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On some properties of near-rings

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Abstract We establish sufficient conditions for 3-prime near-rings to be commutative rings. In particular, for a 3-prime near-ring R with a derivation d , we investigate conditions such as $d([U, V]) \subseteq Z(R)$, $d(U) \subseteq Z(R)$, $x_0 d(R) \subseteq Z(R)$, and $U x_0 \subseteq Z(R)$. As a by-product, we generalize and extend known results related to rings and near-rings. Furthermore, we discuss the converse of a well-known result in rings and near-rings, namely: if $x \in Z(R)$, then $d(x) \in Z(R)$. In addition, we provide useful examples illustrating our results.

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1 Introduction

Throughout this paper, R denotes a left near-ring and $Z(R)$ denotes the multiplicative center of R . A near-ring R is called 3-prime if, for all $x, y \in R$, $xRy = \{0\}$ implies $x = 0$ or $y = 0$. A map $d : R \rightarrow R$ is a derivation on R if d is an additive mapping and $d(xy) = xd(y) + d(x)y$ for all $x, y \in R$. If the map $d : R \rightarrow R$ defined by $d(x) = xr - rx$ or by $d(x) = rx - xr$ for all $x \in R$, where $r \in R$, is a derivation (and it will be if R is a ring), then it is called an inner derivation on R induced by r . An element $x \in R$ is called a right (left) zero divisor in R if there exists a non-zero element $y \in R$, such that $yx = 0$ ($xy = 0$). A zero divisor is a right or a left zero divisor. A near-ring R is called a zero-symmetric near-ring, if $0x = 0$ for all $x \in R$. For subsets $X, Y \subseteq R$, the symbol $[X, Y]$ denotes the set $\{xy - yx | x \in X, y \in Y\}$. We say that U is a semigroup right (left) ideal of a near-ring R if U is a non-empty subset of R satisfying $UR \subseteq U$ ($RU \subseteq U$). We say that U is a semigroup ideal if it is both a semigroup right and left ideal. For further information about near-rings, see [12, 13].

Studying when a near-ring is a ring or a commutative ring is one of the most important questions in the theory of near-rings, which has been the subject of many investigations since the 70s of the foregoing century. For example, among many others, Ligh in [11] and Bell in [2–4] have established some conditions that force a near-ring to be a ring. Later on, Bell and Mason in [7] have studied many properties of 3-prime near-rings that make them commutative rings. Beidar, Fong, and Wang [1] have generalized some results about rings to the setting of near-rings, while Wang [15] has established some beautiful properties of near-rings. On the other hand, Bell in [5] has generalized several results of [7] using one (two) sided semigroup ideals of near-rings. Subsequently, many authors, for instance in [8, 9] and [10], have investigated various properties of near-rings that make of them commutative rings.

In this paper, we establish some new results in this direction. In Sect. 3, we begin with Theorem 3.1 which is a generalization of Theorem 2.3 of [6] about prime rings that satisfy $d([x, y]) \in Z(R)$ for all $x, y \in R$.

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We show that the result remains valid when we replace R by one of its non-zero ideals. One of the well-known results about derivations on rings and near-rings asserts that if $x \in Z(R)$, then $d(x) \in Z(R)$. We study its converse for 3-prime near-rings, in Theorem 3.2, and we get that if $d(x), d(x^2) \in Z(R)$, such that $2x \neq 0 \neq d(x)$, then $x \in Z(R)$. We use this result to prove other results such as Theorem 3.3 (which is a partial generalization of Theorem 3.1) and Theorem 3.4. Note that Theorem 3.4 is one of our main theorems in this paper, namely that it provides a full description of the case $\{0\} \neq d(U) \subseteq Z(R)$ where U is a subsemigroup of $(R, \cdot)$ which provides an interesting extension of many known results (see [5, 7, 9, 10]), where we show that U is commutative, $V = \{u^2 | u \in U\}$ is a subsemigroup of R contained in $Z(R)$ and either $2R = \{0\}$ or $U \subseteq Z(R)$. Moreover, we study three cases: $nU = \{0\}$, $nU \neq \{0\}$, and R is of characteristic zero. We also give an example illustrating these results. Also, one of the main theorems in this paper is Theorem 3.5, where we give a generalization of the condition $x_o d(R) = \{0\}$ or $d(R)x_o = \{0\}$, (which is a very useful tool in proofs), on near-rings and rings. Bell in [5] has proved if R is a 3-prime near-ring with a non-zero semigroup right (left) ideal U , such that $Ux = \{0\}$ ($xU = \{0\}$) for some $x \in R$, then one should have $x = 0$. Theorem 3.6 generalizes Bell's result to the case $Ux \subseteq Z(R)$ ($xU \subseteq Z(R)$).

2 Preliminaries

In this section, we recall some well-known results on derivations and commutativity of near-rings. These will often be used, mostly without explicit mention.

Lemma 2.1 [10, Theorem 2.7] *A near-ring R is zero-symmetric if and only if R admits a derivation.*

Lemma 2.2 [15, Proposition 1] *Let R be a near-ring with a derivation d . Then, $xd(y) + d(x)y = d(x)y + xd(y)$ for all $x, y \in R$.*

Lemma 2.3 [7, Lemma 1] *(The partial distributive law) Let R be a near-ring and d a derivation on R . For all $x, y, z \in R$, we have $(xd(y) + d(x)y)z = xd(y)z + d(x)yz$.*

The following results are very useful in the sequel.

Lemma 2.4 [15, Lemma 2] *Let R be a near-ring with a derivation d . If $x \in Z(R)$, then $d(x) \in Z(R)$.*

Lemma 2.5 [7, Lemma 3(i)] *Let R be a 3-prime near-ring and $z \in Z(R) - \{0\}$. Then, z is not a zero divisor.*

Lemma 2.6 [9, Corollary 3.2] *Let R be a 3-prime near-ring with a non-zero semigroup right (left) ideal U and a non-zero semigroup ideal V . If R admits a non-zero derivation d , such that $d([V, U]) = \{0\}$, then R is a commutative ring.*

Lemma 2.7 [5, Lemma 1.2(iii)] *Let R be a 3-prime near-ring and $x \in Z(R) - \{0\}$. If either yx or xy is in $Z(R)$, then $y \in Z(R)$.*

Lemma 2.8 [5, Lemma 1.4] *Let R be a 3-prime near-ring with a non-zero semigroup ideal U and a non-zero derivation d on R . If $x \in R$ and $d(U)x = \{0\}$ ($xd(U) = \{0\}$), then $x = 0$.*

Lemma 2.9 [9, Proposition 2.8] *Let R be a 3-prime near-ring with a non-zero semigroup ideal U and a non-zero derivation d . Then, for every integer $n \geq 2$, the following statements are equivalent:*

- (i) $d(nR) = \{0\}$.
- (ii) $d(nU) = \{0\}$.
- (iii) $nU = \{0\}$.
- (iv) $nR = \{0\}$.

Lemma 2.10 [5, Theorem 2.1] *Let R be a 3-prime near-ring with a non-zero derivation d and a non-zero semigroup right (left) ideal U of R . If $d(U) \subseteq Z(R)$, then R is a commutative ring.*

Lemma 2.11 [14, Lemma 2.14] *Let R be a 3-prime zero-symmetric near-ring, such that:*

- (i) $x_1 x_2 \dots x_n \in Z(R) - \{0\}$ for some $x_1, x_2, \dots, x_n \in R$, where $n \in \mathbb{N}$. Then, x_n is not a left zero divisor and x_1 is not a right zero divisor.
- (ii) $x^n \in Z(R) - \{0\}$ for some $x \in R, n \in \mathbb{N}$. Then, x is not a zero divisor.

Lemma 2.12 [5, Lemma 1.5] *Let R be a 3-prime near-ring with a non-zero semigroup right (left) ideal U . If $U \subseteq Z(R)$, then R is a commutative ring.*

3 Main results

In this section, we will study some properties on rings and near-rings. We will begin with a result which is a generalization of Theorem 2.3 of [6].

Theorem 3.1 *Let R be a prime ring with a non-zero ideals U and V of R . If R admits a non-zero derivation d , such that $d([U, V]) \subseteq Z(R)$, then R is commutative or $2R = \{0\}$.*

Proof If $d([U, V]) = \{0\}$, then R is commutative by Lemma 2.6. Therefore, suppose $d([U, V]) \neq \{0\}$. For all $u \in U$, $y, w \in V$, we have that $d([u, y], w) \in Z(R)$. Thus:

$$[u, y]d(w) + d([u, y])w - wd([u, y]) - d(w)[u, y] \in Z(R).$$

Since $d([u, y]) \in Z(R)$, we get that:

$$[u, y]d(w) - d(w)[u, y] = [[u, y], d(w)] \in Z(R). \quad (1)$$

Replacing u by $d(v'v)u$, w by $v'v$ and y by $d(v'v)$, where $v', v \in V$, we have:

$$\begin{aligned} & [[d(v'v)u, d(v'v)], d(v'v)] \\ &= d(v'v)[[u, d(v'v)], d(v'v)] \in Z(R) \end{aligned}$$

after simple calculations. Using (1), for each $v'' \in V^2$, either $d(v'') \in Z(R)$ or $[[u, d(v'')], d(v'')] = 0$ for all $u \in U$ by Lemma 2.7. Therefore, in all cases, $[[u, d(v'')], d(v'')] = 0$ for all $u \in U$, $v'' \in V^2$. Now, for each $v \in V^2$, define $d_{d(v)} : R \rightarrow R$ as follows:

$$d_{d(v)}(x) = xd(v) - d(v)x = [x, d(v)].$$

Then, $d_{d(v)}$ is an inner derivation on R induced by $d(v)$. Therefore:

$$d_{d(v)}^2(u) = [d_{d(v)}(u), d(v)] = 0,$$

and hence, $d_{d(v)}^2(U) = \{0\}$. Therefore, for all $u, u' \in U$, we have:

$$0 = d_{d(v)}^2(uu') = 2d_{d(v)}(u)d_{d(v)}(u') = d_{d(v)}(u)2d_{d(v)}(u').$$

From Lemma 2.8, we get that $d_{d(v)} = 0$ or $2d_{d(v)}(U) = \{0\}$. Therefore, either $d(v) \in Z(R)$ or $2R = \{0\}$ by Lemma 2.9. If $2R \neq \{0\}$, then $d(v) \in Z(R)$ for all $v \in V^2$ and R is commutative by Lemma 2.10 since V^2 is a semigroup ideal of R . Therefore, $d([R, R]) = \{0\}$, a contradiction. Therefore, $2R = \{0\}$. $\square$

The following example shows that " $2R = \{0\}$ " in the previous theorem is not redundant.

Example 1 Let $R = M_2(\mathbb{Z}_2)$. Then, R is a prime ring. Define a map d on R to be the inner derivation induced by: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Therefore, for all $a, b, c, e \in \mathbb{Z}_2$:

$$d\left(\begin{bmatrix} a & b \\ c & e \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & e \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & e \end{bmatrix} = \begin{bmatrix} c & a+e \\ 0 & c \end{bmatrix}.$$

Then, clearly, it follows that:

$$d(R) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}.$$

Also, $d^2(R) = \{0\}$. In fact, $d([x, y]) \in Z(R)$ for all $x, y \in R$ as follows: for all $a, b, c, e, x, y, z, w \in \mathbb{Z}_2$, we have:

$$\begin{aligned} & d\left(\begin{bmatrix} a & b \\ c & e \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} + \begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} a & b \\ c & e \end{bmatrix}\right) \\ &= \begin{bmatrix} cx + ez + za + wc & 0 \\ 0 & cx + ez + za + wc \end{bmatrix} \in \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} = Z(R). \end{aligned}$$

Observe that $2R = \{0\}$ and R is not commutative as:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

We know from Lemma 2.4 that if $x \in Z(R)$, then $d(x) \in Z(R)$. In the following result, we discuss its converse for 3-prime near-rings.

Theorem 3.2 *Let R be a 3-prime near-ring with a derivation d , such that $d(u^i) \in Z(R)$, where $i \in \{1, 2\}$, and $d(u) \neq 0$ for some $u \in R$. Then, either $2u = 0$ or $u \in Z(R) - \{0\}$. Moreover, if $d(u^3) \in Z(R)$, then $u^2 \in Z(R)$.*

Proof Observe that:

$$d(u^2) = d(u)(2u) \in Z(R),$$

which implies that $2u \in Z(R)$ as $d(u) \neq 0$. Since

$$rd(u^2) = d(u^2)r$$

for all $r \in R$, we have:

$$d(u)(2ru) = d(u)(2ur).$$

Thus, $d(u)(2ru - 2ur) = 0$ and hence:

$$2ru = 2ur. \quad (2)$$

Since $2u \in Z(R)$, we get that $r(2u) = (2u)r$ for all $r \in R$. However, using (2), we deduce that:

$$r(2u) = 2ru = 2ur.$$

Therefore:

$$(2u)r = 2ur \quad (3)$$

for all $r \in R$. Replacing r by ur in (2), we get:

$$2uru = 2uur,$$

and using (3) on each side of the last equation, we have:

$$(2u)ru = (2u)ur,$$

and hence:

$$(2u)(ru - ur) = 0.$$

Therefore, either $2u = 0$ or $u \in Z(R) - \{0\}$ by Lemma 2.5.

To complete the proof, we will discuss the two cases. If $u \in Z(R) - \{0\}$, then $u^2 \in Z(R)$. On the other hand, if $2u = 0$, then $d(u^2) = 0$, and hence, $d(u^3) = d(u)u^2 \in Z(R)$ which implies that $u^2 \in Z(R)$. $\square$

In the following result, we will prove a similar result of Theorem 3.1 for 3-prime near-rings.

Theorem 3.3 *Let R be a 2-torsion-free 3-prime near-ring. If R admits a non-zero derivation d , such that $d([x, y]), d([x, y]^2) \in Z(R)$ for all $x, y \in R$, then R is a commutative ring.*

Proof From Theorem 3.2, for all $x, y \in R$, either $[x, y] \in Z(R) - \{0\}$ or $d([x, y]) = 0$. If $d([x, y]) = 0$ for all $x, y \in R$, then R is a commutative ring by Lemma 2.6. Now, suppose $d([R, R]) \neq \{0\}$. Therefore, there are $a, b \in R$, such that $d([a, b]) \neq 0$. Thus, $[a, b] \in Z(R) - \{0\}$, and for all $x, y \in R$, such that $d([x, y]) = 0$, we have:

$$\begin{aligned} d([a, b]x, y) &= d([a, b][x, y]) \\ &= d([a, b])[x, y] \in Z(R), \end{aligned}$$

and hence, $[x, y] \in Z(R)$ also by lemma 2.7. Therefore, $[R, R] \subseteq Z(R)$. Observe that $[a, ab] = a[a, b] \in Z(R)$ implies that $a \in Z(R)$, and then, $[a, b] = 0$, a contradiction. Therefore, $d([R, R]) = \{0\}$. $\square$

In Theorem 2.1 of [5], Bell has showed that, if $d(U) \subseteq Z(R)$, then R is a commutative ring, where U is a non-zero semigroup right (left) ideal of R . The following result extends his result over subsemigroups and also generalizes Theorem 2 of [7].

Theorem 3.4 *Let R be a 3-prime near-ring with a non-zero derivation d and a subsemigroup U of $(R, \cdot)$, such that $\{0\} \neq d(U) \subseteq Z(R)$. Then, U is commutative, $S = \{u \in U \mid d(u) = 0\}$ and $V = \{u^2 \mid u \in U\}$ are subsemigroups of R contained in $Z(R)$ and either $2R = \{0\}$ or $U \subseteq Z(R)$. Moreover, for any positive integer n , we have the following cases:*

- (i) *If $nU = \{0\}$, then $nR = \{0\}$, and in this case: (the characteristic of R is odd and $U \subseteq Z(R)$) or $2R = \{0\}$.*
- (ii) *If $nU \neq \{0\}$, then R is n -torsion-free and either ($2R = \{0\}$ and n is odd) or $U \subseteq Z(R)$.*
- (iii) *If R is of characteristic zero, then $U \subseteq Z(R)$.*

Proof As $d(U) \neq \{0\}$, then there exists $v' \in U$, such that $d(v') \in Z(R) - \{0\}$. For all $u \in U$, we have:

$$v'd(uv') = d(uv')v',$$

and hence:

$$d(v')(v'u - uv') = 0.$$

Using Lemma 2.5, we get that v' centralizes U . Again, for all $u, w \in U$, we have:

$$rd(uv') = d(uv')r,$$

and by the same way above, we will get that u centralizes U for all $u \in U$. Hence, U is commutative. Now, for all $u \in U$, either $d(u) = 0$ or not.

If $d(u) = 0$, then:

$$d(uv') = ud(v') \in Z(R),$$

which implies that $u \in Z(R)$. Therefore, $S = \{u \in U \mid d(u) = 0\}$ is a subsemigroup of R contained in $Z(R)$.

If $d(u) \neq 0$, then $u^2 \in Z(R)$ by Theorem 3.2. Consequently, $V = \{u^2 \mid u \in U\}$ is a subsemigroup of R contained in $Z(R)$.

Now, either $d((v')^2) = 0$ or not. If $d((v')^2) = 0$, then for all $r \in R$, we have that:

$$0 = d((v')^2)r = d(v')(2v'r),$$

and hence:

$$0 = 2v'rs = v'r(2s)$$

for all $r, s \in R$. Therefore, $2R = \{0\}$.

If $d((v')^2) \neq 0$, then for all $r \in R$, we have:

$$rd((v')^3) = d((v')^3)r,$$

and hence:

$$d((v')^2)(rv' - v'r) = 0,$$

which implies that $v' \in Z(R)$. Now, for all $r \in R$ and $u \in U$, we have:

$$wd(uv') = d(uv')w,$$

and by the same way above, we will get that $u \in Z(R)$. Therefore, $U \subseteq Z(R)$.

(i) Suppose $nU = \{0\}$. Therefore, $d((v')^n) = 0$, and for all $r \in R$, we have that:

$$d((v')^n)r = d(v')(n(v')^{n-1}r) = 0,$$

which implies that $(v')^{n-1}(nr) = 0$ and hence:

$$(v')^{n-1}(nrs) = (v')^{n-1}r(ns) = 0$$

for all $r, s \in R$. Therefore, $nR = \{0\}$ or $(v')^{n-1} = 0$. If $(v')^{n-1} = 0$, then:

$$d((v')^2)(v')^{n-2} = 0$$

and either $d((v')^2) = 0$ or $(v')^{n-2} = 0$. Continuing on this way, we will get that $d((v')^2) = 0$ in all cases, and by the same discussion above, we have that $2R = \{0\}$. Thus, we have two cases:

Case 1: n is odd which implies that $2U \neq \{0\}$. Therefore, $nR = \{0\}$ and $U \subseteq Z(R)$ from above.

Case 2: n is even and either $2R = \{0\}$ or $U \subseteq Z(R)$. If $2R = \{0\}$, then clearly that $nR = \{0\}$. If $U \subseteq Z(R)$, then for all $r \in R$, we have:

$$0 = r(nv') = v'(nr),$$

which implies that $nR = \{0\}$. If $2R \neq 0$, then from $d((v')^2) \in Z(R)$, we get that $2v' \in Z(R)$. If $2v' = 0$, then $2R = \{0\}$ by the same way above, a contradiction. Therefore, $2v' \in Z(R) - \{0\}$. Suppose $n = 2m$. For all $r \in R$, we have:

$$0 = v'(2mr) = mr(2v') = (2v')(mr),$$

which implies that $mR = \{0\}$. Continuing on this way, we can show that $m_o R = \{0\}$, where m_o is an odd number and $n = 2^k m_o$, for some positive integer k .

(ii) Suppose that $nU \neq \{0\}$. Therefore, there exists $v_o \in U$, such that $nv_o \neq 0$. Let

$$W = \{u \in U \mid d(u) \neq 0\}.$$

If $nw = 0$ for some $w \in W$, then for all $u' \in S$, we have:

$$0 = d(u'(nw)) = d(w(nu')) = d(w)nu'.$$

Thus, $nu' = 0$. Now, for all $u \in U$, we get that:

$$0 = u(nu') = nuu' = u'(nu).$$

If $u' \neq 0$, then $nU = \{0\}$, a contradiction.

Therefore, we have two cases: $S = \{0\}$ or $nw \neq 0$ for all $w \in W$.

Case 1: $S = \{0\}$. For all $w \in W = U - \{0\}$, observe that $d(w^2) \in Z(R)$ implies that $2w \in Z(R)$, and hence, $2U \subseteq Z(R)$. If $2U = \{0\}$, then $2R = \{0\}$ from (i) and n is odd which implies that R is n -torsion-free. If $2U \neq \{0\}$, then there exists $u_o \in U$, such that $2u_o \in Z(R) - \{0\}$. For all $u \in U$, we get that:

$$u(2u_o) = 2uu_o \in Z(R),$$

and hence, $u \in Z(R)$. Therefore, $U \subseteq Z(R)$. Suppose that $nr = 0$ for some $r \in R$. From $d(w^n) \in Z(R)$, we deduce that $nw^{n-1} \in Z(R)$, where $w \in W$. If $nw^{n-1} \neq 0$ for some $w \in W$, then:

$$0 = w^{n-1}(nr) = r(nw^{n-1}),$$

and then, $r = 0$. Therefore, R is n -torsion-free.

On the other hand, if $nw^{n-1} = 0$ for all $w \in W$, then $nw = 0$ as $w \in Z(R) - \{0\}$. Therefore, $nW = \{0\}$, and hence, $nU = \{0\}$, a contradiction.

Case 2: There exists $u_1 \in S$, such that $u_1 \neq 0 \neq nw$ for all $w \in W$. If $2w = 0$ for some $w \in W$, then n is odd, and for all $u \in U - W$, we have:

$$0 = d(u(2w)) = d(w)2u,$$

and hence, $2u = 0$. Now, for all $v \in U$, observe that:

$$0 = v(2u_1) = u_1(2v),$$

and then, $2U = 0$. That implies that $2R = \{0\}$ from (i) and hence R is n -torsion-free. If $2w \neq 0$ for all $w \in W$, then $w \in Z(R)$ by Theorem 3.2. Therefore, $U \subseteq Z(R)$, and by the same way in case 1, we can prove that R is n -torsion-free.

(iii) Since $2R \neq \{0\}$, then $U \subseteq Z(R)$. □

The following example shows that:

- (i) It is possible in Theorem 3.2 to find a prime ring with a derivation d , such that $d(u^i) \in Z(R)$, where $i \in \{1, 2, 3\}$, and $d(u) \neq 0$ for some $u \in R$, and $2u = 0$, but $u \notin Z(R)$.
- (ii) $2R = \{0\}$ is not redundant in Theorem 3.4.

Example 2 Let $R = M_2(\mathbb{Z}_2)$. Then, R is a prime ring. Define the same inner derivation d on R as defined in Example 1. Take $u = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then, $d(u^i) \in Z(R)$, where $i \in \{1, 2, 3\}$, $d(u) \neq 0$ and $2u = 0$ but $u \notin Z(R)$, since:

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

This shows that (i) is true. For (ii), observe that:

$$U = \left\{ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

is a subsemigroup and $\{0\} \neq d(U) \subseteq Z(R)$. However, $U \not\subseteq Z(R)$.

Bell and Mason have proved in [7] that, when R is a 3-prime near-ring, then $x_o d(R) = \{0\}$ ($d(R)x_o = \{0\}$) implies that $x_o = 0$ or $d = 0$, which is a very useful tool in proofs. The following result generalizes their result.

Theorem 3.5 *Let R be a 3-prime near-ring with a non-zero derivation d . If $x_o d(R) \subseteq Z(R)$ ($d(R)x_o \subseteq Z(R)$) for some $x_o \in R$, then $x_o \in Z(R)$. Moreover, If $x_o \neq 0$, then R is a commutative ring.*

Proof Suppose $x_o d(R) \subseteq Z(R)$ (Case 2: $d(R)x_o \subseteq Z(R)$) for some $x_o \in R$. If $x_o d(R) = \{0\}$, then $x_o = 0 \in Z(R)$ by Lemma 2.8. Therefore, suppose there exists $y_o \in R$, such that $x_o d(y_o) \in Z(R) - \{0\}$. If there exists $c \in Z(R)$, such that $d(c) \neq 0$, then $x_o d(c) \in Z(R)$ implies that $x_o \in Z(R)$.

Now, suppose $d(Z(R)) = \{0\}$. If $x_o d(x_o) = 0$, then:

$$x_o d(x_o y_o) = x_o x_o d(y_o) \in Z(R),$$

which implies that $x_o \in Z(R)$ by Lemma 2.7. Therefore, suppose that $x_o d(x_o) \neq 0$, and hence, $d(x_o)$ is not a left zero divisor (Case 2: is not a right zero divisor) by Lemma 2.11(i). Observe that:

$$\begin{aligned} x_o d(x_o x_o) &= x_o d(x_o) x_o + x_o x_o d(x_o) \\ &= x_o d(x_o) (2x_o) \in Z(R), \end{aligned} \quad (4)$$

which implies that $2x_o \in Z(R)$. From above, we deduce that $d(2x_o) = 0$ and then:

$$\begin{aligned} 0 &= x_o x_o d(2x_o) \\ &= 2x_o x_o d(x_o) \\ &= x_o d(x_o) (2x_o). \end{aligned}$$

Thus, $2x_o = 0$ by Lemma 2.5.

(Case 2:

$$\begin{aligned} &(2x_o) d(x_o) x_o \\ &= d(x_o) x_o (2x_o) \\ &= 2d(x_o) x_o x_o \\ &= 2x_o d(x_o) x_o \\ &= x_o d(x_o) (2x_o) \\ &= (2x_o) x_o d(x_o). \end{aligned}$$

Thus:

$$(2x_o) (d(x_o) x_o - x_o d(x_o)) = 0,$$

and then either $2x_o = 0$ or $d(x_o) x_o = x_o d(x_o)$. If $d(x_o) x_o = x_o d(x_o)$, then:

$$\begin{aligned} &(2x_o) d(x_o) \\ &= d(x_o) (2x_o) \\ &= 2d(x_o) x_o \\ &= 2x_o d(x_o) \end{aligned}$$

$$\begin{aligned}
 &= x_o(d(2x_o)) \\
 &= 0.
 \end{aligned}$$

However, $d(x_o)$ is not a right zero divisor. That implies $2x_o = 0$. Therefore, in both cases, we have that $2x_o = 0$.)

Hence, $x_o d(x_o x_o) = 0$ from (4). Therefore:

$$x_o d(x_o x_o x_o) = x_o x_o x_o d(x_o) \in Z(R),$$

which gives us $x_o^2 \in Z(R)$. If $x_o^2 = 0$, then $x_o^2 d(x_o) = 0$ and, hence, $x_o = 0$, a contradiction with $x_o d(x_o) \neq 0$. Therefore, $x_o^2 \in Z(R) - \{0\}$ and x_o is not a zero divisor by Lemma 2.11(ii). From $x_o^2 d(y) = x_o d(y) x_o$ for all $y \in R$, we deduce that:

$$x_o(x_o d(y) - d(y)x_o) = 0.$$

Since x_o is not a zero divisor, we get that:

$$x_o d(y) = d(y)x_o \in Z(R)$$

for all $y \in R$.

(Case 2: By the same way, we can show that $x_o^2 \in Z(R) - \{0\}$. From

$$x_o^2 d(y)x_o = d(y)x_o^3 = x_o d(y)x_o^2 = x_o^3 d(y)$$

for all $y \in R$, we deduce that:

$$x_o^2(d(y)x_o - x_o d(y)) = 0.$$

Therefore,

$$x_o d(y) = d(y)x_o \in Z(R) \quad (5)$$

for all $y \in R$. Therefore, $d(x_o)$ is not a left zero divisor also).

Therefore:

$$\begin{aligned}
 0 &= (2x_o)d(y) \\
 &= d(y)(2x_o) \\
 &= 2d(y)x_o \\
 &= 2x_o d(y) \\
 &= x_o(2d(y))
 \end{aligned}$$

for all $y \in R$ and, hence, $2d(R) = \{0\}$. Therefore, $2R = \{0\}$ by Lemma 2.9.

For all $y \in R$ and using (5), we have:

$$\begin{aligned}
 x_o d(x_o y) &= x_o^2 d(y) + x_o d(x_o) y \\
 &= d(y)x_o^2 + yx_o d(x_o) \\
 &= d(y)x_o x_o + yd(x_o)x_o \\
 &= d(yx_o)x_o \\
 &= x_o d(yx_o),
 \end{aligned}$$

and then, $x_o(d(x_o y) - d(yx_o)) = 0$. Thus, $d(x_o y) = d(yx_o)$ for all $y \in R$. Replacing y by $x_o y$ and using that $d(Z(R)) = \{0\}$, we get:

$$\begin{aligned}
 x_o^2 d(y) &= d(x_o x_o y) = d(x_o y x_o) \\
 &= x_o d(yx_o) + d(x_o)yx_o \\
 &= x_o d(x_o y) + d(x_o)yx_o \\
 &= x_o^2 d(y) + x_o d(x_o)y + d(x_o)yx_o,
 \end{aligned}$$

and hence:

$$0 = x_o d(x_o)y + d(x_o)yx_o$$

$$\begin{aligned} &= d(x_o)x_o y + d(x_o)y x_o \\ &= d(x_o)(x_o y + y x_o). \end{aligned}$$

However, $d(x_o)$ is not a left zero divisor. Therefore:

$$x_o y = -y x_o = y x_o$$

for all $y \in R$ (since $2R = \{0\}$). Therefore, $x_o \in Z(R)$. Therefore, $d(x_o) = 0$ as $d(Z(R)) = \{0\}$, a contradiction with $x_o d(x_o) \neq 0$. Therefore, the case $x_o d(x_o) \neq 0$ is impossible. $\square$

Bell in [5] has proved that, if R is a 3-prime near-ring with a non-zero semigroup right (left) ideal U , such that $Ux = \{0\}$ ($xU = \{0\}$) for some $x \in R$, then $x = 0$. We conclude this paper by a generalization of his result.

Theorem 3.6 *Let R be a 3-prime near-ring, such that $Ux_o \subseteq Z(R)$ ($x_o U \subseteq Z(R)$), where U is a non-zero semigroup right (left) ideal of R and $x_o \in R$. Then, $x_o \in Z(R)$ and either $x_o = 0$ or R is a commutative ring.*

Proof If $Ux_o = \{0\}$, then $x_o = 0 \in Z(R)$ by the primeness of R . Therefore, suppose there exists $u_o \in U$, such that $u_o x_o \in Z(R) - \{0\}$. As $u_o x_o x_o \in Z(R)$, we have that $x_o \in Z(R)$. However, $x_o \neq 0$, so $U \subseteq Z(R)$ and then R is a commutative ring by Lemma 2.12. Similarly, we can write the proof for the case $x_o U \subseteq Z(R)$, but we need to show that if $x_o u_o \in Z(R) - \{0\}$, then $x_o \neq 0$. Indeed, if $x_o = 0$, then:

$$0u_o = (00)u_o = 0u_o 0 = 0$$

a contradiction with $0u_o \neq 0$. Therefore, $x_o \neq 0$. $\square$

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