RESEARCH ARTICLE

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Computing the nonabelian tensor squares of groups of order p^3q

Received: 14 November 2019 / Accepted: 22 October 2020 / Published online: 12 January 2021
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Abstract In this paper, we describe the explicit structures of nonabelian tensor squares of nonabelian groups of order p^3q , where p and q are distinct prime numbers and $p > 2$. Our method is based on determining the structures of their Schur multipliers by applying some well-known results and the presentations of groups, which leads us to obtain the orders of their nonabelian tensor squares.

Mathematics Subject Classification 19C09 · 20D99

1 Introduction

The nonabelian tensor square of a group G denoted by $G \otimes G$, is defined as a group generated by the symbols $g \otimes h$ satisfying the relations

$$gg' \otimes h = ({}^g g' \otimes {}^g h)(g \otimes h) \text{ and } g \otimes hh' = (g \otimes h)({}^h g \otimes {}^h h'),$$

for all $g, g', h, h' \in G$, where ${}^g g' = gg'g^{-1}$. It is a special case of the nonabelian tensor product which has its origins in homotopy theory and was introduced by Brown and Loday in [2]. Nevertheless, a computation of the tensor square itself is desirable because of its close relation to certain homotopy groups as well as to group homology. In particular,

$$\begin{aligned} \pi_3 \Sigma K(G, 1) &\cong \ker(G \otimes G \longrightarrow G) \quad [2] \\ H_2(G) &\cong \ker(G \otimes G \longrightarrow G) \quad [8], \end{aligned}$$

where $\pi_3 \Sigma K(G, 1)$ denotes the suspension of the classifying space of G , and where the nonabelian exterior square $G \wedge G$ is defined by imposing the additional relations $g \otimes g = 1$ on $G \otimes G$, for all $g \in G$, and both maps are induced by the commutator map.

One of the important aspect of the research in to the Schur multipliers and tensor squares of groups includes establishing their structures (Independently, the study of the Schur multiplier $H_2(G)$ dates back to 1904 [11]). In particular, it is an interesting problem to determine them when the order of groups are known. For instance, it has been studied completely in [4, 9, 10] in case of groups of orders p^2q and $8q$ where p and q are distinct prime numbers. The two latter works motivate us to extend their results. In this paper, our aim is to determine the structures of the groups $H_2(G)$ and $G \otimes G$, for all groups G of order p^3q , with $p > 2$.

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In 1899, Western [13] classified the classification of groups of order p^3q , where p and q are distinct prime numbers. He proved that there are 16 types of nonabelian groups of order $8q$, where q is odd and there are 25 types of nonabelian groups of order p^3q , where p is odd.

Theorem 1.1 [13, pp. 258–261] *Let G be a nonabelian group of order p^3q , where p and q are distinct prime numbers and $p > 2$. Then G is one of the following types:*

$$\langle a, b, d \mid a^{p^2} = b^p = d^q = 1, b^{-1}ab = a^{p+1}, ad = da, bd = db \rangle \quad (1)$$

$$\langle a, b, c, d \mid a^p = b^p = c^p = d^q = 1, ab = ba, ac = ca, c^{-1}bc = ab, ad = da, bd = db, cd = dc \rangle. \quad (2)$$

If $q \equiv 1 \pmod{p}$, there are the following:

$$\langle a, d \mid a^{p^3} = d^q = 1, a^{-1}da = d^\alpha \rangle, \quad (3)$$

where α (here and in the next five groups) is any primitive root of $\alpha^p \equiv 1 \pmod{q}$.

$$\langle a, b, d \mid a^{p^2} = b^p = d^q = 1, ab = ba, ad = da, b^{-1}db = d^\alpha \rangle, \quad (4)$$

$$\langle a, b, d \mid a^{p^2} = b^p = d^q = 1, ab = ba, a^{-1}da = d^\alpha, bd = db \rangle, \quad (5)$$

$$\langle a, b, c, d \mid a^p = b^p = c^p = d^q = 1, ab = ba, ac = ca, bc = cb, ad = da, bd = db, c^{-1}dc = d^\alpha \rangle, \quad (6)$$

$$\langle a, b, d \mid a^{p^2} = b^p = d^q = 1, b^{-1}ab = a^{p+1}, ad = da, b^{-1}db = d^\beta \rangle, \quad (7)$$

where $\beta = \alpha$, or $\alpha^2, \dots$, or α^{p-1} .

$$\langle a, b, c, d \mid a^p = b^p = c^p = d^q = 1, ab = ba, ac = ca, ad = da, bd = db, c^{-1}bc = ab, c^{-1}dc = d^\alpha \rangle, \quad (8)$$

$$\langle a, d \mid a^{p^3} = d^q = 1, a^{-1}da = d^\alpha \rangle, \quad (9)$$

where α (here and in the next group) is any primitive root of $\alpha^{p^2} \equiv 1 \pmod{q}$.

$$\langle a, b, d \mid a^{p^2} = b^p = d^q = 1, ab = ba, a^{-1}da = d^\alpha, bd = db \rangle, \quad (10)$$

$$\langle a, d \mid a^{p^3} = d^q = 1, a^{-1}da = d^\alpha \rangle, \quad (11)$$

where α is any primitive root of $\alpha^{p^3} \equiv 1 \pmod{q}$.

When $p \equiv 1 \pmod{q}$, there are the following types (where α, α_2 and α_3 are the primitive q th root of unity modulo p, p^2 and p^3 , respectively):

$$\langle a, d \mid a^{p^3} = d^q = 1, d^{-1}ad = a^{\alpha_3} \rangle, \quad (12)$$

$$\langle a, b, d \mid a^{p^2} = b^p = d^q = 1, ab = ba, ad = da, d^{-1}bd = b^\alpha \rangle, \quad (13)$$

$$\langle a, b, d \mid a^{p^2} = b^p = d^q = 1, ab = ba, d^{-1}ad = a^{\alpha_2}, db = bd \rangle, \quad (14)$$

$$\langle a, b, d \mid a^{p^2} = b^p = d^q = 1, ab = ba, d^{-1}ad = a^{\alpha_2}, d^{-1}bd = b^{\alpha_2^i} \rangle, \quad (15)$$

where $1 \leq i \leq q-1$.

$$\langle a, b, c, d \mid a^p = b^p = c^p = d^q = 1, ab = ba, ac = ca, bc = cb, ad = da, bd = db, d^{-1}cd = c^\alpha \rangle, \quad (16)$$

$$q = 2. \langle a, b, c, d \mid a^p = b^p = c^p = d^2 = 1, ab = ba, ac = ca, bc = cb, ad = da, dbd = b^{-1}, dcd = c^{-1} \rangle$$

$$q > 2. \langle a, b, c, d \mid a^p = b^p = c^p = d^q = 1, ab = ba, ac = ca, bc = cb, ad = da, d^{-1}bd = b^\alpha, d^{-1}cd = c^{\alpha^\lambda} \rangle, \quad (17)$$

where λ represents the different solutions of $xy \equiv 1 \pmod{q}$, in which $b \equiv a^x \pmod{p}$, and a and b are the primitive roots of $q \pmod{p}$.

$$\langle a, b, c, d \mid a^p = b^p = c^p = d^q = 1, ab = ba, ac = ca, bc = cb, d^{-1}ad = a^\alpha, d^{-1}bd = b^{\alpha^x}, d^{-1}cd = c^{\alpha^y} \rangle \quad (18)$$

where $q \equiv 0$ or $\pm 1 \pmod{3}$, and x and y may have any of the values $1, 2, \dots, q-1$.

$$\langle a, b, d \mid a^{p^2} = b^p = d^q = 1, b^{-1}ab = a^{p+1}, bd = db, d^{-1}ad = a^{\alpha^2} \rangle, \quad (19)$$

$$\langle a, b, c, d \mid a^p = b^p = c^p = d^q = 1, ab = ba, ac = ca, ad = da, c^{-1}bc = ab, d^{-1}bd = b^\alpha, d^{-1}cd = c^{\alpha^{q-1}} \rangle, \quad (20)$$

$$\langle a, b, c, d \mid a^p = b^p = c^p = d^q = 1, ab = ba, ac = ca, c^{-1}bc = ab, d^{-1}ad = a^\alpha, db = bd, d^{-1}cd = c^\alpha \rangle, \quad (21)$$

$$\langle a, b, c, d \mid a^p = b^p = c^p = d^q = 1, ab = ba, ac = ca, c^{-1}bc = ab, d^{-1}ad = a^\alpha, d^{-1}bd = b^{\alpha^x}, d^{-1}cd = c^{\alpha^{q+1-x}} \rangle, \quad (22)$$

where $x = 2$ or $3, \dots$, or $\frac{q+1}{2}$ and $q > 2$.

When $p \equiv -1 \pmod{q}$, and $q > 2$, there are the following two types:

$$\langle a, b, c, d \mid a^p = b^p = c^p = d^q = 1, ab = ba, ac = ca, bc = cb, ad = da, d^{-1}bd = c, d^{-1}cd = b^{-1}c^{t^p+t} \rangle, \quad (23)$$

where t (here and in the next group) is any primitive Galoisian root of $t^q \equiv 1 \pmod{p}$.

$$\langle a, b, c, d \mid a^p = b^p = c^p = d^q = 1, ab = ba, ac = ca, c^{-1}bc = ab, ad = da, d^{-1}bd = c, d^{-1}cd = b^{-1}c^{t^p+t} \rangle. \quad (24)$$

And, lastly, when $p^2 + p + 1 \equiv 0 \pmod{q}$, and $q > 3$, there is the following type:

$$\langle a, b, c, d \mid a^p = b^p = c^p = d^q = 1, ab = ba, ac = ca, bc = cb, d^{-1}ad = b, d^{-1}bd = c, d^{-1}cd = ab^{-\lambda^{-1}-\lambda^{-p}-\lambda^{-p^2}}c^{\lambda+\lambda^p+\lambda^{p^2}} \rangle, \quad (25)$$

where λ is a Galois imaginary of the third order, which is a primitive root of $\lambda^q \equiv 1 \pmod{p}$.

In this paper, we use the following notations frequently:

$G^{ab} := G/G'$ is the abelianization of G ;

$e(G)$ is the exponent of G ;

C_p^k is the direct product of k copies of the cyclic group of order p ;

$\mathcal{M}(G)$ is the Schur multiplier of G ;

$E_{p^3}^1$ is the extraspecial p -group of order p^3 and exponent p ;

Φ_i is the isoclinic family of groups of order p^n , where $n \leq 6$ and $p \neq 2$ given in [6].

Our main theorems are the following descriptions of the nonabelian tensor squares of nonabelian groups of order p^3q , $p > 2$.

Theorem A Let G be a group of order p^3q , where p and q are distinct prime numbers with $2 < p < q$.

- (i) if G is any group of types (3), (9) or (11), then $G \otimes G \cong C_{p^3q}$;
- (ii) if G is any group of types (1) or (7), then $G \otimes G \cong C_p^3 \times C_{pq}$;
- (iii) if G is any group of types (4), (5) or (10), then $G \otimes G \cong C_{p^2} \times C_p^2 \times C_{pq}$;
- (iv) if G is any group of types (2) or (8), then $G \otimes G \cong C_p^5 \times C_{pq}$;
- (v) if G is of type (6), then $G \otimes G \cong C_p^8 \times C_{pq}$.

Theorem B Let G be a group of order p^3q , where p and q are distinct prime numbers with $p > q \geq 2$.

- (i) if G is of type (12), then $G \otimes G \cong C_{p^3q}$;

- (ii) if G is any group of types (13), (14) or (19), then $G \otimes G \cong C_{p^2} \times C_{pq}$;
 (iii) if G is any group of types (20) or (24), then $G \otimes G \cong E_{p^3}^1 \times C_q$;
 (iv) if G is of type (15), then

$$G \otimes G \cong \begin{cases} C_{p^2} \times C_{pq}, & \text{when } 1 \leq i \leq q-2; \\ \Phi_2(211)_c \times C_q, & \text{when } i = q-1; \end{cases}$$

- (v) if G is any group of types (17), (21) or (23), then

$$G \otimes G \cong \begin{cases} C_p^2 \times C_{pq}, & \text{when } \mathcal{M}(G) = 1; \\ E_{p^3}^1 \times C_{pq}, & \text{when } \mathcal{M}(G) \cong C_p; \end{cases}$$

- (vi) if G is of type (22), then

$$G \otimes G \cong \begin{cases} E_{p^3}^1 \times C_q, & \text{when } \mathcal{M}(G) = 1; \\ \Phi_3(1^4) \times C_q, & \text{when } \mathcal{M}(G) \cong C_p; \\ \Phi_3(1^5) \times C_q, & \text{when } \mathcal{M}(G) \cong C_p^2; \end{cases}$$

- (vii) if G is any group of types (18) or (25), then

$$G \otimes G \cong \begin{cases} C_p^2 \times C_{pq}, & \text{when } \mathcal{M}(G) = 1; \\ \Phi_2(1^4) \times C_q, & \text{when } \mathcal{M}(G) \cong C_p; \\ \Phi_4(1^5) \times C_q, & \text{when } \mathcal{M}(G) \cong C_p^2; \\ \Phi_{11}(1^6) \times C_q, & \text{when } \mathcal{M}(G) \cong C_p^3; \end{cases}$$

- (viii) if G is of type (16), then $G \otimes G \cong C_p^4 \times C_{pq}$.

2 Preliminaries

For a group G , the following relations hold in $G \otimes G$ for all $g, g', h, h' \in G$ (see [3] for more details):

- (i) ${}^s(g' \otimes h) = {}^s g' \otimes {}^s h$;
 (ii) $[g, h] \otimes h' = (g \otimes h)^{h'}(g \otimes h)^{-1}$;
 (iii) $g' \otimes [g, h] = {}^{g'}(g \otimes h)(g \otimes h)^{-1}$;
 (iv) $[g \otimes h, g' \otimes h'] = [g, h] \otimes [g', h']$,

where $[g, h] = ghg^{-1}h^{-1}$.

Let $\nabla(G)$ denote the subgroup of $G \otimes G$ generated by the elements $g \otimes g$ for all $g \in G$. The nonabelian exterior square $G \wedge G$ of G is then defined as $(G \otimes G)/\nabla(G)$. The commutator map induces homomorphisms $k : G \otimes G \longrightarrow G$ and $k' : G \wedge G \longrightarrow G$ such that $k(g \otimes h) = k'(g \wedge h) = [g, h]$, for all $g, h \in G$. The kernels of k and k' are central and denoted by $J_2(G)$ and $\mathcal{M}(G)$, respectively. The topological interest of $J_2(G)$ [2] is as in the following

$$J_2(G) = \pi_3 \Sigma K(G, 1).$$

Thus, if G' is cyclic, then both $G \otimes G$ and $G \wedge G$ are abelian.

Proposition 2.1 [7, Corollaries 2.2.6, 2.5.3 and Theorem 2.5.5] *Let N be a normal subgroup of a finite group G and $H = G/N$. Then*

- (i) $|\mathcal{M}(H)|$ divides $|\mathcal{M}(G)||G' \cap N|$;
 (ii) if N is central, then $|\mathcal{M}(G)||G' \cap N|$ divides $|\mathcal{M}(H)||\mathcal{M}(N)||H \otimes N|$;
 (iii) if N is a normal Hall subgroup with complement subgroup T , then $\mathcal{M}(G) \cong \mathcal{M}(T) \times \mathcal{M}(N)^T$.

Proposition 2.2 [3, Proposition 8] *If G is a group such that G' has a cyclic complement, then*

$$G \otimes G \cong (G \wedge G) \times G^{ab}.$$

In particular if $\mathcal{M}(G) = 1$ then $G \otimes G \cong G' \times G^{ab}$.

It is known that the groups of order p^3q are solvable. It follows immediately that they are polycyclic. So we can apply the following result to compute their nonabelian tensor squares.

Proposition 2.3 *Let G be a finite polycyclic group with a polycyclic generating sequence $g_1, \dots, g_k$. Then*

- (i) [1, Proposition 20] $G \otimes G = \langle g_j \otimes g_j, g_i \otimes g_j, (g_i \otimes g_j)(g_j \otimes g_i) \rangle$;
- (ii) $G \wedge G = \langle g_i \wedge g_j \rangle$;

in which $1 \leq j < i \leq k$.

The following proposition describes the derived subgroups of groups of order p^3q .

Proposition 2.4 *Let G be a nonabelian group of order p^3q with $p > 2$. Then*

- (i) if G is any group of types (1), (2), (13) or (16), then $G' \cong C_p$;
- (ii) if G is any group of types (3), (4), (5), (6), (9), (10) or (11), then $G' \cong C_q$;
- (iii) if G is any group of types (14) or (19), then $G' \cong C_{p^2}$;
- (iv) if G is any group of types (7) or (8), then $G' \cong C_{pq}$;
- (v) if G is any group of types (17), (21) or (23), then $G' \cong C_p^2$;
- (vi) if G is of type (12), then $G' \cong C_{p^3}$;
- (vii) if G is any group of types (18) or (25), then $G' \cong C_p^3$;
- (viii) if G is of type (15), then $G' \cong C_{p^2} \times C_p$;
- (ix) if G is any group of types (20), (22) or (24), then $G' \cong E_{p^3}^1$.

3 Proof of the main theorems

At the first we establish the structure of the Schur multipliers of groups of order p^3q . Assume that G is an abelian group of order p^3q , where p and q are distinct prime numbers and $p > 2$ stated in Theorem 1.1. The Schur multiplier of G may be obtained immediately by [7, Theorem 2.2.10]. So we proceed by the nonabelian cases.

Proposition 3.1 *Let G be a nonabelian group of order p^3q with $2 < p < q$. Then*

$$\mathcal{M}(G) \cong \begin{cases} 1, & \text{if } G \text{ is any group of types (1), (3), (7), (9) or (11)} \\ C_p, & \text{if } G \text{ is any group of types (4), (5) or (10)} \\ C_p^2, & \text{if } G \text{ is any group of types (2) or (8)} \\ C_p^3, & \text{if } G \text{ is of type (6).} \end{cases}$$

Proof If G is any group of types (1) or (7), then $Z = \langle d \rangle \cong C_q$ is a central subgroup of G . Thus G/Z is a nonabelian group of order p^3 and [7, Theorem 3.3.6] shows that $\mathcal{M}(G/Z) = 1$. Also $G/Z \otimes Z = 1$, because here $\otimes$ is the ordinary tensor product of Abelian groups. Now from Proposition 2.1, it follows that $\mathcal{M}(G) = 1$. If G is of type (3), then since $G^{ab} \cong C_{p^3}$ is cyclic, by [7, Corollary 2.7.7], we have $\mathcal{M}(G) \mid q$ and [7, Theorem 2.1.5] shows that $\mathcal{M}(G) = 1$. For any group of types (9) or (11) we can prove the result similarly.

If G is of type (4), then by Proposition 2.1, we obtain $p \mid |\mathcal{M}(G)|$. On the other hand, since $Z = \langle a \rangle \cong C_{p^2}$ is a central subgroup of G and G/Z is a nonabelian group of order pq , then by [4, Lemma 3.2(i)], it follows that $\mathcal{M}(G/Z) = 1$. As $G/Z \otimes Z \cong C_p$, then from Proposition 2.1 we conclude $|\mathcal{M}(G)| \mid p$, which implies that $\mathcal{M}(G) \cong C_p$. If G is any group of types (5) or (10), then by choosing $Z = \langle b \rangle \cong C_p$ as a central subgroup of G , one can show analogously that $\mathcal{M}(G) \cong C_p$.

If G is of type (2), then by choosing $Z = \langle d \rangle \cong C_q$ as a central subgroup of G , it follows by [7, Theorem 3.3.6] that $\mathcal{M}(G/Z) = C_p^2$ and $G/Z \otimes Z = 1$. Now Proposition 2.1 implies that $\mathcal{M}(G) \cong C_p^2$. If G is of type (8), then by choosing $Z = \langle a \rangle \cong C_p$ as a central subgroup of G and $H = \langle d \rangle$ as a normal subgroup of G , one can deduce that $\mathcal{M}(G) \cong C_p^2$ using Proposition 2.1. Finally, assume G is of type (6). If take $Z = \langle a \rangle$ as a central subgroup of G , we conclude that $\mathcal{M}(G/Z) \cong C_p$ by [4, Lemma 3.2(ii)]. Hence, it follows from Proposition 2.1 that $\mathcal{M}(G) \cong C_p^3$. $\square$

Proposition 3.2 *Let G be a nonabelian group of order p^3q with $p > q \geq 2$. Then*

$$\mathcal{M}(G) \cong \begin{cases} 1, & \text{if } G \text{ is any group of types (12), (13), (14), (19), (20) or (24)} \\ 1, & \text{if } G \text{ is of type (15) and } 1 \leq i \leq q-2 \\ C_p, & \text{if } G \text{ is of type (15) and } i = q-1 \\ C_p, & \text{if } G \text{ is of type (16)} \\ 1 \text{ or } C_p, & \text{if } G \text{ is any group of types (17), (21) or (23)} \\ 1, C_p \text{ or } C_p^2, & \text{if } G \text{ is of type (22)} \\ 1, C_p, C_p^2 \text{ or } C_p^3, & \text{if } G \text{ is any group of types (18) or (25).} \end{cases}$$

Proof If G is of type (12), then by Proposition 2.1(iii) we get $\mathcal{M}(G) = 1$. Also utilizing the method used for the type (4), one can show that $\mathcal{M}(G) = 1$ for the type (13).

If G is of type (14), then by choosing $Z = \langle b \rangle \cong C_p$ as a central subgroup of G , we can see by [4, Lemma 3.2(iii)] that $\mathcal{M}(G/Z) = 1$. Hence, $\mathcal{M}(G) = 1$ by Proposition 2.1. If G is of type (19), then by setting $H = \langle a, b \rangle$ as a normal subgroup of G and using Proposition 2.1 it follows that $\mathcal{M}(G) = 1$.

If G is any group of types (20), (24) or (16), the result follows by applying a same method used for the type (14). If G is of type (15), then we observe that $a \wedge b$ belongs to $\mathcal{M}(G)$. Since $[a, b] = 1$, then the relations of $G \otimes G$ imply that the action of G on $a \wedge b$ is trivial. Moreover, we have $a^m \wedge b^n = (a \wedge b)^{mn}$ for any integers m, n . So $a \wedge b = {}^d(a \wedge b) = {}^d a \wedge {}^d b = a^{\alpha_2} \wedge b^{\alpha_2^i} = (a \wedge b)^{\alpha_2^{i+1}}$, which implies that $a \wedge b = a \wedge b^{\alpha_2^{i+1}}$, where $1 \leq i \leq q-1$. Let $1 \leq i \leq q-2$. Since $p \nmid \alpha_2^{i+1}$, then $\alpha_2^{i+1} = kp + r$ for some integers k and r with $2 \leq r \leq p-1$. Thus $a \wedge b = a \wedge b^{kp+r} = a \wedge b^r = (a \wedge b)^r$, whence $(a \wedge b)^{r-1} = 1$. On the other hand $1 = a \wedge b^p = (a \wedge b)^p$. Therefore, $a \wedge b = 1$ and we deduce that $\mathcal{M}(G) = 1$. For the case $i = q-1$, we can see that $\mathcal{M}(G) \cong C_p$.

If G is the first presentation of type (17), where $q = 2$, then $Z = \langle a \rangle \cong C_p$ is a central subgroup of G so that $\mathcal{M}(G/Z) = 1$ or $\mathcal{M}(G/Z) \cong C_p$ by [4, Lemma 3.2(iii)]. So $\mathcal{M}(G) = 1$ or $|\mathcal{M}(G)| \mid p$ respectively, by Proposition 2.1. For the second case, by the exact sequence given in [7, Theorem 2.5.6(ii)] we have $p \mid |\mathcal{M}(G)|$. Thus, $\mathcal{M}(G) \cong C_p$. For the second presentation of type (17), where $q > 2$, the result follows easily. A similar argument can be applied to the type (23) too.

If G is of type (21), then $H = \langle a, b, c \rangle$ is a normal subgroup of G . We conclude from [7, Theorem 3.3.6] that $\mathcal{M}(H) \cong C_p^2$. Hence Proposition 2.1 yields $\mathcal{M}(G) \cong 1, C_p$ or C_p^2 . On the other hand $H = \langle a, c, d \rangle$ is a normal subgroup of G of order p^2q so that by [4, Lemma 3.2(iii)] we obtain $\mathcal{M}(H) = 1$ or $\mathcal{M}(H) \cong C_p$. Thus [7, Corollary 2.7.7] shows that $|\mathcal{M}(G)| \mid q$ or $|\mathcal{M}(G)| \mid pq$ whence $\mathcal{M}(G) = 1$ or $\mathcal{M}(G) \cong C_p$. For any group of types (18), (22) and (25), the result may be obtained by Proposition 2.1. $\square$

Remark 3.3 Using GAP [12], if G is of type (21), we have $\mathcal{M}(G) = 1$ when $p = 7, q = 3$ and $\alpha = 2$, and $\mathcal{M}(G) \cong C_p$ when $p = 3, q = 2$ and $\alpha = 2$. If G is of type (22), then

$$\mathcal{M}(G) \cong \begin{cases} 1, & \text{if } p = 11, q = 5, x = 3 \text{ and } \alpha = 3 \\ C_p, & \text{if } p = 11, q = 5, x = 2 \text{ and } \alpha = 3 \\ C_p^2, & \text{if } p = 7, q = 3, x = 2 \text{ and } \alpha = 2. \end{cases}$$

For the type (18), we have

$$\mathcal{M}(G) \cong \begin{cases} 1, & \text{if } p = 11, q = 5, x = 1, y = 3 \text{ and } \alpha = 3 \\ C_p, & \text{if } p = 11, q = 5, x = 2, y = 3 \text{ and } \alpha = 3 \\ C_p^2, & \text{if } p = 7, q = 3, x = 1, y = 2 \text{ and } \alpha = 2 \\ C_p^3, & \text{if } p = 3, q = 2, x = 1, y = 1 \text{ and } \alpha = 2. \end{cases}$$

Now the following result give the order of nonabelian tensor square of groups of order p^3q .

Corollary 3.4 *Let G be a nonabelian group of order p^3q .*

(i) If $2 < p < q$, then

$$|G \otimes G| = \begin{cases} p^3q, & \text{if } G \text{ is any group of types (3), (9) or (11)} \\ p^4q, & \text{if } G \text{ is any group of types (1) or (7)} \\ p^5q, & \text{if } G \text{ is any group of types (4), (5) or (10)} \\ p^6q, & \text{if } G \text{ is any group of types (2) or (8)} \\ p^9q, & \text{if } G \text{ is of type (6).} \end{cases}$$

(ii) If $p > q \geq 2$, then

$$|G \otimes G| = \begin{cases} p^3q, & \text{if } G \text{ is any group of types (12), (13), (14), (19), (20) or (24)} \\ p^3q, \text{ or } p^4q, & \text{if } G \text{ is any group of types (15), (17), (21) or (23)} \\ p^3q, p^4q, \text{ or } p^5q, & \text{if } G \text{ is of type (22)} \\ p^5q, & \text{if } G \text{ is of type (16)} \\ p^3q, p^4q, p^5q, \text{ or } p^6q, & \text{if } G \text{ is any group of types (18) or (25).} \end{cases}$$

Proof It follows from Propositions 2.2, 3.1 and 3.2, [7, Theorem 2.2.10] and [5, Lemma 2.3]. $\square$

We are ready to prove our main theorems. Note that if G is an abelian group of order p^3q , where p and q are distinct prime numbers with $p > 2$ given in [13], then the tensor square of G may be computed by [3, Proposition 11].

- Proof of Theorem A* (i) If G is any group of types (3) or (9), then from the natural epimorphism $G \otimes G \rightarrow G^{ab} \otimes G^{ab}$ and Corollary 3.4 we obtain $p^3 \mid e(G \otimes G)$. Since $G \otimes G$ is abelian, the result holds. If G is of type (11), as G' has the cyclic complement $G^{ab} \cong C_{p^3}$, then Proposition 2.2 implies that $G \otimes G \cong C_{p^3q}$.
- (ii) If G is of type (1), then clearly $G \otimes G \cong G^{ab} \otimes G^{ab} \cong C_p^3 \times C_{pq}$. Also for the type (7), the result follows from Corollary 3.4 and the epimorphism $G \otimes G \rightarrow G^{ab} \otimes G^{ab} \cong C_p^4$.
- (iii) If G is of type (4), then from the epimorphism $G \otimes G \rightarrow G^{ab} \otimes G^{ab}$ and Corollary 3.4 we obtain $p^2 \mid e(G \otimes G)$. Assume $H = \langle a^p \rangle \cong C_p$. Then by [4, Theorem B(ii)] we conclude $G/H \otimes G/H \cong C_p^4 \times C_q$. Now the natural epimorphism $G \otimes G \rightarrow G/H \otimes G/H$ shows that $G \otimes G \cong C_{p^2} \times C_p^3 \times C_q$. For any group of types (5) and (10) the result follows by the same argument.
- (iv) If G is of type (2), then $G \otimes G$ is abelian. Now consider the central subgroup $Z = \langle d \rangle$. As $G/Z \cong E_{p^3}^1$, then [5, Corollary 2.4] yields $G/Z \otimes G/Z \cong C_p^6$. Thus, it follows from the exact sequence given in [3, Proposition 9] that $G \otimes G \cong C_p^5 \times C_{pq}$. If G is of type (8), then by choosing the normal subgroup $H = \langle d \rangle$, and using the natural epimorphism $G \otimes G \rightarrow G/H \otimes G/H \cong C_p^6$, one could prove the assertion.
- (v) If G is of type (6), then by the same way used for type (7) it follows that $G \otimes G \cong C_p^8 \times C_{pq}$. $\square$

Proof of Theorem B The cases (i), (ii) and (iii) follow by Proposition 2.2.

(iv) If $M(G) = 1$, then by Proposition 2.2, $G \otimes G \cong C_{p^2} \times C_{pq}$. In the case $M(G) \cong C_p$ as every element g in G may be presented by $g = a^r b^s d^t$ for some integers r, s, t , then one can easily show that $G \wedge G = \langle a \wedge b, b \wedge d, a \wedge d \rangle$. Now the epimorphism $k' : G \wedge G \rightarrow G'$ implies that the generators are independent. Also it follows by induction on any integer n that $a^n \wedge d = (a \wedge d)^n$. So $|a \wedge d| \mid p^2$. As $p^2 \mid |a \wedge d|$, then $|a \wedge d| = p^2$. Similarly $|b \wedge d| = p$. On the other hand, we know $|G \wedge G| = |M(G)||G'| = p^4$ which implies that $|a \wedge b| = p$. Now the relation $[a \wedge d, b \wedge d] = [a, d] \wedge [b, d] = (a \wedge b)^{(\alpha_2-1)^2} \neq 1$ implies $(G \wedge G)' = \langle (a \wedge b)^{(\alpha_2-1)^2} \rangle \cong C_p$. It follows immediately that $G \wedge G \cong \Phi_2(211)c$. Because by [6], a group G of order p^4 with derived subgroup of order p lies in the family Φ_2 . On the other hand, in this family the group $\Phi_2(211)c$ is the only 2-generated group with generators of orders p^2 and p . Therefore Proposition 2.2 yields $G \otimes G \cong \Phi_2(211)c \times C_q$.

(v) If $M(G) = 1$, then $G \otimes G \cong C_p^2 \times C_{pq}$ by Proposition 2.2. If $M(G) \cong C_p$, then by considering the groups of types (17), (21) or (23), we proceed the proof. Suppose G is of type (17), then $|G \wedge G| = p^3$. Set $N = \langle a \rangle$. Obviously $(G/N)' \cong C_p^2$ and $(G/N)^{ab} \cong C_q$. By proof of Proposition 2.3 we know that in this case $M(G/N) \cong C_p$. Hence it follows from Theorem C4 that $G/N \otimes G/N \cong C_q \times H$ where H

is an extraspecial p -group of order p^3 . Now, Proposition 2.3 implies that $G/N \wedge G/N \cong H$. Therefore $G \wedge G \cong G/N \wedge G/N$. As $b \wedge c, b \wedge d$ and $c \wedge d$ are non-trivial independent generators of orders p , then $G \wedge G = \langle b \wedge c, b \wedge d, c \wedge d \rangle \cong E_{p^3}^1$ and so $G \otimes G \cong E_{p^3}^1 \times C_{pq}$ by Proposition 2.3. For the second presentation of type (17), where $q > 2$, one may follow the same argument.

Assume G is of type (21) and $\mathcal{M}(G) \cong C_p$. We know that $|G \wedge G| = p^3$. The polycyclic presentation of G is as follows:

$$\langle a_1, a_2, a_3, a_4 \mid a_1^q = a_2^p = a_3^p = a_4^p = 1, a_2^{a_1} = a_2^\alpha, a_3^{a_1} = a_3, a_4^{a_1} = a_4^\alpha, a_3^{a_2} = a_3 a_4, a_4^{a_2} = a_4, a_4^{a_3} = a_4 \rangle.$$

The above generating set form polycyclic generating sequence so that Proposition 2.3 provides a generating set $\{a_2 \wedge a_1, a_3 \wedge a_1, a_4 \wedge a_1, a_3 \wedge a_2, a_4 \wedge a_2, a_4 \wedge a_3\}$ for $G \wedge G$. Obviously $a_4 \wedge a_3 = a_3 \wedge a_1 = 1$. We claim that $a_3 \wedge a_2$ can be generated by $a_4 \wedge a_2$ and $a_4 \wedge a_1$. First by induction observe that for any integer n , $a_3 \wedge a_2^n = (a_3 \wedge a_2)^n (a_4 \wedge a_2)^{\binom{n}{2}}$. Furthermore $a_3^{-1} \wedge a_2^n = (a_4 \wedge a_2)^{-\binom{n}{2}} (a_3 \wedge a_2)^{-n}$, which implies that $a_3^{-1} \wedge a_2^{\alpha-1} = (a_4 \wedge a_2)^{-\binom{\alpha-1}{2}} (a_3 \wedge a_2)^{1-\alpha}$. On the other hand $a_3^{-1} \wedge a_2^{\alpha-1} = a_3^{-1} \wedge [a_2, a_1] = (a_2 \wedge a_4)^{\alpha-1} (a_4 \wedge a_1)$. So the claim holds by equating the last two equalities. Therefore $G \wedge G = \langle a_4 \wedge a_2, a_4 \wedge a_1, a_2 \wedge a_1 \rangle$. The epimorphism $G \wedge G \rightarrow G'$ implies that $a_4 \wedge a_1$ and $a_2 \wedge a_1$ are non-trivial independent generators whose orders are divided by p . On the other hand $a_4^p \wedge a_1 = (a_4 \wedge a_1)^p = 1$ and $a_2^p \wedge a_1 = (a_2 \wedge a_1)^p = 1$. Hence $|a_4 \wedge a_1| = |a_2 \wedge a_1| = p$. As $[a_4 \wedge a_1, a_2 \wedge a_1] = \langle a_4 \wedge a_2 \rangle = \mathcal{M}(G)$, then $G \wedge G \cong E_{p^3}^1$ and $G \otimes G \cong E_{p^3}^1 \times C_{pq}$ by Proposition 2.2.

Assume G is of type (23). By putting $Z = \langle a \rangle \leq Z(G)$ it follows from Proposition 2.1 that $\mathcal{M}(G/Z) \cong C_p$. Since $(G/Z)'$ has the cyclic complement $(G/Z)^{ab} \cong C_q$, then $G/Z \otimes G/Z \cong (G/Z \wedge G/Z) \times C_q$ by Proposition 2.2. Hence [4, Theorem C(iii)] yields $G/Z \wedge G/Z \cong E_{p^3}^1$. On the other hand $|G \wedge G| = p^3$ which implies that $G \wedge G \cong G/Z \wedge G/Z$. Therefore $G \otimes G \cong E_{p^3}^1 \times C_{pq}$.

(vi) Suppose G is of type (22). When $M(G) = 1$, Proposition 2.2 shows that $G \otimes G \cong E_{p^3}^1 \times C_q$. Assume $\mathcal{M}(G) \cong C_p$. The polycyclic presentation of G is as follows:

$$\langle a_1, a_2, a_3, a_4 \mid a_1^q = a_2^p = a_3^p = a_4^p = 1, a_2^{a_1} = a_2^{c_3}, a_3^{a_1} = a_3^{c_2}, a_4^{a_1} = a_4^\alpha, a_3^{a_2} = a_3 a_4, a_4^{a_2} = a_4, a_4^{a_3} = a_4 \rangle,$$

in which $c_3 = \alpha^{q+1-x} \bmod p$ and $c_2 = \alpha^x \bmod p$. Since $a_2, a_3, a_4 \in G'$, we get $a_2 \otimes a_2 = a_3 \otimes a_3 = a_4 \otimes a_4 = 1$. On the other hand

$$a_4 \otimes a_2 = [a_3, a_2] \otimes a_2 = (a_3 \otimes a_2)^{a_2} (a_3 \otimes a_2)^{-1} = (a_3 \otimes a_2) (a_3 a_4 \otimes a_2)^{-1} = (a_4 \otimes a_2)^{-1},$$

which shows that $(a_4 \otimes a_2)^2 = 1$. As $1 = a_4^p \otimes a_2 = (a_4 \otimes a_4)^p$, we must have $a_4 \otimes a_2 = 1$.

Now we claim that $a_3 \otimes a_4$ can be generated by $a_4 \otimes a_1$ and $a_4 \otimes a_3$. First observe that for any integer m , $a_3 \otimes a_2^m = (a_3 \otimes a_2)^m$. Hence

$$\begin{aligned} a_3^{-1} \otimes a_2^m &= a_3^{-1} (a_3 \otimes a_2^m)^{-1} \\ &= a_3^{-1} (a_3 \otimes a_2)^{-m} \\ &= (a_3 \otimes a_2 a_4)^{-m} \\ &= [(a_3 \otimes a_2) (a_3 a_4 \otimes a_4)]^{-m} \\ &= (a_3 \otimes a_2)^{-m} (a_4 \otimes a_3)^m, \end{aligned}$$

which implies that

$$a_3^{-1} \otimes a_2^{c_3-1} = (a_3 \otimes a_2)^{1-c_3} (a_4 \otimes a_3)^{c_3-1}.$$

Also for any integer n , $a_3^n \otimes a_2 = (a_3 \otimes a_2)^n (a_3 \otimes a_4)^{\binom{n}{2}} = (a_3 \otimes a_2)^n (a_4 \otimes a_3)^{-\binom{n}{2}}$. Hence $a_3^n \otimes a_2^m = (a_3 \otimes a_2)^{nm} (a_3 \otimes a_4)^{m \binom{n}{2}} = (a_3 \otimes a_2)^{nm} (a_4 \otimes a_3)^{-m \binom{n}{2}}$.

On the other hand

$$\begin{aligned} a_3^{-1} \otimes a_2^{c_3-1} &= a_3^{-1} \otimes [a_2, a_1] \\ &= a_3^{-1} (a_2 \otimes a_1)(a_2 \otimes a_1)^{-1} \\ &= (a_2 a_4 \otimes a_1 a_3^{c_2-1})(a_2 \otimes a_1)^{-1} \\ &= a_2 (a_4 \otimes a_1)(a_2 \otimes a_1)^{a_1} [a_2 (a_4 \otimes a_3^{c_2-1})(a_2 \otimes a_3^{c_2-1})] (a_2 \otimes a_1)^{-1} \\ &= (a_4 \otimes a_1)(a_2 \otimes a_1)(a_4 \otimes a_3)^{c_2-1} (a_2^{c_3} \otimes a_3^{c_2(c_2-1)})(a_2 \otimes a_1)^{-1} \\ &= (a_4 \otimes a_1)(a_4 \otimes a_3)^{c_2-1+c_3(c_2(c_2-1))} (a_3 \otimes a_2)^{c_3 c_2(c_2-1)}. \end{aligned}$$

Thus

$$(a_3 \otimes a_2)^{1-c_3(1+c_2(c_2-1))} = (a_4 \otimes a_1)(a_4 \otimes a_3)^{c_2-c_3((c_2(c_2-1))+1)}.$$

Therefore $G \wedge G = \langle a_2 \wedge a_1, a_3 \wedge a_1, a_4 \wedge a_1, a_4 \wedge a_3 \rangle$ by Proposition 2.3. It is readily seen that $|a_2 \wedge a_1| = |a_3 \wedge a_1| = |a_4 \wedge a_1| = |a_4 \wedge a_3| = p$. Moreover we have

$$(G \wedge G)' = \langle (a_4 \wedge a_3)^{(\alpha-1)(c_2-1)}, a_3^{c_2-1} \wedge a_2^{c_3-1} \rangle,$$

from which we deduce that $|(G \wedge G)'| = p^2$. So $G \wedge G \cong \Phi_3(1^4)$ by [6] and then $G \otimes G \cong \Phi_3(1^4) \times C_q$.

If $\mathcal{M}(G) \cong C_p^2$, since $(G \wedge G)/Z(G \wedge G)$ is a nonabelian group of order p^4 we get $G \wedge G \cong \Phi_3(1^5)$ by [6]. Hence, $G \otimes G \cong \Phi_3(1^5) \times C_q$ by Proposition 2.2.

(vii) Assume G is of type (18). Then by Proposition 3.2, $M(G) \cong 1, C_p, C_p^2$ or C_p^3 . If $M(G) = 1$, then $G \wedge G \cong G'$ and the result holds by Proposition 2.2. Suppose that $M(G) \cong C_p$. It is readily seen that $|G \wedge G| = p^4$. The polycyclic presentation of G is as follows:

$$\langle a_1, a_2, a_3, a_4 \mid a_1^q = a_2^p = a_3^p = a_4^p = 1, a_2^{a_1} = a_2^{\alpha}, a_3^{a_1} = a_3^{c_3}, a_4^{a_1} = a_4^{c_4}, a_3^{a_2} = a_3, a_4^{a_2} = a_4, a_4^{a_3} = a_4 \rangle,$$

in which $c_3 = \alpha^x \bmod p$ and $c_4 = \alpha^y \bmod p$. By Proposition 2.3 we get

$$G \wedge G = \langle a_2 \wedge a_1, a_3 \wedge a_1, a_4 \wedge a_1, a_3 \wedge a_2, a_4 \wedge a_2, a_4 \wedge a_3 \rangle,$$

and $\mathcal{M}(G) = \langle a_3 \wedge a_2, a_4 \wedge a_2, a_4 \wedge a_3 \rangle$. It follows from the epimorphism $G \wedge G \longrightarrow G'$ that $a_2 \wedge a_1, a_3 \wedge a_1$ and $a_4 \wedge a_1$ are non-trivial independent generators such that p divides their orders. On the other hand $a_2^p \wedge a_1 = (a_2 \wedge a_1)^p = 1, a_3^p \wedge a_1 = (a_3 \wedge a_1)^p = 1$, and $a_4^p \wedge a_1 = (a_4 \wedge a_1)^p = 1$. Hence $|a_2 \wedge a_1| = |a_3 \wedge a_1| = |a_4 \wedge a_1| = p$. Moreover

$$\begin{aligned} [a_2 \wedge a_1, a_3 \wedge a_1] &= [a_2, a_1] \wedge [a_3, a_1] = a_2^{\alpha-1} \wedge a_3^{c_3-1} = (a_2 \wedge a_3)^{(\alpha-1)(c_3-1)}, \\ [a_2 \wedge a_1, a_4 \wedge a_1] &= [a_2, a_1] \wedge [a_4, a_1] = a_2^{\alpha-1} \wedge a_4^{c_4-1} = (a_2 \wedge a_4)^{(\alpha-1)(c_4-1)}, \\ [a_3 \wedge a_1, a_4 \wedge a_1] &= [a_3, a_1] \wedge [a_4, a_1] = a_3^{c_3-1} \wedge a_4^{c_4-1} = (a_3 \wedge a_4)^{(c_3-1)(c_4-1)} (a_2 \wedge a_3)^{(c_4-1)(c_3-1)}. \end{aligned}$$

Thus

$$(G \wedge G)' = \langle (a_2 \wedge a_3)^{(\alpha-1)(c_3-1)}, (a_2 \wedge a_4)^{(\alpha-1)(c_4-1)}, (a_3 \wedge a_4)^{(c_3-1)(c_4-1)} (a_2 \wedge a_3)^{(c_4-1)(c_3-1)} \rangle,$$

and consequently $(G \wedge G)' = \mathcal{M}(G)$. Therefore by applying [12] we conclude that $G \wedge G \cong \Phi_2(1^4)$ and $G \otimes G \cong \Phi_2(1^4) \times C_q$ by Proposition 2.2. For the case $\mathcal{M}(G) \cong C_p^2$, as $(G \wedge G)/Z(G \wedge G)$ is an abelian group of order p^3 and by applying [6], likewise above we deduce that $G \wedge G \cong \Phi_4(1^5)$ and $G \otimes G \cong \Phi_4(1^5) \times C_q$. Finally, when $\mathcal{M}(G) \cong C_p^3$, the result holds by a same method.

Assume G is of type (25). Then the polycyclic presentation of G is as follows:

$$\langle a_1, a_2, a_3, a_4 \mid a_1^q = a_2^p = a_3^p = a_4^p = 1, a_2^{a_1} = a_3, a_3^{a_1} = a_4, a_4^{a_1} = a_2 a_3^s a_4^t, a_3^{a_2} = a_3, a_4^{a_2} = a_4, a_4^{a_3} = a_4 \rangle,$$

where $s = -\lambda^{-1} - \lambda^{-p} - \lambda^{-p^2} \bmod p$ and $t = \lambda + \lambda^p + \lambda^{p^2} \bmod p$. It is clear from its presentation that $\mathcal{M}(G) = \langle a_3 \otimes a_2, a_4 \otimes a_2, a_4 \otimes a_3 \rangle$. The relation $(a_3 \otimes a_2)^p = a_3^p \otimes a_2 = 1$ implies that $|a_3 \otimes a_2| = p$.

Also $|a_4 \otimes a_2| = |a_4 \otimes a_3| = p$ similarly. As $\mathcal{M}(G) \cong 1, C_p, C_p^2$ or C_p^3 by Proposition 3.2, we proceed by considering four cases.

Case 1. If $\mathcal{M}(G) = 1$, since G' has the cyclic complement $G^{ab} \cong C_q$ we get $G \otimes G \cong C_p^2 \times C_{pq}$ by Proposition 2.2.

Case 2. If $\mathcal{M}(G) = \langle a_3 \otimes a_2 \rangle \cong C_p$, then Proposition 2.4 implies that $|G \wedge G| = p^4$. We shall determine the structure of $G \wedge G$. As $a_2, a_3, a_4 \in G'$, we have $a_2 \otimes a_2 = a_3 \otimes a_3 = a_4 \otimes a_4 = 1$. Furthermore by [5, Lemma 2.3], it follows that $\nabla(G) = \langle a_1 \otimes a_1 \rangle$. Hence

$$G \wedge G = \langle a_3 \wedge a_2, a_2 \wedge a_1, a_3 \wedge a_1, a_4 \wedge a_1 \rangle$$

by Proposition 2.3. First we determine the generators of $(G \wedge G)'$. The all possible commutators are as follows:

$$\begin{aligned} [a_2 \wedge a_1, a_3 \wedge a_1] &= [a_2, a_1] \wedge [a_3, a_1] = a_2^{-1} a_3 \wedge a_3^{-1} a_4 = (a_3 \wedge a_2)^{-1} (a_4 \wedge a_3)^{-1} (a_4 \wedge a_2), \\ [a_2 \wedge a_1, a_4 \wedge a_1] &= [a_2, a_1] \wedge [a_4, a_1] = a_2^{-1} a_3 \wedge a_4^{-1} a_2 a_3^s a_4^t = (a_4 \wedge a_3)^{1-t} (a_4 \wedge a_2)^{t-1} (a_3 \wedge a_2)^{s+1}, \\ [a_3 \wedge a_1, a_4 \wedge a_1] &= [a_3, a_1] \wedge [a_4, a_1] = a_3^{-1} a_4 \wedge a_4^{-1} a_2 a_3^s a_4^t = (a_4 \wedge a_3)^{s+t+1} (a_4 \wedge a_2) (a_3 \wedge a_2)^{-1}. \end{aligned}$$

Therefore $(G \wedge G)' = \langle a_3 \wedge a_2, a_4 \wedge a_2, a_4 \wedge a_3 \rangle = \mathcal{M}(G)$. On the other hand in this case we have $\mathcal{M}(G) = \langle a_3 \wedge a_2 \rangle \cong C_p$, hence $(G \wedge G)' = \mathcal{M}(G) \cong C_p$. Thus $G \wedge G$ is a nonabelian group of order p^4 with derived subgroup of order p . From the classification of p -groups given in [6], we deduce $G \wedge G \cong \Phi_2(1^4)$. Therefore from Proposition 2.2, it follows that $G \otimes G \cong \Phi_2(1^4) \times C_q$.

Case 3. If $\mathcal{M}(G) = \langle a_3 \wedge a_2, a_4 \wedge a_2 \rangle \cong C_p^2$, then by a similar method as above it is readily seen that $G \wedge G = \langle a_3 \wedge a_2, a_4 \wedge a_2, a_2 \wedge a_1, a_3 \wedge a_1, a_4 \wedge a_1 \rangle$. So $|G \wedge G| = p^5$ and $Z(G \wedge G) = (G \wedge G)' = \mathcal{M}(G) \cong C_p^2$. Thus [6] shows that $G \wedge G \cong \Phi_4(1^5)$ and consequently $G \otimes G \cong \Phi_4(1^5) \times C_q$.

Case 4. If $\mathcal{M}(G) = \langle a_3 \wedge a_2, a_4 \wedge a_2, a_4 \wedge a_3 \rangle \cong C_p^3$, then likewise above it follows from [6] that $G \wedge G \cong \Phi_{11}(1^6)$ and hence $G \otimes G \cong \Phi_{11}(1^6) \times C_q$ by Proposition 2.2.

(viii) We know $|G \otimes G| = p^5 q$ by Corollary 3.4. On the other hand, $c \otimes d \notin J_2(G)$ and so $|c \otimes d| = p$. Since $\langle c \otimes d \rangle \trianglelefteq G \otimes G$, then $G \otimes G = J_2(G) \times \langle c \otimes d \rangle$ which implies that $G \otimes G \cong C_p^4 \times C_{pq}$.

The proof is complete. $\square$

Remark 3.5 The final summary table of groups determined by Western [13] has a group missing in the case that $q \equiv 1 \pmod{p}$. The following missing group appears in Western analysis in Section 13:

$$G = \langle a, b, d \mid a^{p^2} = b^p = d^q = 1, b^{-1}ab = a^{p+1}, a^{-1}da = d^\alpha, bd = db \rangle \quad (26)$$

where α is any primitive root of $\alpha^p \equiv 1 \pmod{q}$. It is clear that the subgroup $\langle d \rangle$ is a normal Hall subgroup with complement subgroup T , where T is the extraspecial p -group of order p^3 and exponent p^2 . So Proposition 2.1 together with [7, Theorem 3.3.6] implies that $\mathcal{M}(G) \cong 1$, whence $G \wedge G \cong G' \cong C_{pq}$. As $|G \otimes G| = p^4 q$, it follows from the epimorphism $G \otimes G \rightarrow G^{ab} \otimes G^{ab} \cong C_p^4$ that $G \otimes G \cong (C_p)^4 \times C_q$.

The following result determines the tensor centers of all nonabelian groups of order $p^3 q$, with $p > 2$. Recall that $Z^\otimes(G) = \{g \in G \mid g \otimes x = 1; \forall x \in G\}$ is the largest subgroup of G such that $G \otimes G \cong G/Z^\otimes(G) \otimes G/Z^\otimes(G)$.

Corollary 3.6 *Let G be a nonabelian group of order $p^3 q$, where p and q are distinct prime numbers and $p > 2$. Then $Z^\otimes(G) \cong C_p$ if G is any group of types (1), (7), (20), (24) or (26). In other cases $Z^\otimes(G) = 1$.*

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References

1. Blyth, R.D.; Morse, R.F.: Computing the nonabelian tensor squares of polycyclic groups. *J. Algebra* **321**, 2139–2148 (2009)
2. Brown, R.; Loday, J.-L.: Van Kampen theorems for diagrams of spaces. *Topology* **26**, 311–335 (1987)
3. Brown, R.; Johnson, D.L.; Robertson, E.F.: Some computations of nonabelian tensor products of groups. *J. Algebra* **111**, 177–202 (1987)
4. Jafari, S.H.; Niroomand, P.; Erfanian, A.: The nonabelian tensor square and Schur multiplier of group of order p^2q and p^2qr . *Algebra Colloq.* **19**, 1083–1088 (2012)
5. Jafari, S.H.; Saeedi, F.; Khamseh, E.: Characterization of finite p -groups by their nonabelian tensor square. *Commun. Algebra* **41**, 1954–1963 (2013)
6. James, R.: The groups of order p^6 (p an odd prime). *Math. Comput.* **34**, 613–637 (1980)
7. Karpilovsky, G.: The Schur Multiplier, LMS Monographs New Series 2. Oxford University Press, New York (1987)
8. Miller, C.: The second homology of a group. *Proc. Amer. Math. Soc.* **3**, 588–595 (1952)
9. Rashid, S.; Sarmin, N.H.; Erfanian, A.; Mohd Ali, N.M.: On the Schur multiplier of groups of order $8q$. *Int. J. Appl. Math. Stat.* **28**, 18–22 (2012)
10. Rashid, S.; Sarmin, N.H.; Erfanian, A.; Mohd Ali, N.M.; Zainal, R.: On the nonabelian tensor square and capability of groups of order $8q$. *Indag. Math. (N.S.)* **24**(3), 581–588 (2013)
11. Schur, I.: Über die Darstellungen der endlichen Gruppen durch gebrochene lineare Substitutionen. *J. Reine Angew. Math.* **127**, 20–50 (1904)
12. The GAP Group.: GAP-Groups, Algorithms and Programming, Version 4.7.6 (2014). <http://www.gap-system.org/>
13. Western, A.E.: Groups of order p^3q . *Proc. Lond. Math. Soc.* **30**, 209–263 (1899)

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