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Equality of graphs up to complementation

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Abstract We prove the following: Let G and G' be two graphs on the same set V of v vertices, and let k be an integer, $4 \leq k \leq v - 4$. If for all k-element subsets K of V, the induced subgraphs $G_{\restriction K}$ and $G'_{\restriction K}$ have the same numbers of 3-homogeneous subsets, the same numbers of P_4's, and the same numbers of claws or co-claws, then G' is equal to G or to the complement $\overline{G}$ of G. We give also a similar result whenever the same numbers are modulo a prime.

Mathematics Subject Classfication 05C50 · 05C60

1 Introduction and main results

Our notations and terminology follow [2]. A *symmetric graph* (or more simply *graph*) is an ordered pair $G := (V, \mathcal{E})$, where $\mathcal{E}$ is a subset of $[V]^2$, the set of pairs $\{x, y\}$ of distinct elements of V. Elements of V are the *vertices* of G and elements of $\mathcal{E}$ its *edges*. If K is a subset of V, the *restriction* of G to K, also called the *induced graph* on K is the graph $G_{\restriction K} := (K, [K]^2 \cap \mathcal{E})$. The *complement* of G is the graph $\overline{G} := (V, [V]^2 \setminus \mathcal{E})$. We denote by $V(G)$ the vertex set of a graph G, by $E(G)$ its edge set and by $e(G) := |E(G)|$ the number of edges. Let $G := (V, E)$ be a graph. A 3-element subset T of V such that all pairs belong to $E(G)$ is a *triangle* of G. A 3-element subset of V which is a triangle of G or of $\overline{G}$ is a 3-*homogeneous subset* of G. We denote by K_3 the complete graph on 3 vertices and by $K_{1,3}$ the graph made of a vertex linked to a $\overline{K_3}$. The graph $K_{1,3}$ is called a *claw*, the graph $\overline{K_{1,3}}$ a *co-claw*. For these graphs, see Fig. 1.

In memory of Gérard Lopez with all our affection and admiration. Gérard Lopez died on April 8, 2019; this work was at the end of writing.

Deceased: Gérard Lopez

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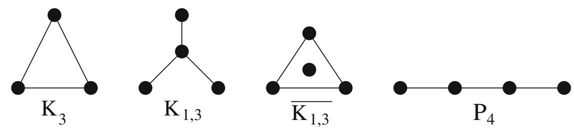

Fig. 1 K_3, $K_{1,3}$, $\overline{K_{1,3}}$, P_4

Given two graphs $G := (V, E)$ and $G' := (V', E')$. A bijection f from V onto V' is an *isomorphism* from G onto G' provided that for any $x, y \in V$, $\{x, y\} \in E$ if and only if $\{f(x), f(y)\} \in E'$. The graphs G and G' are *isomorphic*, which is denoted by $G \simeq G'$, if there exists an isomorphism from one onto the other, otherwise $G \not\simeq G'$.

Let G, G' be two graphs on the same vertex set V. We say that G and G' are *equal up to complementation* if $G' = G$ or $G' = \overline{G}$. If G' has the same number of edges as G or $\overline{G}$, we say that G and G' have *the same number of edges up to complementation*. We say that G and G' are *isomorphic up to complementation* if G' is isomorphic to G or to $\overline{G}$. Let k be a non-negative integer, G and G' are *k-hypomorphic up to complementation* if for every k-element subset K of V, the induced subgraphs $G_{\restriction K}$ and $G'_{\restriction K}$ are isomorphic up to complementation.

Ulam's Reconstruction Conjecture [3,12] asserts that two graphs G and G' on the same finite set V of v vertices, $v \geq 3$, are isomorphic provided that the restrictions $G_{\restriction K}$ and $G'_{\restriction K}$ of G and G' to the $(v-1)$-element subsets K of V are isomorphic. If this latter condition holds for the k-element subsets of V for some k, $2 \leq k \leq v-2$, then $G' = G$. This conclusion does not require the isomorphy of $G_{\restriction K}$ and $G'_{\restriction K}$, but only requires that $G_{\restriction K}$ and $G'_{\restriction K}$ have the same number of edges for all k-element subsets K of V. In [4], a work has been done where the conditions are $G'_{\restriction K}$ and $G_{\restriction K}$ are isomorphic up to complementation for all k-element subsets K of V, or $G'_{\restriction K}$ and $G_{\restriction K}$ have the same number of edges up to complementation for all k-element subsets K of V. Let v be a non negative integer and $\vartheta(v) := 4l$ if $v \in \{4l+2, 4l+3\}$, $\vartheta(v) := 4l-3$ if $v \in \{4l, 4l+1\}$. The following theorems was obtained in [4].

Theorem 1.1 [4] *Let v, k be two integers with $4 \leq k \leq \vartheta(v)$. Then, for every pair of graphs G and G' on the same set V of v vertices, if $G_{\restriction K}$ and $G'_{\restriction K}$ have the same number of edges, up to complementation, and the same number of* 3*-homogeneous subsets, for all k-element subsets K of V, then $G' = G$ or $G' = \overline{G}$.*

Theorem 1.2 [4] *Let k be an integer, $4 \leq k \leq v-2$, $k \equiv 0 \pmod 4$. Let G and G' be two graphs on the same set V of v vertices.*

We assume that $e(G_{\restriction K})$ has the same parity as $e(G'_{\restriction K})$ for all k-element subsets K of V.

Then $G' = G$ or $G' = \overline{G}$.

It is also shown in [4] that two graphs G and G' on the same set of v vertices are equal up to complementation whenever they are k-hypomorphic up to complementation and $4 \leq k \leq v-4$ (for the case $k = v-3$, the conclusion obtained is $G' \simeq G$ or $G' \simeq \overline{G}$ [5]). Later, an extension of this result was obtained for uniform hypergraphs [9].

Theorem 1.3 [9] *Let h be a non-negative integer. There are two non-negative integers k and t, $k \leq t$ such that two h-uniform hypergraphs $\mathcal{H}$ and $\mathcal{H}'$ on the same set V of vertices, $|V| \geq t$, are equal up to complementation whenever $\mathcal{H}$ and $\mathcal{H}'$ are k-hypomorphic up to complementation.*

In this paper, we look for similar results on graphs if the conditions on the restrictions $G_{\restriction K}$ and $G'_{\restriction K}$ are that they have the same numbers of 3-homogeneous subsets, the same numbers of P_4's, and the same numbers of claws or co-claws, we obtain Theorem 1.4. Whenever the same numbers (cited above) are modulo a prime, we obtain Theorem 1.7. This last theorem is a consequence of Theorem 1.4 and Proposition 4.3 which is inspired by a result of Pouzet on monomorphic relations (Proposition 4.2).

Let $G := (V, E)$ be a graph. We set:

$$H^{(3)}(G) := \{\{a, b, c\} \subset V : G_{\restriction\{a,b,c\}} \simeq K_3 \text{ or } G_{\restriction\{a,b,c\}} \simeq \overline{K_3}\}$$
$$P^{(4)}(G) := \{\{a, b, c, d\} \subset V : G_{\restriction\{a,b,c,d\}} \simeq P_4\}$$
$$S^{(4)}(G) := \{\{a, b, c, d\} \subset V : G_{\restriction\{a,b,c,d\}} \simeq K_{1,3} \text{ or } G_{\restriction\{a,b,c,d\}} \simeq \overline{K_{1,3}}\}$$
$$h^{(3)}(G) :=\mid H^{(3)}(G) \mid,\ p^{(4)}(G) :=\mid P^{(4)}(G) \mid,\ s^{(4)}(G) :=\mid S^{(4)}(G) \mid.$$

Our first result is:

Theorem 1.4 *Let G, G' be two graphs on the same finite set V of $v \geq 4$ vertices.*

(1) *If $H^{(3)}(G) = H^{(3)}(G')$, $P^{(4)}(G) = P^{(4)}(G')$ and $S^{(4)}(G) = S^{(4)}(G')$, then $G' = G$ or $G' = \overline{G}$.*
(2) *Let k be a integer, $4 \leq k \leq v - 4$.*
If $h^{(3)}(G_{\restriction K}) = h^{(3)}(G'_{\restriction K})$, $p^{(4)}(G_{\restriction K}) = p^{(4)}(G'_{\restriction K})$ and $s^{(4)}(G_{\restriction K}) = s^{(4)}(G'_{\restriction K})$ for all k-element subsets K of V, then $G' = G$ or $G' = \overline{G}$.

The proof of Theorem 1.4 will be given in Sect. 3. The following propositions show that Theorem 1.4 is optimal.

Proposition 1.5 *For every integer $v \geq 4$ and every integer k, $4 \leq k \leq v$, there are two graphs G and G' on the same set of v vertices, nonisomorphic up to complementation, satisfying $p^{(4)}(G_{\restriction K}) = p^{(4)}(G'_{\restriction K})$ and $s^{(4)}(G_{\restriction K}) = s^{(4)}(G'_{\restriction K})$ for all k-element subsets K of V.*

Proof Let $V := \{1, 2, \ldots, v\}$ with $v \geq 4$. Let G and G' be two graphs on the same vertex set V defined by $G_{\restriction\{2,3,\ldots,v\}}$ and $G'_{\restriction\{2,3,\ldots,v\}}$ are complete graphs, $\{1, x\} \in E(G)$ if $x \neq 2$; $\{1, x\} \in E(G')$ if $x \notin \{2, 3\}$. Clearly $p^{(4)}(G_{\restriction K}) = p^{(4)}(G'_{\restriction K}) = 0$ and $s^{(4)}(G_{\restriction K}) = s^{(4)}(G'_{\restriction K}) = 0$ for all k-element subsets K of V. But G and G' are not isomorphic up to complementation. □

Proposition 1.6 *For every integer $n \geq 4$, there are two graphs G and G' on the same set of $2n$ vertices, nonisomorphic up to complementation, satisfying $h^{(3)}(G) = h^{(3)}(G')$ and $p^{(4)}(G) \neq p^{(4)}(G')$.*

Proof We set $G := M_{2n}$ and $G' := M'_{2n}$ (For the definition of the graphs M_{2n} and M'_{2n}, see Sect. 3, Fig. 3). Clearly, G and G' are not isomorphic up to complementation. We have $h^{(3)}(G) = h^{(3)}(G')$, but $p^{(4)}(G) \neq p^{(4)}(G')$ since $p^{(4)}(G) = n(n-1)(n-2)$ and $p^{(4)}(G') = (n-1)(n-2)^2 + (n-1)^2$. □

Let k, p be positive integers, the decomposition of $k = \sum_{i=0}^{k(p)} k_i p^i$ in the basis p is also denoted $[k_0, k_1, \ldots, k_{k(p)}]_p$ where $k_{k(p)} \neq 0$ if and only if $k \neq 0$.

As an application of Theorem 1.4, our second result is:

Theorem 1.7 *Let G, G' be two graphs on the same set V of v vertices. Let p be a prime number, $p \geq 5$ and $k = [k_0, k_1, \ldots, k_{k(p)}]_p$ be an integer, $4 \leq k \leq v - 4$; ($k_0 = 0$ or $k_0 \geq 4$).*

If $h^{(3)}(G_{\restriction K}) \equiv h^{(3)}(G'_{\restriction K})$ (mod p), $p^{(4)}(G_{\restriction K}) \equiv p^{(4)}(G'_{\restriction K})$ (mod p) and $s^{(4)}(G_{\restriction K}) \equiv s^{(4)}(G'_{\restriction K})$ (mod p) for all k-element subsets K of V, then $G' = G$ or $G' = \overline{G}$.

The proof of Theorem 1.7 will be given in Sect. 4.

2 Sketch of the proofs of Theorems 1.4 and 1.7

Let G, G' be two graphs on the same vertex set V. The *Boolean sum* $G \dot{+} G'$ of G and G' is the graph U on V whose edges are pairs e of vertices such that $e \in E(G)$ if and only if $e \notin E(G')$. Indeed, $G' = G$ or $G' = \overline{G}$ amounts to the fact that U is either the empty graph or the complete graph. The *intersection graph* of G and G' is the graph $G \cap G' := (V, E(G) \cap E(G'))$.

Observation 2.1 [5] *Let G and G' be two graphs on the same finite set V of v vertices, and let $U := G \dot{+} G'$. Then*

(1) $e(U) = e(G) + e(G') - 2e(G \cap G')$.
(2) *$e(G)$ and $e(G')$ have the same parity if and only if $e(U)$ is even.*

To prove (1) of Theorem 1.4, we consider $U := G \dot{+} G'$. From Theorem 1.2, it is sufficient to prove that $e(G_{\restriction X})$ has the same parity as $e(G'_{\restriction X})$ for all 4-element subsets X of V that is, from Observation 2.1, $e(U_{\restriction X})$ is even for all 4-element subsets X of V. For this we proceed by contradiction, assuming that $e(U_{\restriction X})$ is odd. As $H^{(3)}(G) = H^{(3)}(G')$, Theorem 3.1 says that $U_{\restriction X}$ is a path of length 1 or 3. If the length is one, we study two cases. If the length is 3, we conclude using Theorem 3.2.

To prove item (2), and also Theorem 1.7, we will prove that item (1) of Theorem 1.4 holds. For this, we will use linear algebra. The *incidence matrix* $W_{t\,k}(v)$ used by Wilson [13], or more simply $W_{t\,k}$, is defined as follows : Let V be a finite set, with v elements. Given non-negative integers $t \leq k \leq v$, let $W_{t\,k}$ be the $\binom{v}{t}$

by $\binom{v}{k}$ matrix of 0's and 1's, the rows of which are indexed by the t-element subsets T of V, the columns are indexed by the k-element subsets K of V, and where the entry $W_{t\,k}(T, K)$ is 1 if $T \subseteq K$ and is 0 otherwise. The matrix transpose of $W_{t\,k}$ is denoted ${}^tW_{t\,k}$. We denote by $rank_{\mathbb{Q}}\ W_{t\,k}$, the rank of $W_{t\,k}$ over the field $\mathbb{Q}$. Whenever p is a prime, we denote by $rank_p\ W_{t\,k}$, the rank of $W_{t\,k}$ over the field $\mathbb{F}_p$, and by $Ker_p({}^tW_{t\,k})$ the kernel of ${}^tW_{t\,k}$ in $\mathbb{F}_p$.

First, $rank_{\mathbb{Q}}\ W_{t\,k}$ is given by Theorem 2.2 due to Gottlieb [7].

Theorem 2.2 [7] *For $t \leq min(k, v-k)$, the rank of $W_{t\,k}$ over the field $\mathbb{Q}$ of rational numbers is $\binom{v}{t}$ and thus $Ker({}^tW_{t\,k}) = \{0\}$.*

Let $G := (V, E)$ be a graph with v vertices, $v \geq 6$. Let $t \in \{3, 4\}$, and k be an integer, $k \leq v$ and $t \leq min(k, v-k)$.

Let $T_1, T_2, \ldots, T_{\binom{v}{t}}$ be an enumeration of the t-element subsets of V.

Let $K_1, K_2, \ldots, K_{\binom{v}{k}}$ be an enumeration of the k-element subsets of V.

We set:

$$w_G^h := (g_1^h, g_2^h, \ldots, g_{\binom{v}{3}}^h) \text{ where } g_i^h = 1 \quad \text{if} \quad T_i \in H^{(3)}(G), 0 \quad \text{otherwise}\ .$$
$$w_G^p := (g_1^p, g_2^p, \ldots, g_{\binom{v}{4}}^p) \text{ where } g_i^p = 1 \quad \text{if} \quad T_i \in P^{(4)}(G), 0 \quad \text{otherwise}\ .$$
$$w_G^s := (g_1^s, g_2^s, \ldots, g_{\binom{v}{4}}^s) \text{ where } g_i^s = 1 \quad \text{if} \quad T_i \in S^{(4)}(G), 0 \quad \text{otherwise}\ .$$

We have:

$$w_G^h W_{3\,k} = (h^3(G_{\restriction K_1}), h^3(G_{\restriction K_2}), \ldots, h^3(G_{\restriction K_{\binom{v}{k}}})),$$
$$w_G^p W_{4\,k} = (p^{(4)}(G_{\restriction K_1}), p^{(4)}(G_{\restriction K_2}), \ldots, p^{(4)}(G_{\restriction K_{\binom{v}{k}}})).$$
$$w_G^s W_{4\,k} = (s^{(4)}(G_{\restriction K_1}), s^{(4)}(G_{\restriction K_2}), \ldots, s^{(4)}(G_{\restriction K_{\binom{v}{k}}})).$$

Observation 2.3 *Let G and G' be two graphs on the same finite set V of v vertices.*

(1) *If $h^{(3)}(G_{\restriction K_i}) = h^{(3)}(G'_{\restriction K_i})$ (resp. $h^{(3)}(G_{\restriction K_i}) \equiv h^{(3)}(G'_{\restriction K_i})$ (mod p)) for all $i \in \{1, 2, \ldots, \binom{v}{k}\}$, then $w_G^h - w_{G'}^h \in Ker_{\mathbb{Q}}({}^tW_{3\,k})$ (resp. $w_G^h - w_{G'}^h \in Ker_p({}^tW_{3\,k})$).*
(2) *If $p^{(4)}(G_{\restriction K_i}) = p^{(4)}(G'_{\restriction K_i})$ (resp. $p^{(4)}(G_{\restriction K_i}) \equiv p^{(4)}(G'_{\restriction K_i})$ (mod p)) for all $i \in \{1, 2, \ldots, \binom{v}{k}\}$, then $w_G^p - w_{G'}^p \in Ker_{\mathbb{Q}}({}^tW_{4\,k})$ (resp. $w_G^p - w_{G'}^p \in Ker_p({}^tW_{4\,k})$).*
(3) *If $s^{(4)}(G_{\restriction K_i}) = s^{(4)}(G'_{\restriction K_i})$ (resp. $s^{(4)}(G_{\restriction K_i}) \equiv s^{(4)}(G'_{\restriction K_i})$ (mod p)) for all $i \in \{1, 2, \ldots, \binom{v}{k}\}$, then $w_G^s - w_{G'}^s \in Ker_{\mathbb{Q}}({}^tW_{4\,k})$ (resp. $w_G^s - w_{G'}^s \in Ker_p({}^tW_{4\,k})$).*

Now for the end of the proof of item (2), from Observation 2.3, $w_G^h - w_{G'}^h \in Ker_{\mathbb{Q}}({}^tW_{3\,k})$. From Theorem 2.2, $Ker_{\mathbb{Q}}({}^tW_{3\,k}) = \{0\}$, then $w_G^h = w_{G'}^h$. Thus, $H^{(3)}(G) = H^{(3)}(G')$. For the other parameters, we do the same. Thus, we get the hypotheses of item (1).

For Theorem 1.7, there are two cases according to the value of k_0. If $k_0 \geq 4$, we proceed as above. If $k_0 = 0$, we use Theorem 4.1 which gives the dimension and a basis of $Ker_p({}^tW_{t\,k})$.

3 Proof of Theorem 1.4

A description of the Boolean sum $G \dot{+} G'$, of graphs G and G' having the same 3-element homogeneous subsets, is given by Theorem 3.1 below.

We denote by P_9 the Paley graph on 9 vertices (cf. Fig. 2). Note that P_9 is isomorphic to its complement $\overline{P_9}$.

Theorem 3.1 [10] *Let U be a graph. The following properties are equivalent:*

(1) *There are two graphs G and G' having the same 3-element homogeneous subsets such that $U := G \dot{+} G'$;*
(2) *Either (i) U is an induced subgraph of P_9, or (ii) the connected components of U, or of its complement $\overline{U}$, are cycles of even length or paths.*

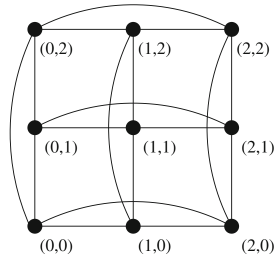

Fig. 2 P_9

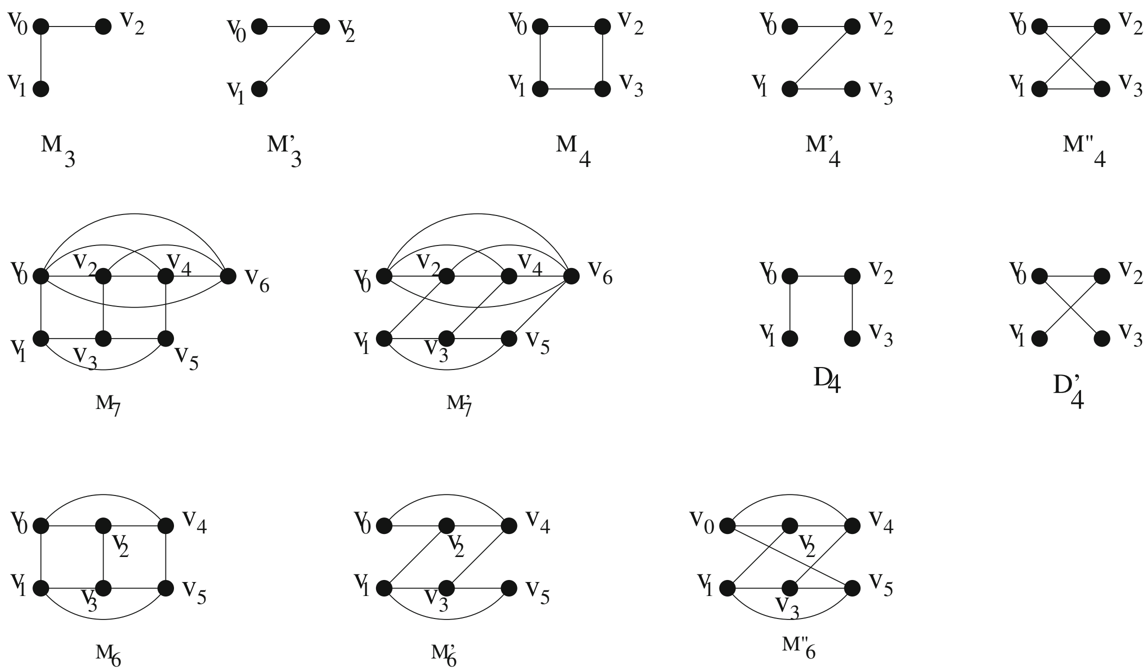

Fig. 3 M_n, M_n', M_n'', D_4, D_4'

For graphs G and G' having the same 3-element homogeneous subsets, Theorem 3.2 below gives the form of their restrictions on a connected component of $G\dot{+}G'$.

Theorem 3.2 [5] *Let G and G' be two graphs on the same vertex set V and $U := G\dot{+}G'$. We assume $H^{(3)}(G) = H^{(3)}(G')$ and U not connected. If C is a connected component of U of cardinality n, then the pair $\{G_{\restriction V(C)}, G'_{\restriction V(C)}\}$ is one of the following:*

(1) $\{M_n, M_n'\}$, $\{\overline{M_n}, \overline{M_n'}\}$, *if C is a path.*
(2) $\{M_n, M_n''\}$, $\{\overline{M_n}, \overline{M_n''}\}$, $\{D_4, D_4'\}$, $\{\overline{D_4}, \overline{D_4'}\}$, *if C is a cycle.*

The graphs mentioned in the conclusion of Theorem 3.2 are defined as follows (see Fig. 3).

Let $n \geq 2$. Let X_n be an n-element set, $v_0, \cdots, v_{n-1}$ be an enumeration of X_n, $X_n^0 := \{v_i \in X_n : i \equiv 0 \ (mod\ 2)\}$ and $X_n^1 := X_n \setminus X_n^0$. Set $R_n := [X_n^0]^2 \cup [X_n^1]^2$, $S_n := \{\{v_{2i}, v_{2i+1}\} : 2i+1 < n\}$, $S_n' := \{\{v_{2i+1}, v_{2i+2}\} : 2i+2 < n\}$. Let M_n and M_n' be the graphs with vertex set X_n and edge sets $E(M_n) := R_n \cup S_n$ and $E(M_n') := R_n \cup S_n'$, respectively. Let $M_n'' := (X_n, R_n \cup S_n' \cup \{\{v_0, v_{n-1}\}\})$ for n even, $n \geq 4$. Finally, let $D_4 := (X_4, \{\{v_0, v_1\}, \{v_0, v_2\}, \{v_2, v_3\}\})$ and $D_4' := (X_4, \{\{v_0, v_2\}, \{v_0, v_3\}, \{v_1, v_2\}\})$.

Let $G = (V, E)$ be a graph. For $x \neq y \in V$, $x \sim_G y$ means $\{x, y\} \in E$, $x \nsim_G y$ means $\{x, y\} \notin E$. For $X, Y \subseteq V$, $X \sim_G Y$ signifies that for every $x \in X$ and $y \in Y$, $x \sim_G y$. Similarly, $X \nsim_G Y$ signifies that for every $x \in X$ and $y \in Y$, $x \nsim_G y$. Whenever $X = \{x\}$, $X \sim_G Y$ and $X \nsim_G Y$ are, respectively, denoted $x \sim_G Y$ and $x \nsim_G Y$.

Now we prove Theorem 1.4.

(1) Let $U := G \dot{+} G'$. Using Theorem 1.2, it is sufficient to prove that $e(G_{\restriction X})$ has the same parity as $e(G'_{\restriction X})$ for all 4-element subsets X of V that is, from Observation 2.1, $e(U_{\restriction X})$ is even for all 4-element subsets X of V.
Let $X := \{v_0, v_1, v_2, v_3\}$ be a subset of V. By contradiction, we assume that $e(U_{\restriction X})$ is odd, then $e(U_{\restriction X}) \in \{1, 3, 5\}$. As $H^{(3)}(G) = H^{(3)}(G')$ then, by Theorem 3.1, $E(U_{\restriction X}) = \{\{v_0, v_1\}, \{v_1, v_2\}, \{v_2, v_3\}\}$ or $\{\{v_0, v_1\}\}$.
- If $E(U_{\restriction X}) = \{\{v_0, v_1\}, \{v_1, v_2\}, \{v_2, v_3\}\}$, we apply (1) of Theorem 3.2. So $\{G_{\restriction X}, G'_{\restriction X}\} = \{M_4, M'_4\}$ or $\{\overline{M_4}, \overline{M'_4}\}$. We get a contradiction with $P^{(4)}(G) = P^{(4)}(G')$ since $M_4 \notin P^{(4)}(G) \cup P^{(4)}(G')$ and $M'_4 \in P^{(4)}(G) \cap P^{(4)}(G')$.
- If $E(U_{\restriction X}) = \{\{v_0, v_1\}\}$, we can suppose $v_0 \sim_G v_1$ and $v_0 \nsim_{G'} v_1$. Without loss of generality, we assume $v_0 \sim_G v_2$. As $v_0 \nsim_U v_2$, $v_0 \sim_{G'} v_2$. We have $v_0 \sim_G \{v_1, v_2\}$, $v_0 \nsim_{G'} v_1$ and $v_0 \sim_{G'} v_2$. As $H^3(G) = H^3(G')$, $v_2 \nsim_G v_1$. Since $v_2 \nsim_U v_1$ then $v_2 \nsim_{G'} v_1$.

Case 1. $v_0 \sim_G v_3$. Since $v_0 \nsim_U v_3$ then $v_0 \sim_{G'} v_3$. Consider $\{v_0, v_1, v_3\}$, as $H^{(3)}(G) = H^{(3)}(G')$ we have $v_1 \nsim_G v_3$, thus $v_1 \nsim_{G'} v_3$.
If $v_2 \nsim_G v_3$, then $v_2 \nsim_{G'} v_3$. So $\{v_0, v_1, v_2, v_3\} \in S^{(4)}(G)$ and $\{v_0, v_1, v_2, v_3\} \notin S^{(4)}(G')$. That contradicts $S^{(4)}(G) = S^{(4)}(G')$. If $v_2 \sim_G v_3$, then $v_2 \sim_{G'} v_3$. So $\{v_0, v_1, v_2, v_3\} \in S^{(4)}(G')$ and $\{v_0, v_1, v_2, v_3\} \notin S^{(4)}(G)$. That contradicts $S^{(4)}(G) = S^{(4)}(G')$.
Case 2. $v_0 \nsim_G v_3$. Since $v_0 \nsim_U v_3$ then $v_0 \nsim_{G'} v_3$. Consider $\{v_0, v_1, v_3\}$, as $H^{(3)}(G) = H^{(3)}(G')$ we have $v_1 \sim_{G'} v_3$, thus $v_1 \sim_G v_3$.
If $v_2 \nsim_G v_3$, then $v_2 \nsim_{G'} v_3$. Then $\{v_0, v_1, v_2, v_3\} \in P^{(4)}(G)$ and $\{v_0, v_1, v_2, v_3\} \notin P^{(4)}(G')$. That contradicts $P^{(4)}(G) = P^{(4)}(G')$. If $v_2 \sim_G v_3$, then $v_2 \sim_{G'} v_3$. Then $\{v_0, v_1, v_2, v_3\} \notin P^{(4)}(G)$ and $\{v_0, v_1, v_2, v_3\} \in P^{(4)}(G')$. That contradicts $P^{(4)}(G) = P^{(4)}(G')$.

(2) We will prove $H^{(3)}(G) = H^{(3)}(G')$, $P^{(4)}(G) = P^{(4)}(G')$, $S^{(4)}(G) = S^{(4)}(G')$ and conclude using (1).
From Observation 2.3, $w^h_G - w^h_{G'} \in Ker_{\mathbb{Q}}({}^tW_{3\,k})$. From Theorem 2.2, $Ker_{\mathbb{Q}}({}^tW_{3\,k}) = \{0\}$, then $w^h_G = w^h_{G'}$. Thus $H^{(3)}(G) = H^{(3)}(G')$.
From Observation 2.3, $w^p_G - w^p_{G'} \in Ker_{\mathbb{Q}}({}^tW_{4\,k})$ and $w^s_G - w^s_{G'} \in Ker_{\mathbb{Q}}({}^tW_{4\,k})$. From Theorem 2.2, $Ker_{\mathbb{Q}}({}^tW_{4\,k}) = \{0\}$, then $w^p_G = w^p_{G'}$ and $w^s_G = w^s_{G'}$. Thus $P^{(4)}(G) = P^{(4)}(G')$ and $S^{(4)}(G) = S^{(4)}(G')$.

$\square$

4 Proof of Theorem 1.7

The following result is one of the keys to our proof.

Theorem 4.1 [1] *Let p be a prime number. Let v, t and k be non-negative integers, $k \leq v$, $k = [k_0, k_1, \ldots, k_{k(p)}]_p$, $t = [t_0, t_1, \ldots, t_{t(p)}]_p$, $t \leq \min(k, v-k)$. We have:*

(1) *$k_j = t_j$ for all $j < t(p)$ and $k_{t(p)} \geq t_{t(p)}$ if and only if $Ker_p({}^tW_{t\,k}) = \{0\}$.*
(2) *$t = t_{t(p)}p^{t(p)}$ and $k = \sum_{i=t(p)+1}^{k(p)} k_i p^i$ if and only if $dim\ Ker_p({}^tW_{t\,k}) = 1$ and $\{(1, 1, \ldots, 1)\}$ is a basis of $Ker_p({}^tW_{t\,k})$.*

Let $k \geq 1$ be an integer and G be a graph. We say that G is *k-monomorphic* (resp. *k-monomorphic up to complementation*) if $G_{\restriction X} \simeq G_{\restriction Y}$ (resp. $G_{\restriction X} \simeq G_{\restriction Y}$ or $G_{\restriction X} \simeq \overline{G}_{\restriction Y}$) for all k-element subsets X and Y of V. The notion of monomorphy was introduced by Fraïssé [6].

The following result on monomorphy, due to Pouzet, is very useful since it is the origin of Proposition 4.3, which is a key in the proof of Theorem 1.7.

Proposition 4.2 [8] *Let v, t, k be integers, $t \leq \min(k, v-k)$ and G be a graph on a set V of v vertices. If G is k-monomorphic then G is t-monomorphic.*

We give a similar result (Proposition 4.3) in the case of the monomorphy up to complementation.

Proposition 4.3 *Let v, t, k be integers, $t \leq \min(k, v-k)$ and G be a graph on a set V of v vertices. If G is k-monomorphic up to complementation then G is t-monomorphic up to complementation.*

Proof Let $T_1, T_2, \ldots, T_{\binom{v}{t}}$ be the t-element subsets of V, and $K_1, K_2, \ldots, K_{\binom{v}{k}}$ be the k-element subsets of V. Let $H := G_{\restriction T_1}$. Set $Isc(H, G) := \{L \subseteq V : G_{\restriction L} \simeq H$ or $G_{\restriction L} \simeq \overline{H}\}$ and $w_{H,G}$ the row vector indexed by the t-element subsets $T_1, T_2, \ldots, T_{\binom{v}{t}}$ of V whose coefficient of T_i is 1 if $T_i \in Isc(H, G)$ and 0 otherwise.

We have $w_{H,G}W_{t\,k} = \left(\alpha_1, \alpha_2, \ldots, \alpha_{\binom{v}{k}}\right)$ where

$$\alpha_j := \left|\{T_i : T_i \subseteq K_j,\ 1 \le i \le \binom{v}{t},\ T_i \in Isc(H, G)\}\right|.$$

Since G is k-monomorphic up to complementation, $\alpha_1 = \alpha_2 = \cdots = \alpha_{\binom{v}{k}}$. Thus $w_{H,G}W_{t\,k} = \alpha_1(1, 1, \cdots, 1)$. Now $(1, 1, \ldots, 1)W_{t\,k} = (\binom{k}{t}, \binom{k}{t}, \ldots, \binom{k}{t}) = \binom{k}{t}(1, 1, \ldots, 1)$. It follows that $\binom{k}{t}w_{H,G}W_{t\,k} = \alpha_1(1, 1, \ldots, 1)W_{t\,k}$. From Theorem 2.2, $Ker_{\mathbb{Q}}({}^tW_{t\,k}) = \{0\}$. Thus $\binom{k}{t}w_{H,G} = \alpha_1(1, 1, \ldots, 1)$. Since $H = G_{\restriction T_1}$, then $w_{H,G} \neq (0, 0, \ldots, 0)$, this implies $\alpha_1 \neq 0$. Then from $w_{H,G} = \frac{\alpha_1}{\binom{k}{t}}(1, 1, \ldots, 1)$, we deduce that $w_{H,G} = (1, 1, \ldots, 1)$. So $T_i \in Isc(H, G)$ for all $i \in \{1, 2, \ldots, \binom{v}{t}\}$. Thus G is t-monomorphic up to complementation. □

Now we prove Theorem 1.7.

We have $w^h_G W_{3\,k} = w^h_{G'}W_{3\,k}$, $w^p_G W_{4\,k} = w^p_{G'}W_{4\,k}$ and $w^s_G W_{4\,k} = w^s_{G'}W_{4\,k}$.

By Theorem 1.4, it is sufficient to prove that $H^{(3)}(G) = H^{(3)}(G')$, $P^{(4)}(G) = P^{(4)}(G')$ and $S^{(4)}(G) = S^{(4)}(G')$. Let $t \in \{3, 4\}$. According to the value of k_0, we have two cases.

Case 1. $k_0 \ge 4$. As $t \in \{3, 4\}$ and $p \ge 5$ then $t(p) = 0$, $t = t_0 \le 4 \le k_0$. From (1) of Theorem 4.1, $\mathrm{Ker}_p({}^tW_{3\,k}) = \{0\}$ and $\mathrm{Ker}_p({}^tW_{4\,k}) = \{0\}$. Then from Observation 2.3, we obtain $H^{(3)}(G) = H^{(3)}(G')$, $P^{(4)}(G) = P^{(4)}(G')$ and $S^{(4)}(G) = S^{(4)}(G')$.

Case 2. $k_0 = 0$. As $t = t_0 \neq 0$ then, from (2) of Theorem 4.1, $\dim \mathrm{Ker}_p({}^tW_{t\,k}) = 1$ and $(1, 1, \ldots, 1)$ is a basis of $Ker_p({}^tW_{t\,k})$. By Observation 2.3, there are $\lambda_h, \lambda_p, \lambda_s \in \{0, 1, -1\}$ such that $w^h_G - w^h_{G'} = \lambda_h(1, 1, \ldots, 1)$, $w^p_G - w^p_{G'} = \lambda_p(1, 1, \ldots, 1)$, $w^s_G - w^s_{G'} = \lambda_s(1, 1, \ldots, 1)$.

If $\lambda_h = 1$, then $w^h_G = (1, 1, \ldots, 1)$ and $w^h_{G'} = (0, 0, \ldots, 0)$. Then in G' there is no 3-homogeneous set, that is impossible according to Ramsey's theorem [11] which ensures that Ramsey's number $r(3, 3)$ is equal to 6. If $\lambda_h = -1$, $w^h_G = (0, 0, \ldots, 0)$ and we conclude as above. Then $\lambda_h = 0$ and thus $H^{(3)}(G) = H^{(3)}(G')$.

If $\lambda_p = 1$, then $w^p_G = (1, 1, \ldots, 1)$. So for all 4-element subsets X of V, $G_{\restriction X} \simeq P_4$, thus G is 4-monomorphic. Then by Proposition 4.2, G is 2-monomorphic, that implies G is the empty graph or the complete graph, so G does not contain a subgraph isomorphic to P_4, a contradiction. If $\lambda_p = -1$, then $w^p_{G'} = (1, 1, \ldots, 1)$ and we conclude as above. Then $\lambda_p = 0$ and thus $P^{(4)}(G) = P^{(4)}(G')$.

If $\lambda_s = 1$, then $w^s_G = (1, 1, \ldots, 1)$. So for all 4-element subsets X of V, $G_{\restriction X} \simeq K_{1,3}$ or $G_{\restriction X} \simeq \overline{K_{1,3}}$, thus G is 4-monomorphic up to complementation. Then by Proposition 4.3, G is 3-monomorphic up to complementation. Let $v_0, v_1, v_2, v_3 \in V$ such that $G_{\restriction\{v_0,v_1,v_2,v_3\}}$ or $\overline{G}_{\restriction\{v_0,v_1,v_2,v_3\}}$ is the claw $(\{v_0, v_1, v_2, v_3\}, \{\{v_0, v_1\}, \{v_0, v_2\}, \{v_0, v_3\}\})$. We have $G_{\restriction\{v_0,v_1,v_2\}} \not\simeq G_{\restriction\{v_1,v_2,v_3\}}$ and $G_{\restriction\{v_0,v_1,v_2\}} \not\simeq \overline{G}_{\restriction\{v_1,v_2,v_3\}}$, this contradicts the fact that G is 3-monomorphic up to complementation. If $\lambda_s = -1$, then $w^s_{G'} = (1, 1, \ldots, 1)$, we conclude as above. Then $\lambda_s = 0$ and thus $S^{(4)}(G) = S^{(4)}(G')$. □

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