dy.$$

Letting n goes to infinity, we obtain

$$\frac{l_0^2}{\|u\|_{E_\alpha}^2} [u]_\alpha^2 + b \frac{l_0^4}{\|u\|_{E_\alpha}^4} [u]_\alpha^4 + \frac{l_0^2}{\|u\|_{E_\alpha}^2} \int_{\mathbb{R}^N} V(x, y) u^2 dx dy = \int_{\mathbb{R}^N} f\left(\frac{l_0 u}{\|u\|_{E_\alpha}}\right) \frac{l_0 u}{\|u\|_{E_\alpha}} dx dy,$$

which yields that $\frac{l_0 u}{\|u\|_{E_\alpha}} \in \mathcal{N}$ and $t_u = \frac{l_0}{\|u\|_{E_\alpha}}$. Consequently $\bar{\Phi}(u_n) \rightarrow \bar{\Phi}(u)$, and so $\bar{\Phi}$ is continuous. The proof of Lemma 2.3 is completed. $\square$

Consider the functionals $\bar{\Psi} : E_\alpha \setminus \{0\} \rightarrow \mathbb{R}$ and $\Psi : S_1 \rightarrow \mathbb{R}$ given by

$$\bar{\Psi}(u) = I(\bar{\Phi}(u)) \quad \text{and} \quad \Psi = \bar{\Psi}|_{S_1}.$$

In view of Lemma 2.3 and with the help of [32], we have the following result.

Lemma 2.4 Assume that (V_0) and (f₁)–(f₄) hold. Then,

(i) $\bar{\Psi} \in C^1(E_\alpha \setminus \{0\}, \mathbb{R})$ and for all $u \in E_\alpha \setminus \{0\}$, $v \in E_\alpha$

$$\langle \bar{\Psi}'(u), v \rangle = \frac{\|\bar{\Phi}(u)\|_{E_\alpha}}{\|u\|_{E_\alpha}} \langle I'(\bar{\Phi}(u)), v \rangle;$$

(ii) $\Psi \in C^1(S_1, \mathbb{R})$ and $\langle \Psi'(u), v \rangle = \|\Phi(u)\|_{E_\alpha} \langle I'(\Phi(u)), v \rangle$ for all $v \in T_u S_1$, where

$$T_u S_1 := \{v \in E_\alpha : (u, v)_{E_\alpha} = 0\};$$

(iii) If $\{u_n\}$ is a (PS) sequence for Ψ , then $\{\Phi(u_n)\}$ is a (PS) sequence for I . Moreover, if $\{u_n\} \subset \mathcal{N}$ is a bounded (PS) for I , then $\{\Phi^{-1}(u_n)\}$ is a (PS) sequence for Ψ ;

(iv) u is a critical point of Ψ if and only if $\Phi(u)$ is a nontrivial critical point for I . Moreover, the corresponding critical values coincide and

$$\inf_{u \in S_1} \Psi(u) = \inf_{u \in \mathcal{N}} I(u).$$

Proof The proof is similar to that of [32, Proposition 9, Corollary 10]. $\square$

Remark 2.5 We notice that

$$c_{\infty}^{\mathcal{N}} := \inf_{u \in \mathcal{N}} I(u) = \inf_{u \in E_{\alpha} \setminus \{0\}} \max_{t > 0} I(tu) = \inf_{u \in S_1} \max_{t > 0} I(tu) > 0. \quad (2.6)$$

For $u \in E_{\alpha}$ with $u^{\pm} \neq 0$, we define the function $\varphi_u : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ by

$$\varphi_u(t, s) = I(tu^+ + su^-).$$

Its gradient is given by

$$\begin{aligned} \nabla \varphi_u(t, s) &= (\langle I'(tu^+ + su^-), u^+ \rangle, \langle I'(tu^+ + su^-), u^- \rangle) \\ &= \left(\frac{1}{t} \langle I'(tu^+ + su^-), tu^+ \rangle, \frac{1}{s} \langle I'(tu^+ + su^-), su^- \rangle \right), \quad t, s > 0. \end{aligned} \quad (2.7)$$

Lemma 2.6 Assume that (V_0) and (f_1) – (f_4) hold. Then

- (i) The pair (t, s) with $t, s > 0$ is a critical point of φ_u if and only if $tu^+ + su^- \in \mathcal{M}$;
- (ii) The function φ_u has an unique critical point $(t_+, s_-) = (t_+(u), s_-(u)) \in (0, \infty)^2$, which is its unique global maximum;

Proof (i) By (2.7), for all $t, s > 0$, $\nabla \varphi_u(t, s) = 0$ if and only if

$$\langle I'(tu^+ + su^-), tu^+ \rangle = 0 \quad \text{and} \quad \langle I'(tu^+ + su^-), su^- \rangle = 0.$$

j So that, $tu^+ + su^- \in \mathcal{M}$.

(ii) Let $s \geq 0$ be fixed. Consider the function ψ_1 defined on $[0, \infty)$ by

$$\psi_1(t) = \varphi_u(t, s).$$

Similar to Lemma 2.3, there exists a unique $t(u, s) > 0$ satisfying

$$\psi_1'(t(u, s)) = 0, \quad \psi_1'(t) > 0 \text{ for } t \in (0, t(u, s)) \text{ and } \psi_1'(t) < 0 \text{ for } t \in (t(u, s), \infty). \quad (2.8)$$

Then, the function $\xi_1 : [0, \infty) \rightarrow (0, \infty)$ given by $\xi_1(s) = t(u, s)$ is well defined. For all $s \geq 0$, we have

$$\frac{\partial \varphi_u}{\partial t}(\xi_1(s), s) = \psi_1'(\xi_1(s)) = 0. \quad (2.9)$$

Thus from (2.7),

$$\begin{aligned} & (1 + b[\xi_1(s)u^+ + su^-]_{\alpha}^2) \\ & \times \left(\int_{\mathbb{R}^{2n+m}} \frac{((\xi_1(s)u^+ + su^-)(x, y) - (\xi_1(s)u^+ + su^-)(z, y))\xi_1(s)(u^+(x, y) - u^+(z, y))}{|x - z|^{n+2\alpha}} dx dz dy \right. \\ & \left. + \int_{\mathbb{R}^N} |\nabla_y(\xi_1(s)u^+)|^2 dx dy \right) + \int_{\mathbb{R}^N} V(x, y)(\xi_1(s)u^+)^2 dx dy \\ & = \int_{\mathbb{R}^N} f(\xi_1(s)u^+) \xi_1(s)u^+ dx dy. \end{aligned} \quad (2.10)$$

Note that

$$[\xi_1(s)u^+ + su^-]_{\alpha}^2 = \xi_1(s)^2[u^+]_{\alpha}^2 + s^2[u^-]_{\alpha}^2 + 2s\xi_1(s)Z(u^+, u^-) \quad (2.11)$$

and

$$\begin{aligned} & \int_{\mathbb{R}^{2n+m}} \frac{((\xi_1(s)u^+ + su^-)(x, y) - (\xi_1(s)u^+ + su^-)(z, y))\xi_1(s)(u^+(x, y) - u^+(z, y))}{|x - z|^{n+2\alpha}} dx dz dy \\ & + \int_{\mathbb{R}^N} |\nabla_y(\xi_1(s)u^+)|^2 dx dy = \xi_1(s)^2[u^+]_{\alpha}^2 + 2s\xi_1(s)Z(u^+, u^-), \end{aligned} \quad (2.12)$$

where

$$Z(u^+, u^-) = - \int_{\mathbb{R}^{2n+m}} \frac{u^+(x, y)u^-(z, y) + u^-(x, y)u^+(z, y)}{|x - z|^{n+2\alpha}} dx dz dy \geq 0.$$

So, from (2.10) we entail

$$\begin{aligned} & (1 + b(\xi_1(s)^2[u^+]_\alpha^2 + s^2[u^-]_\alpha^2 + 2s\xi_1(s)Z(u^+, u^-)))(\xi_1(s)^2[u^+]_\alpha^2 + 2s\xi_1(s)Z(u^+, u^-)) \\ & + \int_{\mathbb{R}^N} V(x, y)(\xi_1(s)u^+)^2 dx dy = \int_{\mathbb{R}^N} f(\xi_1(s)u^+)\xi_1(s)u^+ dx dy. \end{aligned} \quad (2.13)$$

□

Claim 2.7 *The function ξ_1 satisfies the following properties:*

- (a₁) ξ_1 is continuous;
- (a₂) $\xi_1(s) < s$ for s large.

Proof of the Claim 2.7 (a₁) Let $s_n \rightarrow s_0$ in $[0, \infty)$. Then $\{\xi_1(s_n)\}$ is bounded. Suppose by contradiction up to a subsequence $\xi_1(s_n) \rightarrow +\infty$. From (2.11)–(2.12), we can write

$$\begin{aligned} & (1 + b[\xi_1(s_n)u^+ + s_n u^-]_\alpha^2)[\xi_1(s_n)u^+ + s_n u^-]_\alpha^2 \\ & \geq (1 + b[\xi_1(s_n)u^+ + s_n u^-]_\alpha^2) \\ & \times \left(\int_{\mathbb{R}^{2n+m}} \frac{((\xi_1(s_n)u^+ + s_n u^-)(x, y) - (\xi_1(s_n)u^+ + s_n u^-)(z, y))\xi_1(s_n)(u^+(x, y) - u^+(z, y))}{|x - z|^{n+2\alpha}} dx dz dy \right. \\ & \left. + \int_{\mathbb{R}^N} |\nabla_y(\xi_1(s_n)u^+)|^2 dx dy \right). \end{aligned} \quad (2.14)$$

Combining (2.13) and (2.14), we get

$$\begin{aligned} & [\xi_1(s_n)u^+ + s_n u^-]_\alpha^2 + b[\xi_1(s_n)u^+ + s_n u^-]_\alpha^4 + \int_{\mathbb{R}^N} V(x, y)(\xi_1(s_n)u^+)^2 dx dy \\ & \geq \int_{\mathbb{R}^N} f(\xi_1(s_n)u^+)\xi_1(s_n)u^+ dx dy. \end{aligned} \quad (2.15)$$

Thereby,

$$\begin{aligned} & \frac{[\xi_1(s_n)u^+ + s_n u^-]_\alpha^2}{\xi_1(s_n)^4} + b \frac{[\xi_1(s_n)u^+ + s_n u^-]_\alpha^4}{\xi_1(s_n)^4} + \frac{1}{\xi_1(s_n)^2} \int_{\mathbb{R}^N} V(x, y)(u^+)^2 dx dy \\ & \geq \int_{\mathbb{R}^N} \frac{f(\xi_1(s_n)u^+)}{(\xi_1(s_n)u^+)^3} (u^+)^4 dx dy. \end{aligned} \quad (2.16)$$

According to (f₃)–(f₄), Fatou’s lemma and taking account that $\xi_1(s_n) \rightarrow \infty$, and $s_n \rightarrow s_0 < \infty$, passing to the limit as $n \rightarrow \infty$ in (2.16), we obtain

$$b[u^+]_\alpha^4 \geq \int_{\mathbb{R}^N} \liminf_{n \rightarrow \infty} \frac{f(\xi_1(s_n)u^+)}{(\xi_1(s_n)u^+)^3} (u^+)^4 dx dy = +\infty,$$

which is impossible and hence $\{\xi_1(s_n)\}$ is bounded. Then, up to a subsequence $\xi_1(s_n) \rightarrow t_0$. By (2.13), we have

$$\begin{aligned} & \xi_1(s_n)^2[u^+]_\alpha^2 + 2s_n \xi_1(s_n)Z(u^+, u^-) \\ & + b(\xi_1(s_n)^2[u^+]_\alpha^2 + s_n^2[u^-]_\alpha^2 + 2s_n \xi_1(s_n)Z(u^+, u^-))(\xi_1(s_n)^2[u^+]_\alpha^2 + 2s_n \xi_1(s_n)Z(u^+, u^-)) \\ & + \int_{\mathbb{R}^N} V(x, y)(\xi_1(s_n)u^+)^2 dx dy = \int_{\mathbb{R}^N} f(\xi_1(s_n)u^+)\xi_1(s_n)u^+ dx dy. \end{aligned}$$

Letting $n \rightarrow \infty$,

$$\begin{aligned} & t_0^2[u^+]_\alpha^2 + 2s_0 t_0 Z(u^+, u^-) \\ & + b(t_0^2[u^+]_\alpha^2 + s_0^2[u^-]_\alpha^2 + 2s_0 t_0 Z(u^+, u^-))(t_0^2[u^+]_\alpha^2 + 2s_0 t_0 Z(u^+, u^-)) \end{aligned}$$

$$+ \int_{\mathbb{R}^N} V(x, y)(t_0 u^+)^2 dx dy = \int_{\mathbb{R}^N} f(t_0 u^+) t_0 u^+ dx dy,$$

which implies $\psi'_1(t_0) = \frac{\partial \varphi_u}{\partial t}(t_0, s_0) = 0$, and by the uniqueness of $\xi_1(s_0)$, we deduce $\xi_1(s_0) = t_0$. The continuity of ξ_1 is achieved.

(a₂) For s large enough, we have $\xi_1(s) < s$. If not, there is $s_n \rightarrow \infty$ such that $\xi_1(s_n) \geq s_n$ for all n . Then $\xi_1(s_n) \rightarrow \infty$. Therefore, using (2.16), we arrive at a contradiction and consequently (a₂) follows. The claim is proved. $\square$

In the same way, for $t \geq 0$ fixed, we define ψ_2 on $[0, \infty)$ by $\psi_2(s) = \varphi_u(t, s)$, and so we can find $\xi_2 : [0, \infty) \rightarrow (0, \infty)$ satisfying (a₁)–(a₂) and

$$\psi'_2(\xi_2(t)) = \frac{\partial \varphi_u}{\partial s}(t, \xi_2(t)) = 0 \text{ for all } t \geq 0. \quad (2.17)$$

In view of (a₂), there exists $A_1 > 0$ such that for all $t, s \geq A_1$,

$$\xi_1(s) \leq s \text{ and } \xi_2(t) \leq t.$$

Set

$$A_2 = \max \left\{ \max_{s \in [0, A_1]} \xi_1(s), \max_{t \in [0, A_1]} \xi_2(t) \right\}$$

and $A = \max\{A_1, A_2\}$. Let $\xi : [0, A] \times [0, A] \rightarrow (0, \infty)^2$ defined by

$$\xi(t, s) = (\xi_1(s), \xi_2(t)).$$

Obviously, ξ is continuous since ξ_1 and ξ_2 are continuous by Claim 2.7. Moreover, we have

$$\xi([0, A] \times [0, A]) \subset [0, A] \times [0, A].$$

Applying Brouwer's fixed point theorem, there exists $(t_+, s_-) \in [0, A]^2$ such that

$$\xi(t_+, s_-) = (\xi_1(s_-), \xi_2(t_+)) = (t_+, s_-)$$

Since ξ_1 and ξ_2 are positive functions, $t_+, s_- > 0$. In addition, $\nabla \varphi_u(t_+, s_-) = 0$, hence (t_+, s_-) is a critical point of φ_u .

Let us now prove the uniqueness of the critical points of φ_u .

Claim 2.8 For all $v \in \mathcal{M}$, we have $(1, 1)$ is a critical point of φ_v . Moreover, if (t_0, s_0) with $t_0, s_0 > 0$ is a critical point of φ_v , then $t_0 = s_0 = 1$.

Proof of the Claim 2.8 Since $v = v^+ + v^- \in \mathcal{M}$, $\nabla \varphi_u(1, 1) = 0$. Thus $(1, 1)$ is a critical point of φ_v . Let (t_0, s_0) be a critical point of φ_v . Without loss of generality, we may assume that $0 < t_0 \leq s_0$. Then

$$\langle I'(t_0 v^+ + s_0 v^-), t_0 v^+ \rangle = 0 \text{ and } \langle I'(t_0 v^+ + s_0 v^-), s_0 v^- \rangle = 0. \quad (2.18)$$

The first part of (2.18) can be written

$$\begin{aligned} & (t_0^2 [v^+]_\alpha^2 + 2t_0 s_0 Z(v^+, v^-)) (1 + b(t_0^2 [v^+]_\alpha^2 + s_0^2 [v^-]_\alpha^2 + 2t_0 s_0 Z(v^+, v^-))) \\ & + \int_{\mathbb{R}^N} V(x, y)(t_0 v^+)^2 dx dy = \int_{\mathbb{R}^N} f(t_0 v^+) t_0 v^+ dx dy. \end{aligned}$$

Dividing by t_0^4 , we obtain

$$\begin{aligned} & \frac{1}{t_0^2} [v^+]_\alpha^2 + b[v^+]_\alpha^4 + b \frac{s_0^2}{t_0^2} [v^+]_\alpha^2 [v^-]_\alpha^2 + 2 \left(\frac{s_0}{t_0^3} + 2b \frac{s_0}{t_0} [v^+]_\alpha^2 + b \frac{s_0^3}{t_0^3} [v^-]_\alpha^2 \right) Z(v^+, v^-) \\ & + 4b \frac{s_0^2}{t_0^2} Z(v^+, v^-)^2 + \frac{1}{t_0^2} \int_{\mathbb{R}^N} V(x, y)(v^+)^2 dx dy = \int_{\mathbb{R}^N} \frac{f(t_0 v^+)}{t_0^3} v^+ dx dy. \end{aligned}$$

We have $0 < t_0 \leq s_0$ and $Z(v^+, v^-) \geq 0$, thus

$$\begin{aligned} & \frac{1}{t_0^2} [v^+]_\alpha^2 + b[v^+]_\alpha^4 + b[v^+]_\alpha^2 [v^-]_\alpha^2 + 2 \left(\frac{1}{t_0^2} + 2b[v^+]_\alpha^2 + b[v^-]_\alpha^2 \right) Z(v^+, v^-) \\ & + 4bZ(v^+, v^-)^2 + \frac{1}{t_0^2} \int_{\mathbb{R}^N} V(x, y)(v^+)^2 dx dy \leq \int_{\mathbb{R}^N} \frac{f(t_0 v^+)}{(t_0 v^+)^3} (v^+)^4 dx dy. \end{aligned} \quad (2.19)$$

On the other hand, $v^+ + v^- \in \mathcal{M}$ implies

$$\begin{aligned} & [v^+]_\alpha^2 + b[v^+]_\alpha^4 + b[v^+]_\alpha^2 [v^-]_\alpha^2 + 2(1 + 2b[v^+]_\alpha^2 + b[v^-]_\alpha^2) Z(v^+, v^-) \\ & + 4bZ(v^+, v^-)^2 + \int_{\mathbb{R}^N} V(x, y)(v^+)^2 dx dy = \int_{\mathbb{R}^N} \frac{f(v^+)}{(v^+)^3} (v^+)^4 dx dy. \end{aligned} \quad (2.20)$$

From (2.19) to (2.20), we deduce

$$\begin{aligned} & \left(1 - \frac{1}{t_0^2} \right) \left([v^+]_\alpha^2 + \int_{\mathbb{R}^N} V(x, y)(v^+)^2 dx dy \right) + 2 \left(1 - \frac{1}{t_0^2} \right) Z(v^+, v^-) \\ & \geq \int_{\mathbb{R}^N} \left(\frac{f(v^+)}{(v^+)^3} - \frac{f(t_0 v^+)}{(t_0 v^+)^3} \right) (v^+)^4 dx dy. \end{aligned} \quad (2.21)$$

If $t_0 < 1$, by (f₄) and (2.21) we derive a contradiction. Then $s_0 \geq t_0 \geq 1$.

Similarly, using the second part of (2.18), we get

$$\begin{aligned} & \left(\frac{1}{s_0^2} - 1 \right) \left([v^-]_\alpha^2 + \int_{\mathbb{R}^N} V(x, y)(v^-)^2 dx dy \right) + 2 \left(\frac{1}{s_0^2} - 1 \right) Z(v^+, v^-) \\ & \geq \int_{\mathbb{R}^N} \left(\frac{f(s_0 v^-)}{(s_0 v^-)^3} - \frac{f(v^-)}{(v^-)^3} \right) (v^-)^4 dx dy. \end{aligned} \quad (2.22)$$

This and (f₄) imply $t_0 \leq s_0 \leq 1$. So, we conclude that $t_0 = s_0 = 1$. The claim is proved. $\square$

Proof Now, we turn to prove (ii). Let (t_1, s_1) and (t_2, s_2) be two critical points of φ_u with $t_1, s_1, t_2, s_2 > 0$. By (i), we have

$$v_1 := t_1 u^+ + s_1 u^- \in \mathcal{M} \quad \text{and} \quad v_2 := t_2 u^+ + s_2 u^- \in \mathcal{M}.$$

Then $v_1^\pm \neq 0$ and

$$\frac{t_2}{t_1} v_1^+ + \frac{s_2}{s_1} v_1^- = v_2 \in \mathcal{M}.$$

Therefore, as a consequence of (i), $\left(\frac{t_2}{t_1}, \frac{s_2}{s_1} \right)$ is a critical point of φ_{v_1} . From Claim 2.8, we obtain $\frac{t_2}{t_1} = \frac{s_2}{s_1} = 1$, that is $t_1 = t_2$ and $s_1 = s_2$, and hence the uniqueness is achieved.

Next, we will show that φ_u admits a global maximum. Let $\Omega_0^\pm := \{(x, y) \in \mathbb{R}^N : u^\pm(x, y) \neq 0\}$. Then $|\Omega_0^\pm| > 0$ and

$$\begin{aligned} \varphi_u(t, s) &= \frac{1}{2} \|tu^+ + su^-\|_{E_\alpha}^2 + \frac{b}{4} [tu^+ + su^-]_\alpha^4 - \int_{\mathbb{R}^N} F(tu^+ + su^-) dx dy \\ &= \frac{t^2}{2} \|u^+\|_{E_\alpha}^2 + \frac{s^2}{2} \|u^-\|_{E_\alpha}^2 + tsZ(u^+, u^-) + \frac{b}{4} [tu^+ + su^-]_\alpha^4 - \int_{\mathbb{R}^N} F(tu^+ + su^-) dx dy \\ &= \frac{t^2}{2} \|u^+\|_{E_\alpha}^2 + \frac{s^2}{2} \|u^-\|_{E_\alpha}^2 + tsZ(u^+, u^-) + \frac{b}{4} (t^2[u^+]_\alpha^2 + s^2[u^-]_\alpha^2 + 2tsZ(u^+, u^-))^2 \\ &\quad - \int_{\Omega_0^+} F(tu^+) dx dy - \int_{\Omega_0^-} F(su^-) dx dy \end{aligned}$$

$$\leq (t^2 + s^2) \left(\frac{1}{2} \|u^+\|_{E_\alpha}^2 + \frac{1}{2} \|u^-\|_{E_\alpha}^2 + Z(u^+, u^-) \right) + \frac{b(t^2 + s^2)^2}{4} ([u^+]_\alpha^2 + [u^-]_\alpha^2 + Z(u^+, u^-))^2 \\ - \int_{\Omega_0^+} F(tu^+) dx dy - \int_{\Omega_0^-} F(su^-) dx dy.$$

Thus

$$\frac{\varphi_u(t, s)}{(t^2 + s^2)^2} \leq \frac{1}{t^2 + s^2} \left(\frac{1}{2} \|u^+\|_{E_\alpha}^2 + \frac{1}{2} \|u^-\|_{E_\alpha}^2 + Z(u^+, u^-) \right) + \frac{b}{4} ([u^+]_\alpha^2 + [u^-]_\alpha^2 + Z(u^+, u^-))^2 \\ - \frac{t^4}{(t^2 + s^2)^2} \int_{\Omega_0^+} \frac{F(tu^+)}{(tu^+)^4} (u^+)^4 dx dy - \frac{s^4}{(t^2 + s^2)^2} \int_{\Omega_0^-} \frac{F(su^-)}{(su^-)^4} (u^-)^4 dx dy. \quad (2.23)$$

Observing that

$$\frac{t^4}{(t^2 + s^2)^2} \geq \frac{1}{4} \text{ if } t \geq s \quad \text{and} \quad \frac{s^4}{(t^2 + s^2)^2} \geq \frac{1}{4} \text{ if } t \leq s,$$

it follows from (2.23), (f₃) and Fatou's Lemma that

$$\limsup_{\substack{|(t,s)| \rightarrow \infty \\ t \geq s}} \frac{\varphi_u(t, s)}{(t^2 + s^2)^2} \leq \frac{b}{4} ([u^+]_\alpha^2 + [u^-]_\alpha^2 + Z(u^+, u^-))^2 \\ - \frac{1}{4} \int_{\Omega_0^+} \liminf_{t \rightarrow +\infty} \frac{F(tu^+)}{(tu^+)^4} (u^+)^4 dx dy = -\infty$$

and

$$\limsup_{\substack{|(t,s)| \rightarrow \infty \\ s \geq t}} \frac{\varphi_u(t, s)}{(t^2 + s^2)^2} \leq \frac{b}{4} ([u^+]_\alpha^2 + [u^-]_\alpha^2 + Z(u^+, u^-))^2 \\ - \frac{1}{4} \int_{\Omega_0^-} \liminf_{s \rightarrow +\infty} \frac{F(su^-)}{(su^-)^4} (u^-)^4 dx dy = -\infty.$$

Hence, $\varphi_u(t, s) \rightarrow -\infty$ as $|(t, s)| \rightarrow \infty$. So, by the continuity, we conclude that φ_u admits a global maximum $(\hat{t}, \hat{s}) \in [0, \infty)^2$. We notice that $[tu^+]_\alpha^2 + [su^-]_\alpha^2 \leq [tu^+ + su^-]_\alpha^2$ and

$$\int_{\mathbb{R}^N} F(tu^+ + su^-) dx dy = \int_{\mathbb{R}^N} F(tu^+) dx dy + \int_{\mathbb{R}^N} F(su^-) dx dy.$$

Therefore

$$\varphi_u(t, 0) + \varphi_u(0, s) = I(tu^+) + I(su^-) \leq I(tu^+ + su^-) = \varphi_u(t, s) \text{ for all } t, s \geq 0.$$

This together with the fact that $\max_{t \geq 0} \varphi_u(t, 0), \max_{s \geq 0} \varphi_u(0, s) > 0$ leads to

$$\max_{t \geq 0} \varphi_u(t, 0) < \max_{t, s \geq 0} \varphi_u(t, s) = \varphi_u(\hat{t}, \hat{s}) \quad \text{and} \quad \max_{s \geq 0} \varphi_u(0, s) < \max_{t, s \geq 0} \varphi_u(t, s) = \varphi_u(\hat{t}, \hat{s}),$$

which shows that $\hat{t}, \hat{s} > 0$. Consequently $(\hat{t}, \hat{s})$ is a critical point with positive coordinates. By the uniqueness of (t_+, s_-) , we deduce that $(\hat{t}, \hat{s}) = (t_+, s_-)$, and hence (t_+, s_-) is a unique global maximum. This finishes the proof of Lemma 2.6. $\square$

Lemma 2.9 Let $\{u_n\} \subset \mathcal{M}$. If $u_n \rightharpoonup u$ weakly in E_α , then $u^\pm \neq 0$.

Proof For all $v \in \mathcal{M}$, we have

$$\|v^\pm\|_{E_\alpha}^2 \leq \int_{\mathbb{R}^N} f(v^\pm) v^\pm dx dy. \quad (2.24)$$

Therefore, using (f₁)–(f₂) and Lemma 2.2, for $\varepsilon > 0$, there exists C_ε , such that

$$\|v^\pm\|_{E_\alpha}^2 \leq \varepsilon \|V^{-1}\|_\infty \|v^\pm\|_{E_\alpha}^2 + CC_\varepsilon \|v^\pm\|_{E_\alpha}^{p^*}.$$

Letting $\varepsilon < \frac{1}{\|V^{-1}\|_\infty}$, we get

$$\|v^\pm\|_{E_\alpha} \geq \delta := \left(\frac{1 - \varepsilon \|V^{-1}\|_\infty}{CC_\varepsilon} \right)^{\frac{1}{p^*-2}}.$$

Let $u_n \rightharpoonup u$ weakly in E_α . It follows from (2.24) that

$$\int_{\mathbb{R}^N} f(u_n^\pm) u_n^\pm dx dy \geq \delta \text{ for all } n \in \mathbb{N}.$$

By (f₁)–(f₂), we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} f(u_n^\pm) u_n^\pm dx dy = \int_{\mathbb{R}^N} f(u^\pm) u^\pm dx dy, \quad (2.25)$$

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} F(u_n^\pm) dx dy = \int_{\mathbb{R}^N} F(u^\pm) dx dy. \quad (2.26)$$

Then

$$\int_{\mathbb{R}^N} f(u^\pm) u^\pm dx dy \geq \delta > 0,$$

which implies that $u^\pm \neq 0$. □

Remark 2.10 We have

$$c_\infty^\mathcal{M} := \inf_{u \in \mathcal{M}} I(u) \geq c_\infty^\mathcal{N},$$

being $\mathcal{M} \subset \mathcal{N}$.

Proof of Theorem 1.1 Let $\{u_n\} \subset \mathcal{M}$ be a minimizing sequence of I , that is

$$\lim_{n \rightarrow \infty} I(u_n) = \inf_{v \in \mathcal{M}} I(v) = c_\infty^\mathcal{M} > 0. \quad (2.27)$$

Let us first prove that $\{u_n\}$ is bounded in E_α . Suppose that up to a subsequence $\|u_n\|_{E_\alpha} \rightarrow \infty$ and set $v_n = \frac{u_n}{\|u_n\|_{E_\alpha}}$. Then $\{v_n\}$ is bounded in E_α . So, in view of Lemma 2.2, we may assume that

$$v_n \rightharpoonup v \text{ in } E_\alpha, \quad v_n \rightarrow v \text{ a.e. } \mathbb{R}^N \text{ and } v_n \rightarrow v \text{ in } L^q(\mathbb{R}^N) \text{ for } q \in [2, p^*]. \quad (2.28)$$

Since $\|u_n\|_{E_\alpha} v_n^+ + \|u_n\|_{E_\alpha} v_n^- = u_n \in \mathcal{M}$, by Lemma 2.6,

$$t_+(v_n) = s_-(v_n) = \|u_n\|_{E_\alpha}.$$

Therefore, for all $t > 0$ and $n \in \mathbb{N}$,

$$\begin{aligned} I(u_n) &= I(\|u_n\|_{E_\alpha} v_n) = I(t_+(v_n) v_n^+ + s_-(v_n) v_n^-) = \varphi_{v_n}(t_+(v_n), s_-(v_n)) \\ &\geq \varphi_{v_n}(t, t) = I(tv_n) = \frac{t^2}{2} + \frac{bt^4}{4} [v_n]_\alpha^4 - \int_{\mathbb{R}^N} F(tv_n) dx dy \\ &\geq \frac{t^2}{2} - \int_{\mathbb{R}^N} F(tv_n) dx dy \end{aligned} \quad (2.29)$$

If $v = 0$, then (2.27)–(2.29) imply

$$c_{\infty}^{\mathcal{M}} \geq \frac{t^2}{2} \text{ for all } t > 0, \quad (2.30)$$

which is a contradiction. Consequently $v \neq 0$. Then, $|\Omega_0| > 0$, where $\Omega_0 = \{(x, y) : v(x, y) \neq 0\}$.

Dividing $I(u_n)$ by $\|u_n\|_{E_{\alpha}}^4$ and using (2.28), (f₃) and Fatou's lemma, we get

$$0 = \lim_{n \rightarrow \infty} \frac{I(u_n)}{\|u_n\|_{E_{\alpha}}^4} \leq \frac{b}{4} - \int_{\Omega_0} \liminf_{n \rightarrow \infty} \frac{F(\|u_n\|_{E_{\alpha}} v_n)}{(\|u_n\|_{E_{\alpha}} v_n)^4} v_n^4 dx dy = -\infty.$$

This is impossible and hence $\{u_n\}$ is bounded in E_{α} . Then $u_n \rightharpoonup u$ in E_{α} . From Lemma 2.9, we infer that $u^{\pm} \neq 0$. Applying Lemma 2.6, there exist $t_+, s_- > 0$ such that $t_+u^+ + s_-u^- \in \mathcal{M}$ and

$$\langle I'(t_+u^+ + s_-u^-), u^{\pm} \rangle = 0.$$

Since $u_n \rightharpoonup u$ and $u_n \in \mathcal{M}$, it follows from (2.25) and Fatou's lemma that

$$\langle I'(u), u^{\pm} \rangle \leq \liminf_{n \rightarrow \infty} \langle I'(u_n), u_n^{\pm} \rangle = 0.$$

Similar to the proof of Claim 2.8, we see that $t_+, s_- \leq 1$. It follows from (f₄) and (2.25)–(2.26) that

$$\begin{aligned} c_{\infty}^{\mathcal{M}} &\leq I(t_+u^+ + s_-u^-) \\ &= I(t_+u^+ + s_-u^-) - \frac{1}{4} \langle I'(t_+u^+ + s_-u^-), t_+u^+ + s_-u^- \rangle \\ &= \frac{1}{4} \|t_+u^+ + s_-u^-\|_{E_{\alpha}}^2 + \int_{\mathbb{R}^N} \left(\frac{1}{4} f(t_+u^+) t_+u^+ - F(t_+u^+) \right) dx dy \\ &\quad + \int_{\mathbb{R}^N} \left(\frac{1}{4} f(s_-u^-) s_-u^- - F(s_-u^-) \right) dx dy \\ &\leq \frac{1}{4} \|u\|_{E_{\alpha}}^2 + \int_{\mathbb{R}^N} \left(\frac{1}{4} f(u^+) u^+ - F(u^+) \right) dx dy \\ &\quad + \int_{\mathbb{R}^N} \left(\frac{1}{4} f(u^-) u^- - F(u^-) \right) dx dy \\ &= \frac{1}{4} \|u\|_{E_{\alpha}}^2 + \int_{\mathbb{R}^N} \left(\frac{1}{4} f(u) u - F(u) \right) dx dy \\ &\leq \liminf_{n \rightarrow \infty} \left[\frac{1}{4} \|u_n\|_{E_{\alpha}}^2 + \int_{\mathbb{R}^N} \left(\frac{1}{4} f(u_n) u_n - F(u_n) \right) dx dy \right] \\ &= \liminf_{n \rightarrow \infty} \left[I(u_n) - \frac{1}{4} \langle I'(u_n), u_n \rangle \right] = c_{\infty}^{\mathcal{M}}. \end{aligned}$$

This yields $I(t_+u^+ + s_-u^-) = c_{\infty}^{\mathcal{M}}$. Arguing as in the proof of Claim 2.8, we derive that $t_+ = s_- = 1$, and so

$$I(u) = \inf_{v \in \mathcal{M}} I(v).$$

Now, we show that $I'(u) = 0$. Suppose by contradiction that $I'(u) \neq 0$. Then, by the continuity of I' , there exist $v, \varrho > 0$ such that

$$\|I'(v)\| \geq v \text{ for all } v \in E_{\alpha} \text{ with } \|v - u\|_{E_{\alpha}} \leq 3\varrho. \quad (2.31)$$

Let $0 < a_1 < 1 < a_2$ with $a_2 - a_1 < \frac{\varrho}{\sqrt{2}\|u\|_{E_{\alpha}}}$, and $D = (a_1, a_2) \times (a_1, a_2)$. Define the function $\gamma_u : \overline{D} \rightarrow E_{\alpha}$ by

$$\gamma_u(t, s) = tu^+ + su^-.$$

Then $I(\gamma_u(1, 1)) = c_{\infty}^{\mathcal{M}}$, and by Lemma 2.6(ii), $I(\gamma_u(t, s)) < c_{\infty}^{\mathcal{M}}$ for all $(t, s) \in \overline{D} \setminus \{(1, 1)\}$. Therefore

$$\varpi := \max_{(t, s) \in \partial D} I(\gamma_u(t, s)) < c_{\infty}^{\mathcal{M}}. \quad (2.32)$$

In view of (2.31) and [34, Lemma 2.3], for $\varepsilon < \min \left\{ \frac{\varrho}{8}, \frac{c_\infty^{\mathcal{M}} - \varpi}{2} \right\}$, there exists $\eta \in C([0, 1] \times E_\alpha, E_\alpha)$ verifying

- $\eta_1)$ $\eta(t, v) = v$ if $v \notin I^{-1}([c_\infty^{\mathcal{M}} - 2\varepsilon, c_\infty^{\mathcal{M}} + 2\varepsilon])$;
- $\eta_2)$ $I(\eta(1, v)) \leq c_\infty^{\mathcal{M}} - \varepsilon$ for all $v \in E_\alpha$ with $\|v - u\|_{E_\alpha} \leq \varrho$ and $I(v) \leq c_\infty^{\mathcal{M}} + \varepsilon$;
- $\eta_3)$ $I(\eta(1, v)) \leq I(\eta(0, v)) = I(v)$ for all $v \in E_\alpha$.

We claim that

$$\max_{(t,s) \in \overline{D}} I(\eta(1, \gamma_u(t, s))) < c_\infty^{\mathcal{M}}. \quad (2.33)$$

Indeed, by Lemma 2.6,

$$I(\gamma_u(t, s)) < c_\infty^{\mathcal{M}} + \varepsilon \text{ for all } (t, s) \in \overline{D}.$$

On the other hand, we have

$$\begin{aligned} \|\gamma_u(t, s) - u\|_{E_\alpha}^2 &= \|(t-1)u^+ + (s-1)u^-\|_{E_\alpha}^2 \\ &\leq 2(a_2 - a_1)^2 \|u\|_{E_\alpha}^2 < \varrho^2, \end{aligned}$$

thus $\|\gamma_u(t, s) - u\|_{E_\alpha} < \varrho$ for all $(t, s) \in \overline{D}$. Therefore, (η_2) implies $I(\eta(1, \gamma_u(t, s))) \leq c_\infty^{\mathcal{M}} - \varepsilon$, and hence (2.33) holds.

Now, we show that

$$\eta(1, \gamma_u(D)) \cap \mathcal{M} \neq \emptyset. \quad (2.34)$$

Let

$$\zeta_u(t, s) = \eta(1, \gamma_u(t, s)),$$

$$\begin{aligned} \phi_0(t, s) &= (\langle I'(\gamma_u(t, s)), u^+ \rangle, \langle I'(\gamma_u(t, s)), u^- \rangle) \\ &=: (\phi_0^1(t, s), \phi_0^2(t, s)) \end{aligned}$$

and

$$\phi_1(t, s) = \left(\frac{1}{t} \langle I'(\zeta_u(t, s)), (\zeta_u(t, s))^+ \rangle, \frac{1}{s} \langle I'(\zeta_u(t, s)), (\zeta_u(t, s))^- \rangle \right).$$

By direct computation

$$\begin{aligned} \frac{\partial \phi_0^1}{\partial t}(1, 1) &= \|u^+\|_{E_\alpha}^2 + 3b[u^+]_\alpha^4 + 8bZ(u^+, u^-)[u^+]_\alpha^2 + b[u^+]_\alpha^2[u^-]_\alpha^2 + 4bZ(u^+, u^-)^2 \\ &\quad - \int_{\mathbb{R}^N} f'(u^+)(u^+)^2 dx dy \end{aligned} \quad (2.35)$$

and

$$\begin{aligned} \frac{\partial \phi_0^1}{\partial s}(1, 1) &= 4bZ(u^+, u^-)[u^+]_\alpha^2 + 6bZ(u^+, u^-)[u^-]_\alpha^2 + 2b[u^+]_\alpha^2[u^-]_\alpha^2 + 8bZ(u^+, u^-)^2 \\ &\quad + 2Z(u^+, u^-). \end{aligned} \quad (2.36)$$

From $\langle I'(u^+ + u^-), u^+ \rangle = 0$, we entail

$$\begin{aligned} \int_{\mathbb{R}^N} f(u^+)u^+ dx dy &= \|u^+\|_{E_\alpha}^2 + b[u^+]_\alpha^4 + 4bZ(u^+, u^-)[u^+]_\alpha^2 + 2bZ(u^+, u^-)[u^-]_\alpha^2 \\ &\quad + b[u^+]_\alpha^2[u^-]_\alpha^2 + 4bZ(u^+, u^-)^2 + 2Z(u^+, u^-). \end{aligned} \quad (2.37)$$

By (f₄), we see that

$$f'(t)t^2 > 3f(t)t \text{ for all } t \neq 0. \quad (2.38)$$

Combining (2.35) and (2.37)–(2.38), we obtain

$$-\frac{\partial \phi_0^1}{\partial t}(1, 1) > 2\|u^+\|_{E_\alpha}^2 + 4bZ(u^+, u^-)[u^+]_\alpha^2 + 6bZ(u^+, u^-)[u^-]_\alpha^2 + 2b[u^+]_\alpha^2[u^-]_\alpha^2 + 8bZ(u^+, u^-)^2 + 6Z(u^+, u^-). \quad (2.39)$$

Similarly, we have

$$\frac{\partial \phi_0^2}{\partial t}(1, 1) = 6bZ(u^+, u^-)[u^+]_\alpha^2 + 4bZ(u^+, u^-)[u^-]_\alpha^2 + 2b[u^+]_\alpha^2[u^-]_\alpha^2 + 8bZ(u^+, u^-)^2 + 2Z(u^+, u^-) \quad (2.40)$$

and

$$-\frac{\partial \phi_0^2}{\partial s}(1, 1) > 2\|u^-\|_{E_\alpha}^2 + 6bZ(u^+, u^-)[u^+]_\alpha^2 + 4bZ(u^+, u^-)[u^-]_\alpha^2 + 2b[u^+]_\alpha^2[u^-]_\alpha^2 + 8bZ(u^+, u^-)^2 + 6Z(u^+, u^-). \quad (2.41)$$

From (2.36) and (2.39)–(2.41), we deduce

$$-\frac{\partial \phi_0^1}{\partial t}(1, 1) > \frac{\partial \phi_0^1}{\partial s}(1, 1) \quad \text{and} \quad -\frac{\partial \phi_0^2}{\partial s}(1, 1) > \frac{\partial \phi_0^2}{\partial t}(1, 1).$$

Therefore,

$$\text{Jac}(\phi_0)(1, 1) > 0.$$

Note that $(1, 1)$ is the unique pre-image of $(0, 0)$ by ϕ_0 , thus $(1, 1)$ is the unique regular point of ϕ_0 . By the Brouwer degree theory,

$$\deg(\phi_0, D, (0, 0)) = \text{sign}(\text{Jac}(\phi_0)(1, 1)) = 1. \quad (2.42)$$

By (2.32) for all $(t, s) \in \partial D$,

$$I(\gamma_u(t, s)) \leq \varpi < c_\infty^{\mathcal{M}} - 2\varepsilon.$$

This together with (η_1) gives $\eta(1, \gamma_u(t, s)) = \gamma_u(t, s)$ for all $(t, s) \in \partial D$. Hence

$$\deg(\phi_1, D, (0, 0)) = \deg(\phi_0, D, (0, 0)) = 1, \quad (2.43)$$

which implies $\phi_1(t_0, s_0) = (0, 0)$ for some $(t_0, s_0) \in D$. Consequently

$$\eta(1, \gamma_u(t_0, s_0)) = \zeta_u(t_0, s_0) \in \mathcal{M},$$

but this contradicts (2.33). We conclude that u is a critical point of I and so it is a sign-changing solution of (1.1).

If f is odd, then I is even. In view of Lemma 2.4 and Remark 2.5, Ψ is bounded below in S_1 . By standard arguments and Lemma 2.3, we see that Ψ satisfies (PS) condition on S_1 . According to [29, Theorem 8.10] and Lemma 2.4, we deduce that I has infinitely many critical points, which completes the proof of Theorem 1.1. $\square$

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Compliance with ethical standards

Conflict of interest The author declares that there is no conflict of interest regarding the publication of this paper.

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