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Halpern–Ishikawa type iterative schemes for approximating fixed points of multi-valued non-self mappings

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Abstract Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to CB(H)$ be a multi-valued Lipschitz pseudocontractive nonself mapping. A Halpern–Ishikawa type iterative scheme is constructed and a strong convergence result of this scheme to a fixed point of T is proved under appropriate conditions. Moreover, an iterative method for approximating a fixed point of a k-strictly pseudocontractive mapping $T : C \to Prox(H)$ is constructed and a strong convergence of the method is obtained without end point condition. The results obtained in this paper improve and extend known results in the literature.

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1 Introduction

Let C be a nonempty subset of a real Hilbert space H. The set C is called *proximinal* if for each $x \in H$ there exists $u \in C$ such that

$$||x - u|| = \inf\{||x - y|| : y \in C\} = d(x, C),$$

where d is the metric on H generated by the inner product. It is well known that any nonempty closed and convex subset of a Hilbert space is proximinal. The family of nonempty proximinal bounded subsets of the set C is denoted by $Prox(C)$.

Let $A, B \in CB(H)$, where $CB(H)$ is the set of nonempty, closed and bounded subsets of H. The Hausdorff distance between A and B, denoted by $D(A, B)$, is defined as

$$D(A, B) = \max\left\{\sup_{x\in B} d(x, A), \sup_{x\in A} d(x, B)\right\}.$$

A multi-valued mapping $T : C \to 2^H$ is said to be *L-Lipschitz* if there exists $L \geq 0$ such that

$$D(Tx, Ty) \leq L||x - y||, \text{ for all} x, y \in C.$$

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If $L = 1$, then the mapping T is called *nonexpansive* mapping. It is immediate from the definition that every nonexpansive mapping is Lipschitz mapping.

A mapping $T : C \to 2^H$ is said to be

(a) *k-strictly pseudocontractive* if there exists $k \in (0, 1)$ such that for each $x, y \in C$,

$$D^2(Tx, Ty) \le ||x - y||^2 + k||x - y - (u - v)||^2, \forall u \in Tx, v \in Ty.$$

(b) *pseudocontractive* if for each $x, y \in C$,

$$D^2(Tx, Ty) \le ||x - y||^2 + ||x - y - (u - v)||^2, \forall u \in Tx, v \in Ty.$$

We observe that the class of multi-valued pseudocontractive mappings includes the class of multi-valued k-strictly pseudocontractive mappings and hence the class of multi-valued nonexpansive mappings.

Given a multi-valued mapping $T : C \to 2^H$, a point $x \in C$ is called a *fixed point* of T if $x \in Tx$. We denote the set of all fixed points of the mapping T by $F(T)$.

If $F(T) \neq \emptyset$ and $D(Tx, Tp) \le ||x - p||, \forall x \in C, \forall p \in F(T)$, then T is said to be *quasi-nonexpansive* mapping. Clearly, every nonexpansive mapping T with $F(T) \neq \emptyset$ is quasi-nonexpansive mapping. But the converse is not necessarily true (see, e.g., [23]).

Several physical problems in differential inclusions, economics, convex optimization, etc. can be transformed into finding fixed points of multi-valued mappings. As a result, researchers have studied the existence of fixed points and their approximations for different types of multi-valued mappings (see, e.g., [1,3–5,12,13,18,19] and the references therein). For approximating fixed points of single-valued mappings, basically three iterative methods are in common use: Mann iteration method, Halpern iteration method and Ishikawa iteration method.

Mann iteration method, initially studied by Mann [17], is given by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \tag{1.1}$$

where the initial guess $x_0 \in C$ is arbitrary, T is single-valued self mapping on C and $\{\alpha_n\} \subseteq [0, 1]$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum \alpha_n = \infty$. This iteration method has been extensively investigated for nonexpansive mappings (see, e.g., [8,20]). However, the Mann iteration scheme provides only weak convergence in an infinite-dimensional Hilbert space (see, e.g., [8]).

In 1967, Halpern [9] studied the following recursive formula:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, n \ge 0, \tag{1.2}$$

where T is single-valued self mapping on C and α_n is a sequence of numbers in $(0, 1)$ satisfying certain conditions. He proved strong convergence of $\{x_n\}$ to a fixed point of T, provided that T is single-valued nonexpansive mapping. Halpern's iterative method has been studied extensively by many authors (see, e.g., [14,21,26] and the references therein).

The Mann and Halpern methods were successful only for approximating fixed points of single-valued nonexpansive mappings. For approximating fixed points of single-valued Lipschitz pseudocontractive self-mapping T, in [10] Ishikawa introduced the following iterative method.

$$\begin{cases} x_0 \in C, \\ y_n = \beta_n x_n + (1 - \beta_n)Tx_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Ty_n, \ n \ge 0, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}, \{\beta_n\}$ are sequences of positive numbers satisfying the conditions:

(i) $0 \le \alpha_n \le \beta_n \le 1$; (ii) $\lim_{n\to\infty} \beta_n = 0$; (iii) $\sum \alpha_n\beta_n = \infty$. Then he showed that the sequence $\{x_n\}$ converges strongly to a fixed point of T, provided that C is compact convex subset of H. Several authors have extended the results of Ishikawa [10] to Banach spaces without compactness assumption on C (see, e.g., [15,30]).

On the other hand, in 2005, Sastry and Babu [22] introduced Mann and Ishikawa-type iterative methods for multi-valued self mappings in a real Hilbert space H as follows.

(i) Mann-type iterative method:

$$x_0 \in C, x_{n+1} = \alpha_n y_n + (1-\alpha_n)x_n, n \geq 0,$$

where $y_n \in Tx_n$ such that $||y_n - p|| = d(p, Tx_n)$ and $\alpha_n \in [0, 1]$.

(ii) Ishikawa-type iterative method:

$$\begin{cases} x_0 \in C, \\ y_n = \beta_n z_n + (1-\beta_n)x_n, \\ x_{n+1} = \alpha_n z'_n + (1-\alpha_n)x_n, n \geq 0, \end{cases} \tag{1.4}$$

where $C \subset H, T : C \to Prox(C)$, $p \in F(T)$, $z_n \in Tx_n$, $z'_n \in Ty_n$ such that $||z_n - p|| = d(p, Tx_n)$, $||z'_n - p|| = d(p, Ty_n)$ and α_n, $\beta_n \in [0, 1]$.

Then they obtained strong convergence of the schemes to points in $F(T)$ assuming that C is compact and convex subset of H, T is nonexpansive mapping with $F(T) \neq \emptyset$ and $\alpha_n, \beta_n \in [0, 1]$ satisfying certain conditions.

In [25], Song and Wang extended the result of Sastry and Babu [22] to uniformly convex Banach spaces assuming that $F(T) \neq \emptyset$ and $Tp = \{p\}, \forall p \in F(T)$.

In [23], Shahzad and Zegeye extended the above results to multi-valued quasi-nonexpansive mappings and relaxed the compactness condition on C. In addition, they introduced the following new iterative scheme in an attempt to remove the end point condition, $Tp = \{p\}, \forall p \in F(T)$, in the result of Song and Wang [25].

Let C be a nonempty, closed and convex subset of a real Banach space E, $T : C \to Prox(C)$ be a multi-valued mapping and $P_T x := \{y \in Tx : ||x - y|| = d(x, Tx)\}$. Let $\{x_n\}$ be a sequence generated from $x_0 \in C$ as follows.

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n z_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n z'_n, n \geq 0, \end{cases} \tag{1.5}$$

where $z_n \in P_T x_n$, $z'_n \in P_T y_n$ and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in [0, 1]. Then they proved that $\{x_n\}$ converges strongly to a fixed point of T under some mild conditions.

In 2016, Tufa and Zegeye [27] pointed out that the above results hold for approximating fixed points of self-mappings which are not always the cases in practical applications. Motivated by the result of Colao and Marino obtained in [6], Tufa and Zegeye introduced and studied Mann-type iterative scheme for multi-valued nonexpansive non-self mappings in a real Hilbert space. They obtained convergence results of the scheme to fixed points of the mappings.

Recently, Zegeye and Tufa [28] constructed a Halpern–Ishikawa type iterative scheme for single-valued Lipschitz pseudocontractive non-self mappings in Hilbert spaces and obtained strong convergence of the scheme to fixed points of the mappings under some mild conditions. Their result mainly extends the result of Colao et al. [7] from k-strictly pseudocontractive to pseudocontractive mapping.

Motivated by the above results, our purpose in this paper is to construct and study Halpern–Ishikawa type iterative schemes for multi-valued Lipschitz pseudocontractive non-self mappings in real Hilbert spaces. Strong convergence of the schemes to fixed points of the mappings are obtained under appropriate conditions. Our results extend and generalize many of the results in the literature.

2 Preliminaries

In this section, we collect some definitions and known results that we may use in the subsequent section.

Let C be a nonempty subset of a real Hilbert space H. A mapping $T : C \to 2^H$ is said to be *inward* if for any $x \in C$, we have

$$Tx \subseteq I_C(x) := \{x + \lambda(w - x) : \text{for some } w \in C \text{ and } \lambda \geq 1\}.$$

The set $I_C(x)$ is called *inward set* of C at x. A mapping $I - T$, where I is an identity mapping on C, is called *demiclosed* at zero if for any sequence $\{x_n\}$ in C such that $x_n \rightharpoonup x$ and $d(x_n, Tx_n) \to 0$ as $n \to \infty$, then $x \in Tx$.

Lemma 2.1 *For any $x, y \in H$, the following inequality holds:*

$$||x + y||^2 \leq ||x||^2 + 2\langle y, x + y\rangle.$$

Lemma 2.2 [2] *Let C be a convex subset of a real Hilbert space H and let $x \in H$. Then $x_0 = P_C x$ if and only if*

$$\langle z - x_0, x - x_0\rangle \leq 0, \forall z \in C,$$

where P_C is the metric projection of H onto C defined by

$$P_C x = \{y \in C : ||x - y|| = \inf ||x - z||, z \in C\}.$$

Lemma 2.3 [32] *Let H be a real Hilbert space. Then for all $x, y \in H$ and $\alpha \in [0, 1]$ the following equality holds:*

$$||\alpha x + (1 - \alpha)y||^2 = \alpha||x||^2 + (1 - \alpha)||y||^2 - \alpha(1 - \alpha)||x - y||^2.$$

Lemma 2.4 [27] *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $T : C \to CB(H)$ be a mapping and $u \in Tx$. Define $h_u : C \to R$ by*

$$h_u(x) = \inf\{\lambda \geq 0 : \lambda x + (1 - \lambda)u \in C\}.$$

Then for any $x \in C$ the following hold:

(1) $h_u(x) \in [0, 1]$ *and* $h_u(x) = 0$ *if and only if* $u \in C$;
(2) *if* $\beta \in [h_u(x), 1]$, *then* $\beta x + (1 - \beta)u \in C$;
(3) *if T is inward, then* $h_u(x) < 1$;
(4) *if* $u \notin C$, *then* $h_u(x)x + (1 - h(x))u \in \partial C$.

Lemma 2.5 [19] *Let E be a real Banach space. If $A, B \in CB(E)$ and $a \in A$, then for every $\gamma > 0$ there exists $b \in B$ such that $||a - b|| \leq D(A, B) + \gamma$.*

Lemma 2.6 [11] *Let E be a real Banach space. If $A, B \in Prox(E)$ and $a \in A$, then there exists $b \in B$ such that $||a - b|| \leq D(A, B)$.*

Lemma 2.7 [29] *Let C be a closed convex nonempty subset of a real Hilbert space H and $T : C \to CB(H)$ be a Lipschitz pseudocontractive mapping. Then $F(T)$ is closed convex subset of C.*

From the method of the proof of Lemma 1 of [24], we obtain the following lemma.

Lemma 2.8 *Let C be a closed and convex subset of a real Hilbert space H and $T : C \to Prox(H)$ be a multi-valued mapping. Define $P_T : C \to Prox(H)$ by $P_T(x) = \{y \in Tx : ||x - y|| = d(x, Tx)\}$. Then the following are equivalent:*

(i) $p \in F(T)$;
(ii) $P_T(p) = \{p\}$;
(iii) $p \in F(P_T)$.

Furthermore, $F(T) = F(P_T)$.

Lemma 2.9 *Let H be a real Hilbert space. Then the following equation holds: if $\{x_n\}$ is a sequence in H such that $x_n \rightharpoonup z \in H$, then*

$$\limsup_{n\to\infty} ||x_n - y||^2 = \limsup_{n\to\infty} ||x_n - z||^2 + ||z - y||^2, \forall y \in H.$$

Lemma 2.10 [31] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\delta_n, n \geq 0,$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\delta_n\} \subset IR$ satisfying the conditions: $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\limsup_{n\to\infty} \delta_n \leq 0$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.11 [16] *Let $\{a_n\}$ be sequences of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in N$. Then there exists a nondecreasing sequence $\{m_k\} \subset N$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in N$:*

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

3 Main results and discussion

Let C be a nonempty, closed and convex subset of a real Hilbert space H. In this section, we introduce a new iterative scheme for a multi-valued non-self mapping $T : C \to CB(H)$ and prove strong convergence results of the scheme with end point condition, $Tp = \{p\}, \forall p \in F(T)$. We also construct an iterative sequence which strongly converges to a fixed point of a multi-valued mapping $T : C \to Prox(H)$ without the end point condition.

3.1 Strong convergence results with end point condition

Let $T : C \to CB(H)$ be a multi-valued inward Lipschitz mapping with Lipschitz constant L and $\beta \in \left(1 - \frac{1}{1+\sqrt{(L+1)^2+1}}, 1\right)$. For a sequence $\{\alpha_n\}$ in $(0, 1)$, we define Halpern–Ishikawa type iterative scheme as follows:

Given $u, x_0 \in C$, let $u_0 \in Tx_0$ and

$$h_{u_0}(x_0) := \inf\{\lambda \geq 0 : \lambda x_0 + (1-\lambda)u_0 \in C\}.$$

Now if we choose $\lambda_0 \in [\max\{\beta, h_{u_0}(x_0)\}, 1)$, then it follows from Lemma 2.4 that $y_0 := \lambda_0 x_0 + (1-\lambda_0)u_0 \in C$.

By Lemma 2.5, we can choose $v_0 \in Ty_0$ such that

$$||u_0 - v_0|| \leq D(Tx_0, Ty_0) + ||x_0 - y_0||.$$

Let $g_{v_0}(y_0) := \inf\{\theta \geq 0 : \theta x_0 + (1-\theta)v_0 \in C\}$. If we choose $\theta_0 \in [\max\{\lambda_0, g_{v_0}(y_0)\}, 1)$, then by Lemma 2.4, $\theta_0 x_0 + (1-\theta_0)v_0 \in C$. Thus, it follows that

$$x_1 := \alpha_0 u + (1-\alpha_0)\big(\theta_0 x_0 + (1-\theta_0)v_0\big) \in C.$$

Hence, by the principle of mathematical induction, we have

$$\begin{cases} \lambda_n \in [\max\{\beta, h_{u_n}(x_n)\}, 1); \\ y_n = \lambda_n x_n + (1-\lambda_n)u_n; \\ \theta_n \in [\max\{\lambda_n, g_{v_n}(y_n)\}, 1); \\ x_{n+1} = \alpha_n u + (1-\alpha_n)\big(\theta_n x_n + (1-\theta_n)v_n\big), \end{cases} \tag{3.1}$$

where $u_n \in Tx_n$ and $v_n \in Ty_n$ such that $||u_n - v_n|| \leq D(Tx_n, Ty_n) + ||x_n - y_n||$, $h_{u_n}(x_n) := \inf\{\lambda \geq 0 : \lambda x_n + (1-\lambda)u_n \in C\}$ and

$g_{v_n}(y_n) := \inf\{\theta \geq 0 : \theta x_n + (1-\theta)v_n \in C\}, \forall n \geq 0$.

Now, we prove our main results.

Lemma 3.1 *Let C be a nonempty, closed and convex subset of a real Hilbert space H, $T : C \to CB(H)$ be L-Lipschitz pseudocontractive inward mapping and let $\{x_n\}$ and $\{y_n\}$ be sequences defined by (3.1) such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum \alpha_n = \infty$. Suppose that $F(T) \neq \emptyset$ with $Tp = \{p\}, \forall p \in F(T)$. Then $\{x_n\}$ and $\{y_n\}$ are bounded.*

Proof Let $p \in F(T)$. Then from (3.1) and Lemma 2.3 and the fact that T is pseudocontractive, we have

$$\begin{aligned} ||x_{n+1} - p||^2 &= ||\alpha_n u + (1-\alpha_n)(\theta_n x_n + (1-\theta_n)v_n) - p||^2 \\ &\leq \alpha_n ||u-p||^2 + (1-\alpha_n)||\theta_n(x_n - p) + (1-\theta_n)(v_n - p)||^2 \\ &= \alpha_n ||u-p||^2 + (1-\alpha_n)\big[\theta_n ||x_n - p||^2 + (1-\theta_n||v_n - p||^2\big] \\ &\quad -(1-\alpha_n)\theta_n(1-\theta_n)||v_n - x_n||^2 \\ &\leq \alpha_n ||u-p||^2 + (1-\alpha_n)\big[\theta_n ||x_n - p||^2 + (1-\theta_n)D^2(Ty_n, p)\big] \\ &\quad -(1-\alpha_n)\theta_n(1-\theta_n)||v_n - x_n||^2 \\ &\leq \alpha_n ||u-p||^2 + (1-\alpha_n)\theta_n ||x_n - p||^2 + (1-\alpha_n)(1-\theta_n) \\ &\quad \times\big[||y_n - p||^2 + ||y_n - v_n||^2\big] - (1-\alpha_n)\theta_n(1-\theta_n)||v_n - x_n||^2 \end{aligned}$$

$$\le \alpha_n||u-p||^2 + (1-\alpha_n)(1-\theta_n)\Big(||y_n-p||^2 + ||y_n-v_n||^2\Big)$$
$$+(1-\alpha_n)\theta_n\Big(||x_n-p||^2 - (1-\theta_n)||v_n-x_n||^2\Big) \tag{3.2}$$

and

$$\begin{aligned} ||y_n-p||^2 &= ||\lambda_n(x_n-p) + (1-\lambda_n)(u_n-p)||^2 \\ &= \lambda_n||x_n-p||^2 + (1-\lambda_n)||u_n-p||^2 \\ &\quad -\lambda_n(1-\lambda_n)||x_n-u_n||^2 \\ &\le \lambda_n||x_n-p||^2 + (1-\lambda_n)D^2(Tx_n,p)^2 \\ &\quad -\lambda_n(1-\lambda_n)||x_n-u_n||^2 \\ &\le \lambda_n||x_n-p||^2 + (1-\lambda_n)\big[||x_n-p||^2 + ||x_n-u_n||^2\big] \\ &\quad -\lambda_n(1-\lambda_n)||x_n-u_n||^2 \\ &= ||x_n-p||^2 + (1-\lambda_n)^2||x_n-u_n||^2. \end{aligned} \tag{3.3}$$

On the other hand, since T is L-Lipschitz, it follows from (3.1) and Lemma 2.3 that

$$\begin{aligned} ||y_n-v_n||^2 &= ||\lambda_n(x_n-v_n) + (1-\lambda_n)(u_n-v_n)||^2 \\ &= \lambda_n||x_n-v_n||^2 + (1-\lambda_n)||u_n-v_n||^2 \\ &\quad -\lambda_n(1-\lambda_n)||x_n-u_n||^2 \\ &\le \lambda_n||x_n-v_n||^2 + (1-\lambda_n)\Big(D(Tx_n,Ty_n) + ||x_n-y_n||\Big)^2 \\ &\quad -\lambda_n(1-\lambda_n)||x_n-u_n||^2 \\ &\le \lambda_n||x_n-v_n||^2 + (1-\lambda_n)(L+1)^2||x_n-y_n||^2 \\ &\quad -\lambda_n(1-\lambda_n)||x_n-u_n||^2 \\ &= \lambda_n||x_n-v_n||^2 + (1-\lambda_n)^2(L+1)^2||x_n-u_n||^2 \\ &\quad -\lambda_n(1-\lambda_n)||x_n-u_n||^2 \\ &= \lambda_n||x_n-v_n||^2 \\ &\quad -(1-\lambda_n)\big(\lambda_n-(L+1)^2(1-\lambda_n)^2\big)||x_n-u_n||^2. \end{aligned} \tag{3.4}$$

Thus, from (3.2), (3.3) and (3.4), we obtain

$$\begin{aligned} ||x_{n+1}-p||^2 &\le \alpha_n||u-p||^2 + (1-\alpha_n)(1-\theta_n)\Big(||x_n-p||^2 \\ &\quad +(1-\lambda_n)^2||x_n-u_n||^2\Big) + (1-\alpha_n)(1-\theta_n)\Big(\lambda_n||x_n-v_n||^2 \\ &\quad -(1-\lambda_n)(\lambda_n-(L+1)^2(1-\lambda_n)^2)||x_n-u_n||^2\Big) \\ &\quad +(1-\alpha_n)\theta_n||x_n-p||^2 - (1-\alpha_n)\theta_n(1-\theta_n)||v_n-x_n||^2 \\ &= \alpha_n||u-p||^2 + (1-\alpha_n)||x_n-p||^2 - (1-\alpha_n)(1-\theta_n)(1-\lambda_n) \\ &\quad \times\Big(1-(L+1)^2(1-\lambda_n)^2 - 2(1-\lambda_n)\Big)||x_n-u_n||^2 \\ &\quad +(1-\alpha_n)(1-\theta_n)(\lambda_n-\theta_n)||v_n-x_n||^2. \end{aligned} \tag{3.5}$$

Since for each $n \ge 0$, $\theta_n \ge \lambda_n$ and

$$1-2(1-\lambda_n)-(L+1)^2(1-\lambda_n)^2 \ge 1-2(1-\beta)-(L+1)^2(1-\beta)^2 > 0, \tag{3.6}$$

inequality (3.5) implies that

$$||x_{n+1} - p||^2 \leq \alpha_n ||u - p||^2 + (1 - \alpha_n)||x_n - p||^2. \tag{3.7}$$

Hence, by induction,

$$||x_{n+1} - p||^2 \leq \max\{||u - p||^2, ||x_0 - p||^2\}, \forall n \geq 0.$$

This implies that the sequence $\{x_n\}$ is bounded which in turn implies that $\{y_n\}$ is bounded. □

Theorem 3.2 *Let C be a nonempty, closed and convex subset of a real Hilbert space H, $T : C \to CB(H)$ be L-Lipschitz pseudocontractive inward mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by (3.1) such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum \alpha_n = \infty$. Suppose that $Tp = \{p\}, \forall p \in F(T)$ and $I - T$ is demiclosed at zero. If there exists $\epsilon > 0$ such that $\theta_n \leq 1 - \epsilon, \forall n \geq 0$, then $\{x_n\}$ converges strongly to a fixed point x^* of T nearest to u in the sense that $x^* = P_{F(T)}(u)$.*

Proof Let $x^* = P_{F(T)}(u)$. Then by (3.1), Lemma 2.1, Lemma 2.3 and pseudocontractivity of T, we have

$$\begin{aligned}
||x_{n+1} - x^*||^2 &= ||\alpha_n u + (1-\alpha_n)\big(\theta_n x_n + (1-\theta_n)v_n\big) - x^*||^2 \\
&= ||\alpha_n(u - x^*) + (1-\alpha_n)\big[\theta_n x_n + (1-\theta_n)v_n - x^*\big]||^2 \\
&\leq (1-\alpha_n)||\theta_n x_n + (1-\theta_n)v_n - x^*||^2 \\
&\quad +2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle \\
&= (1-\alpha_n)\theta_n||x_n - x^*||^2 + (1-\alpha_n)(1-\theta_n)||v_n - x^*||^2 \\
&\quad -(1-\alpha_n)\theta_n(1-\theta_n)||v_n - x_n||^2 + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle \\
&\leq (1-\alpha_n)\theta_n||x_n - x^*||^2 + (1-\alpha_n)(1-\theta_n)D^2(Ty_n, x^*) \\
&\quad -(1-\alpha_n)\theta_n(1-\theta_n)||v_n - x_n||^2 + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle \\
&\leq (1-\alpha_n)\theta_n||x_n - x^*||^2 \\
&\quad +(1-\alpha_n)(1-\theta_n)\big[||y_n - x^*||^2 + ||y_n - v_n||^2\big] \\
&\quad -(1-\alpha_n)\theta_n(1-\theta_n)||v_n - x_n||^2 + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle.
\end{aligned}$$

Moreover, since $x^* \in F(T)$, from (3.3) and (3.4) it follows that

$$||y_n - x^*||^2 \leq ||x_n - x^*||^2 + (1-\lambda_n)^2||x_n - u_n||^2$$

and

$$||y_n - v_n||^2 \leq \lambda_n||x_n - v_n||^2 - (1-\lambda_n)\bigg(\lambda_n - (L+1)^2(1-\lambda_n)^2\bigg)||x_n - u_n||^2.$$

Hence, by substitution, we obtain

$$\begin{aligned}
||x_{n+1} - x^*||^2 &\leq (1-\alpha_n)\theta_n||x_n - x^*||^2 + (1-\alpha_n)(1-\theta_n) \\
&\quad \times\big[||x_n - x^*||^2 + (1-\lambda_n)^2||x_n - u_n||^2\big] + (1-\alpha_n)(1-\theta_n) \\
&\quad \times\big[\lambda_n||x_n - v_n||^2 - (1-\lambda_n)(\lambda_n - (L+1)^2(1-\lambda_n)^2)||x_n - u_n||^2\big] \\
&\quad -(1-\alpha_n)\theta_n(1-\theta_n)||v_n - x_n||^2 + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle \\
&= (1-\alpha_n)||x_n - x^*||^2 - (1-\alpha_n)(1-\theta_n)(1-\lambda_n) \\
&\quad \times[1 - (L+1)^2(1-\lambda_n)^2 - 2(1-\lambda_n)]||x_n - u_n||^2 \\
&\quad +(1-\alpha_n)(1-\theta_n)(\lambda_n - \theta_n)||x_n - v_n||^2 \\
&\quad +2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle &(3.8) \\
&\leq (1-\alpha_n)||x_n - x^*||^2 + 2\alpha_n\langle u - x^*, x_n - x^*\rangle \\
&\quad +2\alpha_n||u - x^*||||x_{n+1} - x_n||. &(3.9)
\end{aligned}$$

Next, we consider two possible cases.

Case 1. Suppose that there exists $n_0 \in N$ such that $\{||x_n - x^*||\}$ is decreasing for all $n \geq n_0$. Then it follows that $\{||x_n - x^*||)\}$ is convergent. Thus, (3.8), (3.6) and the fact that $\theta_n \geq \lambda_n$ and $\lim_{n\to\infty} \alpha_n = 0$ imply that

$$x_n - u_n \to 0 \text{ as } n \to \infty. \tag{3.10}$$

Combining this with (3.1) yields

$$||y_n - x_n|| = (1 - \lambda_n)||x_n - u_n|| \to 0 \text{ as } n \to \infty, \tag{3.11}$$

and so from Lipschitz continuity of T, we have

$$\begin{aligned} ||v_n - x_n|| &\leq ||v_n - u_n|| + ||u_n - x_n|| \\ &\leq D(Ty_n, Tx_n) + ||x_n - y_n|| + ||u_n - x_n|| \\ &\leq (L+1)||y_n - x_n|| + ||u_n - x_n|| \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.12}$$

Thus, from (3.1), it follows that

$$||x_{n+1} - x_n|| \leq \alpha_n ||u - x_n|| + (1 - \alpha_n)(1 - \theta_n)||v_n - x_n|| \to 0. \tag{3.13}$$

On the other hand, since $\{x_n\}$ is bounded and H is reflexive, we can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$x_{n_i} \rightharpoonup w \text{ and } \limsup_{n\to\infty} \langle u - x^*, x_n - x^* \rangle = \lim_{i\to\infty} \langle u - x^*, x_{n_i} - x^* \rangle.$$

Also from (3.1) and (3.10), we have $d(x_n, Tx_n) \leq ||x_n - u_n|| \to 0$. Then since $I - T$ is demiclosed at 0, it follows that $w \in F(T)$. Therefore, by Lemmas 2.7 and 2.2, we obtain

$$\begin{aligned} \limsup_{n\to\infty} \langle u - x^*, x_n - x^* \rangle &= \lim_{i\to\infty} \langle u - x^*, x_{n_i} - x^* \rangle \\ &= \langle u - x^*, w - x^* \rangle \leq 0. \end{aligned} \tag{3.14}$$

Then it follows from (3.9), (3.14) and Lemma 2.10 that $||x_n - x^*|| \to 0$ as $n \to \infty$. Consequently, $x_n \to x^* = P_{F(T)}(u)$.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$||x_{n_i} - x^*|| < ||x_{n_i+1} - x^*||, \forall i \in N.$$

Then by Lemma 2.11, there exists a nondecreasing sequence $\{m_k\} \subset N$ such that $m_k \to \infty$ and

$$||x_{m_k} - x^*|| \leq ||x_{m_k+1} - x^*|| \text{ and } ||x_k - x^*|| \leq ||x_{m_k+1} - x^*||, \forall k \in N. \tag{3.15}$$

Thus, by (3.8) and (3.6), we have $||x_{m_k} - u_{m_k}||] \to 0$ as $k \to \infty$, which implies that

$$d(x_{m_k}, Tx_{m_k}) \to 0 \text{ as } k \to \infty.$$

Then using the methods we used in Case 1, we obtain

$$\limsup_{k\to\infty} \langle u - x^*, x_{m_k} - x^* \rangle \leq 0. \tag{3.16}$$

Now, from (3.9), we have

$$\begin{aligned} ||x_{m_k+1} - x^*||^2 \leq (1 - \alpha_{m_k})||x_{m_k} - x^*||^2 + 2\alpha_{m_k} \langle u - x^*, x_{m_k} - x^* \rangle \\ +2\alpha_{m_k} ||u - x^*|| ||x_{m_k+1} - x_{m_k}||, \end{aligned} \tag{3.17}$$

and hence (3.15) and (3.17) imply that

$$\begin{aligned} \alpha_{m_k} ||x_{m_k} - x^*||^2 &\leq ||x_{m_k} - x^*||^2 - ||x_{m_k+1} - x^*||^2 + 2\alpha_{m_k} \langle u - x^*, x_{m_k} - x^* \rangle \\ &\quad +2\alpha_{m_k} ||u - x^*|| ||x_{m_k+1} - x_{m_k}|| \\ &\leq 2\alpha_{m_k} \langle u - x^*, x_{m_k} - x^* \rangle + 2\alpha_{m_k} ||u - x^*|| ||x_{m_k+1} - x_{m_k}||. \end{aligned}$$

Then since $\alpha_{m_k} > 0$, we have

$$||x_{m_k} - x^*||^2 \le 2\langle u - x^*, x_{m_k} - x^*\rangle + 2||u - x^*||||x_{m_k+1} - x_{m_k}||.$$

Thus, using (3.13) and (3.16), we obtain

$$\limsup_{k\to\infty} ||x_{m_k} - x^*||^2 \le 0 \text{ and hence } ||x_{m_k} - x^*|| \to 0 \text{ as } k \to \infty.$$

This together with (3.17) imply that $||x_{m_k+1} - x^*|| \to 0$ as $k \to \infty$. But, since $||x_k - x^*|| \le ||x_{m_k+1} - x^*||$, for all $k \in N$, it follows that $x_k \to x^* = P_{F(T)}(u)$. Therefore, the above two cases imply that $\{x_n\}$ converges strongly to the fixed point of T nearest to u. □

If T is assumed to be k-strictly pseudocontractive, then T is pseudocontractive and so, we have the following corollary.

Corollary 3.3 *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $T : C \to CB(H)$ be L-Lipschitz k-strictly pseudocontractive inward mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by (3.1) such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum \alpha_n = \infty$. Suppose that $Tp = \{p\}, \forall p \in F(T)$ and $I - T$ is demiclosed at zero. If there exists $\epsilon > 0$ such that $\theta_n \le 1 - \epsilon\ \forall n \ge 0$, then $\{x_n\}$ converges strongly to a fixed point of T nearest to u.*

Definition 3.4 A point $x \in F(T)$ is said to be a minimum norm point of $F(T)$ if $||x|| \le ||y||, \forall y \in F(T)$.

If C contains the zero element, then we have the following theorem for finding a point with minimum-norm in the set of fixed points of a Lipschitz pseudocontractive mapping.

Theorem 3.5 *Let C be a nonempty, closed and convex subset of a real Hilbert space H containing 0, $T : C \to CB(H)$ be L-Lipschitz pseudocontractive inward mapping and let $\{x_n\}$ be a sequence defined by (3.1) with $u = 0$. Suppose that $F(T) \neq \emptyset, Tp = \{p\}, \forall p \in F(T)$ and $I - T$ is demiclosed at zero. If there exists $\epsilon > 0$ such that $\theta_n \le 1 - \epsilon\ \forall n \ge 0$, then $\{x_n\}$ converges strongly to the minimum-norm point in $F(T)$.*

Proof By Theorem 3.2, x_n converges to a fixed point x^* of T nearest to 0. Thus, $||x^*|| = ||x^* - 0|| \le ||x - 0|| = ||x||, \forall x \in C$ and hence the proof. □

3.2 Strong convergence results without end point condition

Before introducing our algorithm, we prove the following lemmas.

Lemma 3.6 *Let C be a nonempty, closed convex subset of a real Hilbert space H and $T : C \to Prox(H)$ be a k-strictly pseudocontractive multi-valued mapping. Then T is Lipschitz mapping.*

Proof Let $x, y \in C$ and $u \in Tx$. Then by Lemma 2.6, there is $v \in Ty$ such that

$$||u - v|| \le D(Tx, Ty).$$

Then since T is k-strictly pseudocontractive, we have

$$\begin{aligned} D^2(Tx, Ty) &\le ||x - y||^2 + k||x - y - (u - v)||^2 \\ &\le \Big(||x - y|| + \sqrt{k}\big(||x - y|| + ||u - v||\big)\Big)^2 \\ &\le \Big(||x - y|| + \sqrt{k}\big(||x - y|| + D^2(Tx, Ty)\big)\Big)^2 \end{aligned}$$

which implies that

$$D(Tx, Ty) \le \frac{1 + \sqrt{k}}{1 - \sqrt{k}}||x - y||.$$

Therefore, T is Lipschitzian with Lipschitz constant $L = \frac{1+\sqrt{k}}{1-\sqrt{k}}$. □

Lemma 3.7 *Let $T : C \to Prox(H)$ be a multi-valued mapping such that P_T is k-strictly pseudocontractive. Then $I - P_T$ is demiclosed at zero.*

Proof Let $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup p$ and $d(x_n, P_T x_n) \to 0$. Let $y \in P_T p$. By Lemma 2.6, for each $n \in N$, there exists $y_n \in P_T x_n$ such that

$$||y_n - y|| \leq D(P_T y_n, P_T p).$$

Also, since $y_n \in P_T x_n$, it follows that

$$||x_n - y_n|| = d(x_n, P_T x_n) \to 0.$$

Now, for each $x \in H$, define $f : H \to [0, \infty]$ by

$$f(x) = \limsup_{n\to\infty} ||x_n - x||^2. \tag{3.18}$$

Then from Lemma 2.9, we obtain

$$f(x) = \limsup_{n\to\infty} ||x_n - p||^2 + ||p - x||^2, \forall x \in H,$$

which implies that

$$f(x) = f(p) + ||p - x||^2, \forall x \in H.$$

Hence, we obtain that

$$f(y) = f(p) + ||p - y||^2. \tag{3.19}$$

In addition, by the definition of k-strictly pseudocontractive mapping, we have

$$\begin{aligned} f(y) &= \limsup_{n\to\infty} ||x_n - y||^2 \\ &= \limsup_{n\to\infty} ||x_n - y_n + y_n - y||^2 \\ &= \limsup_{n\to\infty} ||y_n - y||^2 \\ &\leq \limsup_{n\to\infty} D^2(P_T x_n, P_T p) \\ &\leq \limsup_{n\to\infty} \left(||x_n - p||^2 + k||x_n - y_n - (p - y)||^2 \right) \\ &\leq \limsup_{n\to\infty} \left(||x_n - p||^2 + k\big(||x_n - y_n|| + ||p - y||\big)^2 \right) \\ &= \limsup_{n\to\infty} ||x_n - p||^2 + k||p - y||^2 \\ &= f(p) + k||p - y||^2. \end{aligned} \tag{3.20}$$

Then it follows from (3.19) and (3.20) that $(1-k)||p - y||^2 = 0$ and hence, $p = y \in P_T p$. Therefore, $I - P_T$ is demiclosed at zero. □

Now, we present our algorithm as follows. Let $T : C \to Prox(H)$ be a multi-valued mapping such that P_T is inward Lipschitz mapping with Lipschitz constant L and $\beta \in \left(1 - \frac{1}{1+\sqrt{L^2+1}}, 1\right)$. For a sequence $\{\alpha_n\}$ in $(0, 1)$, we define Halpern–Ishikawa type iterative scheme as follows:

Given $u, x_0 \in C$, let $u_0 \in P_T x_0$ and

$$h_{u_0}(x_0) := \inf\{\lambda \geq 0 : \lambda x_0 + (1 - \lambda)u_0 \in C\}.$$

Now, if we choose $\lambda_0 \in [\max\{\beta, h_{u_0}(x_0)\}, 1)$, then it follows from Lemma 2.4 that

$$y_0 := \lambda_0 x_0 + (1 - \lambda_0) u_0 \in C.$$

By Lemma 2.6, we can choose $v_0 \in P_T y_0$ such that

$$||u_0 - v_0|| \leq D(P_T x_0, P_T y_0).$$

Let $g_{v_0}(y_0) := \inf\{\theta \geq 0 : \theta x_0 + (1-\theta) v_0 \in C\}$. If we choose $\theta_0 \in [\max\{\lambda_0, g_{v_0}(y_0)\}, 1)$, then by Lemma 2.4, $\theta_0 x_0 + (1-\theta_0) v_0 \in C$. Thus, it follows that

$$x_1 := \alpha_0 u + (1-\alpha_0)\big(\theta_0 x_0 + (1-\theta_0) v_0\big) \in C.$$

Inductively, $\{x_n\}$ is defined as

$$\begin{cases} \lambda_n \in [\max\{\beta, h_{u_n}(x_n\}, 1); \\ y_n = \lambda_n x_n + (1-\lambda_n) u_n; \\ \theta_n \in [\max\{\lambda_n, g_{v_n}(y_n)\}, 1); \\ x_{n+1} = \alpha_n u + (1-\alpha_n)\big(\theta_n x_n + (1-\theta_n) v_n\big), n \geq 0, \end{cases} \tag{3.21}$$

where $u_n \in P_T x_n$ and $y_n \in P_T y_n$ such that $||u_n - v_n|| \leq D(P_T x_n, P_T y_n)$,
$h_{u_n}(x_n) := \inf\{\lambda \geq 0 : \lambda x_n + (1-\lambda) u_n \in C\}$ and
$g_{v_n}(y_n) := \inf\{\theta \geq 0 : \theta x_n + (1-\theta) v_n \in C\}$.

Theorem 3.8 *Let C be a nonempty, closed and convex subset of a real Hilbert space H, $T : C \to Prox(H)$ be a multi-valued mapping such that P_T is k-strictly pseudocontractive inward mapping and $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by (3.21) such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum \alpha_n = \infty$. If there exists $\epsilon > 0$ with $\theta_n \leq 1 - \epsilon \; \forall n \geq 0$, then $\{x_n\}$ converges strongly to a fixed point of T nearest to u.*

Proof By Lemma 3.6, P_T is Lipschitz with Lipschitz constant $L = \frac{1+\sqrt{k}}{1-\sqrt{k}}$ and $I - P_T$ is demiclosed at zero by Lemma 3.7. Moreover, by Lemma 2.8, $F(T) = F(P_T)$ and $P_T p = \{p\}$ for all $p \in F(T)$. The rest of the proof is very similar to the proof of Theorem 3.2. □

In Theorem 3.8, if P_T is assumed to be nonexpansive mapping, then P_T is k-strictly pseudocontractive and hence we have the following corollary.

Corollary 3.9 *Let C be a nonempty, closed and convex subset of a real Hilbert space H, $T : C \to Prox(H)$ be a multi-valued mapping such that P_T is nonexpansive inward mapping and $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by (3.21) such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum \alpha_n = \infty$. If there exists $\epsilon > 0$ with $\theta_n \leq 1 - \epsilon \; \forall n \geq 0$, then $\{x_n\}$ converges strongly to a fixed point of T nearest to u.*

The method of the proof of Theorem 3.2 also provides the following result.

Theorem 3.10 *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $T : C \to Prox(H)$ be a multi-valued mapping such that P_T is an inward Lipschitz pseudocontractive mapping. Suppose that $F(T) \neq \emptyset$, $I - P_T$ is demiclosed at 0 and $\{x_n\}$ be a sequence defined by (3.21). If there exists $\epsilon > 0$ such that $\theta_n \leq 1 - \epsilon \; \forall n \geq 0$, then $\{x_n\}$ converges strongly to a fixed point of T nearest to u.*

Remark 3.11 Note that, in Algorithms (3.1) and (3.21), the coefficients λ_n and θ_n can be chosen simply as follows: $\lambda_n = \max\{\beta, h_{u_n}(x_n)\}$ and $\theta_n = \max\{\lambda_n, g_{v_n}(y_n)\}$.

4 Numerical example

Now, we give an example of a nonlinear mapping which satisfies the conditions of Theorem 3.2.

Example 4.1 Let $H = \mathrm{IR}R$ with Euclidean norm. Let $C = [-1, \frac{1}{2}]$ and $T : C \to \mathbb{R}$ be defined by

$$Tx = \begin{cases} \{-x, 0\}, & x \in [-1, 0), \\ x, & x \in [0, \frac{1}{2}]. \end{cases} \tag{4.1}$$

Then we observe that T satisfies the inward condition and $F(T) = [0, \frac{1}{2}]$. We first show that T is Lipschitz pseudocontractive mapping. We consider the following cases.

Case 1: Let $x, y \in [-1, 0)$. Then $Tx = \{-x, 0\}$ and $Ty = \{-y, 0\}$. Thus, we have

$$\begin{aligned} D(Tx, Ty) &= \max\left\{ \sup_{a \in Ty} d(a, Tx), \sup_{b \in Tx} d(b, Ty) \right\} \\ &= \max\{\min\{|x-y|, |y|\}, \min\{|x-y|, |x|\}\} \\ &= \begin{cases} \max\{\min\{|x-y|, |y|\}, |x-y|\}, & \text{if } x \le y, \\ \max\{|x-y|, \min\{|x-y|, |x|\}, & \text{if } y \le x, \end{cases} \\ &= |x-y|. \end{aligned}$$

Case 2: Let $x, y \in [0, \frac{1}{2}]$. Then $Tx = \{x\}$ and $Ty = \{y\}$. Thus, we have

$$\begin{aligned} D(Tx, Ty) &= \max\left\{ \sup_{a \in Ty} d(a, Tx), \sup_{b \in Tx} d(b, Ty) \right\} \\ &= |x-y|. \end{aligned}$$

Case 3: Let $x \in [-1, 0)$ and $y \in [0, \frac{1}{2}]$. Then $Tx = \{-x, 0\}$ and $Ty = \{y\}$. Thus, we have

$$\begin{aligned} D(Tx, Ty) &= \max\left\{ \sup_{a \in Ty} d(a, Tx), \sup_{b \in Tx} d(b, Ty) \right\} \\ &= \max\{\min\{|x+y|, y\}, \max\{|x+y|, y\}\} \\ &\le |x-y|. \end{aligned}$$

From the above cases, it follows that T is L-Lipschitz pseudocontractive mapping with Lipschitz constant $L = 1$. Then $1 - \frac{1}{1+\sqrt{(L+1)^2+1}} = 0.691$. Thus, we can choose $\beta = \frac{5}{6}$ and $\alpha_n = \frac{2}{n+5}$. Now, let $x_0 = -1$ and $u = 0.5$. Then $Tx_0 = \{0, 1\}$. Take $u_0 = 0$. Then we have

$$\begin{aligned} h_{u_0}(x_0) &= \inf\{\lambda \ge 0 : \lambda x_0 + (1-\lambda)u_0 \in C\} \\ &= \inf\left\{\lambda \ge 0 : -\lambda \in C\right\} \\ &= 0. \end{aligned}$$

Let $\lambda_0 = \max\{\beta, h_{u_0}(x_0)\} = \frac{5}{6}$. Then $y_0 = \lambda_0 x_0 + (1-\lambda_0)u_0 = -\frac{5}{6}$ and $Ty_0 = \{0, \frac{5}{6}\}$. If we take $v_0 = 0$, then we get

$$g_{v_0}(y_0) = \inf\{\theta \ge 0 : \theta x_0 + (1-\theta)v_0 \in C\} = 0.$$

If we choose $\theta_0 = \max\{\lambda_0, g_{v_0}(y_0)\} = \frac{5}{6}$, then we have

$$x_1 = \alpha_0 u + (1-\alpha_0)[\theta_0 x_0 + (1-\theta_0)v_0] = -\frac{3}{10} = -0.3.$$

Then $Tx_1 = \{0, \frac{3}{10}\}$. If we choose $u_1 = 0$, the we obtain $h_{u_1}(x_1) = 0$. Now, we can choose $\lambda_1 = \frac{5}{6}$, which yields

$$y_1 = \lambda_1 x_1 + (1-\lambda_1)u_1 = -\frac{1}{4} \text{ and } Ty_1 = \left\{0, \frac{1}{4}\right\}.$$

Again, we can choose $v_1 = 0$ and $\theta_1 = \frac{5}{6}$, which yields $x_2 = 0$. Then $Tx_2 = \{0\}$. In this case $u_2 = Tx_2 = 0$ and hence $h_{u_2}(x_2) = 0$. Thus, we can choose $\lambda_2 = \frac{5}{6}$ which yields $y_2 = 0$ and $x_3 = 0.14$ for $\theta_2 = \frac{5}{6}$. In

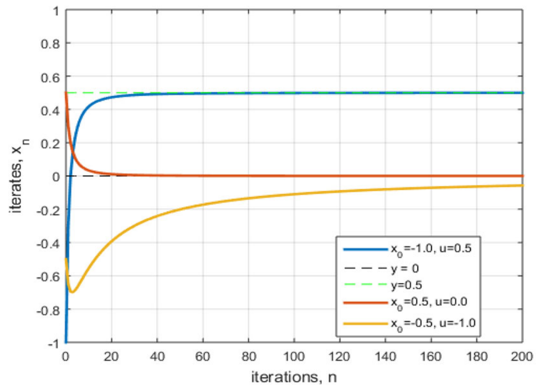

Fig. 1 Convergence of x_n for different values of the initial point x_0 and the constant u

general, we observe that for $x_0 = -1, u = 0.5$ and $\alpha_n = \frac{2}{n+5}$, we can choose $\lambda_n = \theta_n = \frac{5}{6}$. Thus, all the conditions of Theorem 3.2 are satisfied and x_n converges to $0.5 = P_{F(T)}u$ (see Fig. 1).

Similarly, for $x_0 = 0.5$ and $u = 0$, the sequence $\{x_n\}$ converges to $0 = P_{F(T)}u$. Moreover, for $x_0 = -0.5$ and $u = -1$, x_n converges to $0 = P_{F(T)}u$ (see Fig. 1 which is obtained using MATLAB version 8.5.0.197613(R2015a)).

5 Conclusion

In this paper, we have constructed Halpern–Ishikawa type iterative methods for approximating fixed points of multi-valued pseudocontractive non-self mappings in the setting of real Hilbert spaces. Strong convergence results of the scheme to a fixed points of multi-valued Lipschitz pseudocontractive mappings are obtained under appropriate conditions on the iterative parameter and an end point condition on the mappings under consideration. In addition, a Halpern–Ishikawa type iterative method for approximating fixed points of multi-valued k-strictly pseudocontractive mappings is introduced and strong convergence results of the scheme are obtained without the end point condition. Our results extend and generalize many of the results in the literature (see, e.g., [6,7,22,23,25,27–29]). More particularly, Theorem 3.2 extends Theorem 3.2 of Zegeye and Tufa [28] from single-valued mapping to multi-valued mapping. Thus, if we assume that T is single-valued mapping in Theorem 3.2, then we get Theorem 3.2 of Zegeye and Tufa [28]. Theorem 3.8 extends Theorem 8 of Colao et al. [7] from single-valued mapping to multi-valued mapping.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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