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Surjectivity of certain adjoint operators and applications

Received: 9 April 2020 / Accepted: 2 September 2020 / Published online: 15 September 2020
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Abstract This paper is an extension and generalization of some previous works, such as the study of M. Benalili and A. Lansari. Indeed, these authors, in their work about the finite co-dimension ideals of Lie algebras of vector fields, restricted their study to fields X_0 of the form $X_0 = \sum_{i=1}^n (\alpha_i \cdot x_i + \beta_i \cdot x_i^{1+m_i}) \frac{\partial}{\partial x_i}$, where α_i, β_i are positive and m_i are even natural integers. We will first study the sub-algebra U of the Lie-Fréchet space E , containing vector fields of the form $Y_0 = X_0^+ + X_0^- + Z_0$, such as $X_0(x, y) = A(x, y) = (A^-(x), A^+(y))$, with A^- (respectively, A^+) a symmetric matrix having eigenvalues $\lambda < 0$ (respectively, $\lambda > 0$) and Z_0 are germs infinitely flat at the origin. This sub-algebra admits a hyperbolic structure for the diffeomorphism $\psi_{t*} = (\exp \cdot tY_0)_*$. In a second step, we will show that the infinitesimal generator ad_{-X} is an epimorphism of this admissible Lie sub-algebra U . We then deduce, by our fundamental lemma, that $U = E$.

Mathematics Subject Classification 53C05

1 Introduction

The ideals of finite codimension in Lie algebras of vector fields have recently received a lot of attention. Some authors such as Pursel and Shanks [9], by studying the invertibility of the Lie bracket $[X, Y] = ad_X(Y)$ which is an infinitesimal generator of an one-parameter group $t, \gamma_t = (\exp tX)^*$, in Lie algebras containing a germ of vector fields X do not vanish at the origin O , have treated the finite-codimensional ideals of these algebras. This result has been prolonged in the Banach-Lie algebras of vector fields infinitely flat at 0 containing germs which vanish at the origin of the form $X_0 = \sum_{i=1}^n (\alpha_i \cdot x_i + Z_0(x))$, where α_i are of constant signs [2, 3, 8]. Among the motivations and possible applications of these results:

- The properties of the injectivity of the exponential function of a vector field have given rise to the existence of the Fourier series [11].
- The properties of the surjectivity of the directional derivative of the exponential function have given rise to the existence of the inversibility of the exponential function through the Nash–Moser theorem where positive results were obtained first in a Fréchet space and then in a hyperbolic type Fréchet space by integrating the diffeomorphisms in the smooth flows [12].
- Boris Kolev in [7] studied the particular case of a Lie–Poisson canonical structure.

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- M. BENALILI in [1] has studied suitable spectral properties of the adjoint operators induced by appropriate perturbations of some hyperbolic linear vector fields of the form $Y_0 = X_0^+ + X_0^- + Z_0$, where Z_0 is k -flat in the unit ball.

This paper is an extension on the basis of these works where we will first study the sub-algebra U of the Lie–Fréchet space E , containing vector fields of the form $Y_0 = X_0^+ + X_0^- + Z_0$, such as $X_0(x, y) = A(x, y) = (A^-(x), A^+(y))$, with A^- (respectively, A^+) a symmetric matrix having eigenvalues $\lambda < 0$ (respectively, $\lambda > 0$) and Z_0 are germs infinitely flat at the origin. This sub-algebra admits a hyperbolic structure for the diffeomorphism $\psi_{t*} = (\exp \cdot tY_0)_*$. In a second step, we will show that the infinitesimal generator ad_{-X} is an epimorphism of this admissible Lie sub-algebra U . We then deduce, by our fundamental lemma, that $U = E$.

Part I: Admissible hyperbolic-type algebra

2 Definitions

2.1 Fréchet space

Let $\mathbb{R}^n$ be the Euclidean space provided with the scalar product $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ the norm induced by this scalar product.

(a) Let E be the space of vector fields X of class C^∞ on $\mathbb{R}^n$, satisfying:

$\forall r \in \mathbb{N}, \exists M_r > 0$ such that:

$$\begin{cases} \forall x \in \mathbb{R}^n \\ \forall \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n \\ \forall k \in \mathbb{N} \end{cases} \quad \text{and} \quad \begin{cases} |\alpha| = \alpha_1 + \dots + \alpha_n \\ |\alpha| + k \leq r \end{cases}.$$

we have $\|D^\alpha X(x)\| (1 + \|x\|^2)^{k/2} \leq M_r$.

We define on E a graduation of seminorms:

$$\|X\|_r = \sup_{x \in \mathbb{R}^n} \max_{k+|\alpha| \leq r} \|D^\alpha X(x)\| (1 + \|x\|^2)^{k/2}$$

so $(E, \|\cdot\|_r)$ is called a Fréchet space [6].

(b) Let G the Schwartz space, which is the vector space of class C^∞ functions on $\mathbb{R}^n$ satisfying:

$$G = \{f \in C^\infty(\mathbb{R}^n) / \forall p \geq 0, \forall x \in \mathbb{R}^n, \forall \alpha \in \mathbb{N}^n \exists C_p > 0 \text{ satisfying } \|D^\alpha f(x)\| (1 + \|x\|^2)^p \leq C_p\}.$$

G is a space where the Fourier transform exists as well as its inverse.

We define on G a graduation of seminorms as follows:

$$\|f\|_r = \sup_{x \in \mathbb{R}^n} \max_{k+|\alpha| \leq r} \|D^\alpha f(x)\| (1 + \|x\|^2)^{k/2}$$

so $(G, \|\cdot\|_r)$ is called a Fréchet space [6].

2.2 Smooth flow

(a) Adjoint diffeomorphisms

Definition 2.1 For all $X, Y \in E$ such that $Y = \sum_{i=1}^n u_i(x) \frac{\partial}{\partial x_i}$, where $u_i \in C^\infty(\mathbb{R}^n)$, we define the adjoint diffeomorphisms ϕ_t^* and $(\phi_t)_*$ by

$$\begin{cases} (\phi_t)^* Y = (\exp tX)^* \cdot Y = ((D\phi_t)^{-1} \cdot Y) \circ \phi_t = \sum_{j=1}^n e^{-tX} \cdot u_j(e^{tX} \cdot x) \frac{\partial}{\partial x_j} \\ (\phi_t)_* Y = (\exp tX)_* \cdot Y = (D\phi_t \cdot Y) \circ (\phi_t)^{-1} = \sum_{j=1}^n e^{tX} \cdot u_j(e^{-tX} \cdot x) \frac{\partial}{\partial x_j} \end{cases}$$

Proprieties. For all $X \in E$, we associate the X -flow $\phi_t = \exp tX$, so

(i) ad_X (respectively, ad_{-X}) is an infinitesimal generator of the one-parameter group ϕ_t^* (respectively, $(\phi_t)_*$) on E ; that is, ϕ_t^* is a solution of the following dynamic system:

$$\begin{cases} \frac{d}{dt} (\phi_t)^* Y = (\phi_t)^* \cdot ad_X (Y) = \phi_t^* [X, Y] \\ (\phi_0)^* Y = Y \end{cases}$$

and, respectively, $(\phi_t)_*$ is the solution of

$$\begin{cases} \frac{d}{dt} (\phi_t)_* Y = (\phi_t)_* \cdot ad_{-X} (Y) = (\phi_t)_* [Y, X] \\ (\phi_0)_* Y = Y \end{cases}$$

(ii) $(\phi_t)_* = \phi_{-t}^*$; $\phi_t^* = (\phi_{-t})_*$, $\forall t > 0$.

Proof (i) In fact,

$$\begin{aligned} \frac{d}{dt} ((\phi_t)_* Y) (x) &= \frac{d}{dt} D [\phi_t (\phi_{-t} (x))] \cdot Y (\phi_{-t} (x)) + D \phi_t (\phi_{-t} (x)) \cdot \frac{d}{dt} Y (\phi_{-t} (x)) \\ &= D_x X (\phi_t \circ \phi_{-t} (x)) \cdot D \phi_t (\phi_{-t} (x)) \cdot Y (\phi_{-t} (x)) \\ &\quad - D \phi_t (\phi_{-t} (x)) \cdot D_y (Y (\phi_{-t} (x))) \cdot X \circ \phi_{-t} (x) \quad \text{where } y = \phi_{-t} (x) \\ &= D_x X (x) (D \phi_t \cdot Y) (\phi_{-t} (x)) - (D \phi_t \cdot X) (\phi_{-t} (x)) \cdot D_y Y (y) \\ &= D_x X (x) ((\phi_t)_* \cdot Y) (x) - ((\phi_t)_* \cdot X) (x) \cdot D_y Y (y) \\ &= (\phi_t)_* [Y, X] (x) \\ &= (\phi_t)_* ad_{-X} (Y) (x). \end{aligned}$$

That is to say:

$$\frac{d}{dt} (\phi_t)_* Y = (\phi_t)_* [Y, X];$$

so,

$$\frac{d}{dt} ((\phi_t)_* Y) (x) |_{t=0^+} = [Y, X] (x) = ad_{-X} (Y) (x),$$

i.e. that ad_{-X} is an infinitesimal generator of the one-parameter group $(\phi_t)_*$ on E , and by the same reasoning, we will have

$$\frac{d}{dt} (\phi_t)^* Y = (\phi_t)^* [X, Y].$$

(ii) Let us show that $(\phi_t)_* = \phi_{-t}^*$; we put $y = \phi_{-t} (x)$:

$$\begin{aligned} \frac{d}{dt} ((\phi_t)_* Y) (x) &= D_x X (\phi_t \circ \phi_{-t} (x)) \cdot D \phi_t (\phi_{-t} (x)) \cdot Y (\phi_{-t} (x)) \\ &\quad - D \phi_t (\phi_{-t} (x)) \cdot D (Y (\phi_{-t} (x))) \cdot X \circ \phi_{-t} (x) \\ &= D_x X (x) ((D \phi_{-t})^{-1} \cdot Y) (\phi_{-t} (x)) - ((D \phi_{-t})^{-1} \cdot X) (\phi_{-t} (x)) \cdot D_y Y (y) \\ &= D_x X (x) ((\phi_{-t})^* \cdot Y) (x) - ((\phi_{-t})^* \cdot X) (x) \cdot D_y Y (y) \\ &= (\phi_{-t})^* [Y, X] (x) \\ &= (\phi_{-t})^* ad_{-X} (Y) (x). \end{aligned}$$

Now, according to Property (i):

$$\frac{d}{dt} ((\phi_t)_* Y) (x) = (\phi_t)_* ad_{-X} (Y) (x),$$

then $(\phi_t)_* = (\phi_{-t})^*$.

- The same reasoning can be applied in the case: $\phi_t^* = (\phi_{-t})_*$

□.

(b) Smooth flow Let $X \in E$ and the X -flow $\phi_t = \exp tX$. We say that an adjoint flow ϕ_t^* decays tamely on E of degree r and base b if

- (i) ϕ_t^* preserves $E \forall t \geq 0$,
- (ii) for any integer $k \geq b$ there is an integer $l_k = k + r$ and a strictly positive, continuous and decreasing function $C_k(t)$ defined on $[0, \infty)$ satisfying:

$$\|\phi_t^* Z\|_k \leq C_k(t) \|Z\|_{l_k} \quad (\forall t \geq 0; \forall Z \in E),$$

and the improper integral:

$$\int_0^\infty C_k(t) dt \quad \text{converges} \quad \forall k \geq b.$$

Alternately, ϕ_t^* can be replaced by $(\phi_t)_* = \phi_{-t}^*$ according to the asymptotic behaviour of ϕ_t [12].

2.3 Fréchet space with hyperbolic structure

(a) Admissible algebra

Definition 2.2 (*Admissible algebra*) Let U be a Lie–Fréchet sub-algebra of E . U is said to be admissible with respect to the vector field $Y_0 = X_0 + Z_0$ if and only if it satisfies the following conditions:

- (i) Let X_0 be a vector field on E such that

$$X_0 \cdot \begin{pmatrix} x \\ y \end{pmatrix} = A \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A^- & 0 \\ 0 & A^+ \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A^- x \\ A^+ y \end{pmatrix} \\ \forall (x, y) \in K \subset \mathbb{R}^k \times \mathbb{R}^l = \mathbb{R}^n$$

A^- (respectively, A^+) is a real symmetric matrix of type $(k \times k)$ (respectively, $l \times l$) with $k + l = n$, having strictly negative eigenvalues (respectively, strictly positive).

The matrix A^- satisfies: $\forall m \geq 2, \forall i = 1, k : \exists a_R, a_L > 0, \rho_1 > 1$, such that

$$\begin{cases} -a_L \leq \lambda_i \leq -a_R < 0 \\ m \cdot a_R - a_L \geq \rho_1 > 1 \end{cases} \quad (I).$$

And, respectively, for A^+ :

$\forall m \geq 2, \forall i = 1, l : \exists b_R, b_L > 0, \rho_2 > 1$ such that

$$\begin{cases} 0 < b_L \leq \lambda_i \leq b_R \\ m \cdot b_L - b_R \geq \rho_2 > 1 \end{cases} \quad (II).$$

We adopt the following notation for the rest of this paper:

$$X_0 = X_0^- + X_0^+; \quad X_0(x, y) = (X_0^-(x), X_0^+(y)) = (A^- \cdot x, A^+ \cdot y) \in \mathbb{R}^k \times \mathbb{R}^l,$$

from which we have

$$\phi_t(x, y) = (\exp tX_0)(x, y) = ((\exp tX_0^-)(x), (\exp tX_0^+)(y)); \quad \forall (x, y) \in \mathbb{R}^k \times \mathbb{R}^l.$$

- (ii) There exists U_1 (respectively, U_2) a Lie–Fréchet sub-algebra of E containing the vector field $Y_0^- = X_0^- + Z_0^-$ (respectively, $Y_0^+ = X_0^+ + Z_0^+$) such that

$$U = U_1 \oplus U_2,$$

where $Z_0 = Z_0^- + Z_0^+$ is a perturbation of X_0 such that

$$Z_0(x, y) = (Z_0^-(x), Z_0^+(y)) \in \mathbb{R}^k \times \mathbb{R}^l = \mathbb{R}^n; \quad \forall (x, y) \in K \subset \mathbb{R}^k \times \mathbb{R}^l$$

Z_0^- (respectively, Z_0^+) is an infinitely flat germ at the origin, i.e. satisfying the following estimate:

$$\forall x \in \mathbb{R}^n, \begin{cases} \forall \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n \\ |\alpha| = \sum_{i=1}^n \alpha_i \end{cases}$$

There exists $M_\alpha > 0$ such that

$$\|D^\alpha Z_0^-(x)\| \leq M_\alpha \cdot \|x\|^{k_1}, \quad \forall k_1 > 1 \quad \forall x \in \mathbb{R}^k,$$

(resp)

$$\|D^\alpha Z_0^+(y)\| \leq M_\alpha \cdot \|y\|^{k_2}, \quad \forall k_2 > 1 \quad \forall y \in \mathbb{R}^l.$$

(iii) $\forall Y \in U; \exists! Y^i \in U_i (i = 1, 2)$ such that $Y = Y^1 + Y^2$ and $\forall (x, y) \in \mathbb{R}^n = \Omega_1 \cup \Omega_2$, we have

$$\begin{aligned} Y(x, y) &= (Y^1(x), Y^2(y)) \in \Omega_1 \cup \Omega_2; \\ (\exp t Y_0)_* Y(x, y) &= ((\exp t Y_0^-)_* Y^1(x), (\exp t Y_0^+)_* Y^2(y)) \in \Omega_1 \cup \Omega_2, \end{aligned}$$

where

$$\begin{cases} \Omega_1 = \{(x, y) \in \mathbb{R}^k \times \mathbb{R}^l / Y(x, y) = (Y^1(x), 0)\} \\ \Omega_2 = \{(x, y) \in \mathbb{R}^k \times \mathbb{R}^l / Y(x, y) = (0, Y^2(y))\}. \end{cases}$$

(4i) There exists $\delta > 0$ such that $U_i^\delta = \{Y^i \in U_i / \text{supp } Y^i \subseteq \Omega_i^\delta\}$ with $\Omega_i^\delta = \Omega_i + B_\delta$ and $B_\delta \subset \Omega_j, i \neq j$.

(5i) U is locally closed, i.e. for any sequence $(X_m^i)_{m \geq 0}$ in U_i such that $\text{supp } X_m^i \subset \Omega_i$ converges to X^i in Ω_i .

(b) Hyperbolic structure

Definition 2.3 Let E be a Fréchet space of vector fields, $X \in E$ and the X -flow $\psi_t = \exp t X$. E has a tame hyperbolic structure for $(\psi_t)_*$ if and only if $\exists \delta' > 0$ /

- (i) ψ_{t*} is invariant on E ; $\forall t \in \mathbb{R}$,
- (ii) $(\psi_t)_*$ decays on $E_1^{\delta'}$; $\forall t \geq 0$,
- (iii) $(\psi_{-t})_*$ decays on $E_2^{\delta'}$; $\forall t \geq 0$,

where $E_i^{\delta'} = \{Z \in E / \text{supp } Z \subset \Omega_i^{\delta'}\}$ with $\Omega_i^{\delta'} = \Omega_i + B_{\delta'}$, such that $B_{\delta'} \subset \Omega_j, i \neq j$ and $E = E_1 \oplus E_2 = E_1^{\delta'} + E_2^{\delta'}$ [12].

3 Estimations

To show that U has a hyperbolic structure, we need some estimates.

3.1 Estimation of $\exp t Y_0^+$ and $\exp t Y_0^-$

(a) Estimation of $\exp t X_0^+$ and $\exp t X_0^-$

Lemma 3.1 Let $X_0^+ \in U_2$ (respectively, $X_0^- \in U_1$) Then X_0^+ -flow has for estimate, for all $t \geq 0$:

$$\begin{cases} \|y\| e^{b_L t} \leq \|(\exp t X_0^+)(y)\| \leq \|y\| e^{b_R t} \\ \|y\| e^{-b_R t} \leq \|(\exp -t X_0^+)(y)\| \leq \|y\| e^{-b_L t} \end{cases} \quad (\forall y \in \mathbb{R}^l),$$

respectively, for the X_0^- -flow:

$$\begin{cases} \|x\| e^{-a_L t} \leq \|(\exp t X_0^-)(x)\| \leq \|x\| e^{-a_R t} \\ \|x\| e^{a_R t} \leq \|(\exp -t X_0^-)(x)\| \leq \|x\| e^{a_L t} \end{cases} \quad (\forall x \in \mathbb{R}^k).$$

Proof We put $\phi_t^+(y) = (\exp t X_0^+)(y)$, from which

$$\frac{1}{2} \frac{d}{dt} \|\phi_t^+(y)\|^2 = \langle \phi_t^+(y), X_0^+ \circ \phi_t^+(y) \rangle.$$

According to Wintner theorem [10], $\langle y, A^+ y \rangle \leq d \|y\|^2$, $y \in \mathbb{R}^l$, where the constant d is the largest eigenvalue of the symmetric matrix A^+ . We get

$$\inf_{i=1, \dots, l} \lambda_i \cdot \|\phi_t^+(y)\|^2 \leq \frac{1}{2} \frac{d}{dt} \|\phi_t^+(y)\|^2 \leq \sup_{i=1, \dots, l} \lambda_i \cdot \|\phi_t^+(y)\|^2.$$

We finally have

$$\|y\| e^{b_L t} \leq \|\phi_t^+(y)\| \leq \|y\| e^{b_R t}.$$

The same should be applicable to ϕ_{-t}^+ by replacing t by $(-t)$:

$$\|y\| e^{-b_R t} \leq \|\phi_{-t}^+(y)\| \leq \|y\| e^{-b_L t}.$$

On the other hand, applying the same reasoning to X_0^- shows that

$$\begin{cases} \|x\| e^{-a_L t} \leq \|\phi_t^-(x)\| \leq \|x\| e^{-a_R t} \\ \|x\| e^{a_R t} \leq \|\phi_{-t}^-(x)\| \leq \|x\| e^{a_L t} \end{cases}$$

□

(b) Estimation of $\exp t Y_0^+$ and $\exp t Y_0^-$

We put $Y_0^- = X_0^- + Z_0^-$ (respectively, $Y_0^+ = X_0^+ + Z_0^+$) Then the vector form of Y_0^- - flow (respectively, Y_0^+ - flow) will be

$$\psi_t^-(x) = \exp t (X_0^- + Z_0^-)(x) = \phi_t^-(x) + \int_0^t \phi_{t-s}^- \circ Z_0^-(\psi_s^-(x)) ds \quad (1),$$

respectively,

$$\psi_t^+(y) = \exp t (X_0^+ + Z_0^+)(y) = \phi_t^+(y) + \int_0^t \phi_{t-s}^+ \circ Z_0^+(\psi_s^+(y)) ds.$$

Solution of the following dynamic system:

$$\begin{cases} \frac{d}{dt} \psi_t^-(x) = X_0^- \circ \psi_t^-(x) + Z_0^- \circ \psi_t^-(x) \\ \psi_0^-(x) = x, \end{cases}$$

(respectively,

$$\begin{cases} \frac{d}{dt} \psi_t^+(y) = X_0^+ \circ \psi_t^+(y) + Z_0^+ \circ \psi_t^+(y) \\ \psi_0^+(y) = y. \end{cases}$$

Lemma 3.2 Let $Y_0^- \in U_1$ (respectively, $Y_0^+ \in U_2$) Then the vector fields Y_0^- (respectively, Y_0^+) is complete, and the flow ψ_t^- (respectively, ψ_t^+) would satisfy the following estimates:

$$\begin{cases} \|x\| \cdot e^{-a_L t} \leq \|(\exp t Y_0^-)(x)\| \leq \|x\| \cdot e^{-a_R t} \\ \|x\| \cdot e^{a_R t} \leq \|(\exp -t Y_0^-)(x)\| \leq \|x\| \cdot e^{a_L t}, \end{cases} \quad \forall t \geq 0$$

(respectively,

$$\begin{cases} \|y\| \cdot e^{b_L t} \leq \|(\exp t Y_0^+)(y)\| \leq \|y\| \cdot e^{b_R t} \\ \|y\| \cdot e^{-b_R t} \leq \|(\exp -t Y_0^+)(y)\| \leq \|y\| \cdot e^{-b_L t}, \end{cases} \quad \forall t \geq 0).$$

Proof Consider the equation below:

$$\frac{1}{2} \frac{d}{dt} \| \psi_t^-(x) \|^2 = \langle \psi_t^-(x), X_0^-(\psi_t^-(x)) + Z_0^-(\psi_t^-(x)) \rangle.$$

After taking $\zeta = \| \psi_t^-(x) \|$, we have

$$\begin{cases} -a_L \zeta^2 - M_0 \zeta^{k_1+1} \leq \frac{1}{2} \frac{d}{dt} \zeta^2 \leq -a_R \zeta^2 + M_0 \zeta^{k_1+1} \\ \zeta(0) = \| x \| . \end{cases}$$

Then

$$-a_L \zeta^{1-k_1} - M_0 \leq \zeta^{-k_1} \frac{d}{dt} \zeta \leq -a_R \zeta^{1-k_1} + M_0.$$

If we put $z = \zeta^{1-k_1}$, we will have $dz = (1 - k_1) \zeta^{-k_1} d\zeta$ and the system becomes

$$-a_L z - M_0 \leq \frac{1}{1 - k_1} \frac{d}{dt} z \leq -a_R z + M_0$$

which has as solutions

$$b_1 e^{-a_R(1-k_1)t} + \frac{M_0}{a_L} \leq z \leq b_2 e^{-a_L(1-k_1)t} - \frac{M_0}{a_R}$$

such as b_1, b_2 , two functions of x and as $a_L, a_R > 0$, then

$$b_1 e^{-a_R(1-k_1)t} \leq z \leq b_2 e^{-a_L(1-k_1)t}.$$

As $1 - k_1 < 0$, then

$$b_3 e^{-a_L t} \leq \zeta \leq b_4 e^{-a_R t},$$

where b_3, b_4 also two functions of x ,
from which

$$\| x \| \cdot e^{-a_L t} \leq \| \psi_t^-(x) \| \leq \| x \| \cdot e^{-a_R t}.$$

Similarly, we will have

$$\| x \| \cdot e^{a_R t} \leq \| \psi_{-t}^-(x) \| \leq \| x \| \cdot e^{a_L t}.$$

And by a similar reasoning, we come to

$$\begin{cases} \| y \| \cdot e^{b_L t} \leq \| \psi_t^+(y) \| \leq \| y \| \cdot e^{b_R t} \\ \| y \| \cdot e^{-b_R t} \leq \| \psi_{-t}^+(y) \| \leq \| y \| \cdot e^{-b_L t}; \quad \forall t \geq 0 \end{cases}$$

□

(c) The final estimate of the $l' - th$ derivative of Y_0 -flow

(i) Estimation of the first derivative of the Y_0 -flow:

We denote $\eta^-(t, x, v_1) = D\psi_t^-(x) v_1$ and $\eta^+(t, y, v_2) = D\psi_t^+(y) v_2$, where $v_1, v_2 \in \mathbb{R}^k$. The first derivative with respect to x of Y_0^- -flow (respectively, Y_0^+ -flow) is a solution of the dynamic system:

$$\begin{cases} \frac{d}{dt} \eta^-(t, x, v_1) = (D_{\xi_1} X_0^- + D_{\xi_1} Z_0^-) \cdot \eta^-(t, x, v_1) \\ \eta^-(0, x, v_1) = v_1 \end{cases}$$

with $\xi_1 = \psi_t^-(x)$ and, respectively,

$$\begin{cases} \frac{d}{dt} \eta^+(t, y, v_2) = (D_{\xi_2} X_0^+ + D_{\xi_2} Z_0^+) \cdot \eta^+(t, y, v_2) \\ \eta^+(0, y, v_2) = v_2 \end{cases}$$

with $\xi_2 = \psi_t^+(y)$.

Lemma 3.3 *The derivative of Y_0^- -flow (respectively, of Y_0^+ -flow) has the following estimates for all $t \geq 0$:*

$$\begin{cases} e^{-a_L t} \leq \| (D \exp t Y_0^-)(x) \| \leq e^{-a_R t} \\ e^{a_R t} \leq \| (D \exp -t Y_0^-)(x) \| \leq e^{a_L t}, \end{cases}$$

(respectively,

$$\begin{cases} e^{b_L t} \leq \| (D \exp t Y_0^+)(y) \| \leq e^{b_R t} \\ e^{-b_R t} \leq \| (D \exp -t Y_0^+)(y) \| \leq e^{-b_L t} \end{cases}.$$

Proof Let $z_1 = \| \eta^-(t, x, v) \|$, and consider the following equation:

$$\begin{cases} \frac{1}{2} \frac{d}{dt} z_1^2 = \frac{1}{2} \frac{d}{dt} \| \eta^-(t, x, v_1) \|^2 \\ = \langle \eta^-(t, x, v_1), (D_{\xi_1} X_0^- + D_{\xi_1} Z_0^-) \cdot \eta^-(t, x, v_1) \rangle \\ z_1(0) = \| v_1 \| . \end{cases}$$

We will have

$$\begin{cases} z_1^2 (-a_L - M_1 e^{-a_L t k_1}) \leq \frac{1}{2} \frac{d}{dt} z_1^2 \leq z_1^2 \cdot (-a_R + M_1 e^{-a_R t k_1}) \\ z_1(0) = \| v_1 \| . \end{cases}$$

Then

$$(-a_L - M_1 e^{-a_L t k_1}) dt \leq \frac{dz_1}{z_1} \leq (-a_R + M_1 e^{-a_R t k_1}) dt.$$

We will then have

$$-a_L t + \frac{M_1}{a_L k_1} e^{-a_L t k_1} + b_6 \leq \ln z_1 \leq -a_R t - \frac{M_1}{a_R k_1} e^{-a_R t k_1} + b_5,$$

where b_5, b_6 are two functions depending on x .

As $-a_L t + b_6 \leq \ln z_1 \leq -a_R t + b_5$, then $\|v_1\| \cdot e^{-a_L t} \leq z_1 \leq \|v_1\| \cdot e^{-a_R t}$. Taking $\|v_1\| = 1$, we get

$$e^{-a_L t} \leq \| D\psi_t^-(x) \| \leq e^{-a_R t} \quad \forall x \in \mathbb{R}^k, \forall t > 0.$$

We can easily deduce

$$e^{a_R t} \leq \| D\psi_{-t}^-(x) \| \leq e^{a_L t}.$$

And by a similar reasoning, we come to

$$\begin{cases} e^{b_L t} \leq \| D\psi_t^+(y) \| \leq e^{b_R t} \\ e^{-b_R t} \leq \| D\psi_{-t}^+(y) \| \leq e^{-b_L t} \end{cases}.$$

□

(ii) The final estimate of the l' -th derivative of Y_0 -flow

Lemma 3.4 *The l' -th derivative of Y_0^- -flow (resp of Y_0^+ -flow) has the following estimates for all $t \geq 0$, $\forall l' \in \mathbb{N}$:*

$$\begin{cases} \| D^{l'} \psi_t^-(x) \| \leq M_l'' e^{-a_R t} \\ \| D^{l'} \psi_{-t}^-(x) \| \leq M_l'' e^{a_L t} \end{cases} \quad \forall x \in \mathbb{R}^k$$

and

$$\begin{cases} \| D^{l'} \psi_t^+(y) \| \leq M_l'' e^{b_R t} \\ \| D^{l'} \psi_{-t}^+(y) \| \leq M_l'' e^{-b_L t} \end{cases} \quad \forall y \in \mathbb{R}^l.$$

Proof We can generalize the previous result using a recurrence reasoning as follows:

(1) For $j = 0, 1$, the result is verified.

(2) So, let us suppose the statement true until $(j - 1)$ order, i.e.

$$\| D^{l'} \psi_t^-(x) \| \leq M_{l'}' e^{-a_R t}; \quad \forall l' \leq j - 1.$$

(3) Let us show that this last property is true at the order j . Let the j^{th} derivative of Y_0^- -flow

$$\eta_j^-(t, x, v) = D^j \psi_t^-(x) v^j, \quad \forall v \in \mathbb{R}^n$$

solution of the following dynamic system:

$$\begin{cases} \frac{d}{dt} \eta_j^-(t, x, v) = D_{\xi_1} Y_0^- \cdot \eta_j^-(t, x, v) + G_j^-(t, x, v) \\ \eta_j^-(0, x, v, \dots, v) = v \end{cases}$$

with

$$\begin{cases} G_j^-(t, x, v) = \sum_{k'=2}^j D_{\xi_1}^{k'} Z_0^-(\xi_1) \sum_{i_1+i_2+\dots+i_{k'}=j} D_x^{i_1} \psi_t^-(x) v^{i_1} \dots D_x^{i_{k'}} \psi_t^-(x) v^{i_{k'}} \\ \xi_1 = \psi_t^-(x). \end{cases}$$

Using the so-called resolvent transform method discussed, for example, in [4], Y. Domar studied the closed ideals in some Banach algebras [5]), we deduce that

$$\eta_j^-(t, x, v, \dots, v) = D \psi_t^-(x) v + \int_0^t D \psi_{t-s}^-(\psi_s^-(x)) G_j^-(s, x, v) ds.$$

The integral in the preceding expression is well defined at the point $s = 0$ because

$$\lim_{s \rightarrow 0^+} D \psi_{t-s}^-(\psi_s^-(x)) = D \psi_t^-(x)$$

and there are constants $A_l > 0$ such that

$$\lim_{s \rightarrow 0^+} G_j^-(s, x, v) = \sum_{k''=2}^m A_{k''} D_{\xi_1}^{k''} Z_0^-(\xi_1) v^{k''}.$$

Since v is arbitrary, we can choose $\|v\| = 1$, and put

$$I_j = \int_0^t \| D \psi_{t-s}^-(\psi_s^-(x)) \| \cdot \| G_j^-(s, x, v) \| ds.$$

Let $K \subset \mathbb{R}^k$ a compact set, so I_j converges uniformly when t tends to $+\infty$ for all $x \in K$. As

$$\| D^{k'} Z_0^-(x) \| \leq M_{k'}', \quad \forall k' \geq 1, \text{ and } \forall x \in \mathbb{R}^k$$

and using the recurrence hypotheses, we arrive at

$$I_j \leq \sum_{k'=2}^j M_{k'}' \sum_{i_1+i_2+\dots+i_{k'}=j} M_{i_1}' \dots M_{i_{k'}}' \int_0^t e^{-(t-s+sk')a_R} ds \leq M_j' e^{-a_R t}.$$

The integral I_j is then uniformly convergent with respect to x when t tends to $+\infty$. Consequently, there are constants $M_j'' = \sup(1, M_j') > 0$ such that

$$\begin{aligned} \| \eta_j^- \| &\leq \| D \psi_t^-(x) v \| + \int_0^{+\infty} \| D \psi_{t-s}^-(\psi_s^-(x)) \| \cdot \| G_j^-(s, x, v) \| ds \\ &\leq M_j'' e^{-a_R t}. \end{aligned}$$

(4) Conclusion:

$$\|D^{l'}\psi_t^-(x)\| \leq M_l'' e^{-a_R t}, \forall l' \in \mathbb{N}; \quad \forall x \in \mathbb{R}^k.$$

By similar reasoning, we shall have

$$\|D^{l'}\psi_{-t}^-(x)\| \leq M_l'' e^{a_L t}$$

and

$$\begin{cases} \|D^{l'}\psi_t^+(y)\| \leq M_l'' e^{b_R t} \\ \|D^{l'}\psi_{-t}^+(y)\| \leq M_l'' e^{-b_L t} \end{cases} \quad \forall y \in \mathbb{R}^l.$$

□

3.2 Estimation of $(\exp t Y_0^+)_*$ and $(\exp t Y_0^-)_*$

As $\psi_t = \phi_t \circ f_t$, the estimate of ψ_{t_*} is deduced according to the estimates of ϕ_{t_*} and f_{t_*} .

(a) The absorbents N_1^+ and N_2^+

A closed subset N_i^+ is called positive absorbent for the flow ϕ_t , if and only if for any compact K_i in Ω_i , there exists $t_{K_i} > 0$, such that $\phi_t(K_i) \subset N_i$.

Example: Let K_i be a compact of Ω_i , ($i = 1, 2$) Then $\exists \delta > 0$, such that $\delta = \min_{x \in K_i} \|x\|$, we can then have the following absorbents:

$$\begin{cases} N_1^+ = \{(x, y) \in \mathbb{R}^k \times \mathbb{R}^l / Y(x, y) = (Y^1(x), 0) \text{ and } \|x\| \geq \delta\} \subset \Omega_1 \\ N_2^+ = \{(x, y) \in \mathbb{R}^k \times \mathbb{R}^l / Y(x, y) = (0, Y^2(y)) \text{ and } \|y\| \geq \delta\} \subset \Omega_2 \end{cases}$$

(b) Estimation of $(\exp t X_0^+)_*$ and $(\exp t X_0^-)_*$

Lemma 3.5 The diffeomorphism $(\phi_t^-)_*$ (respectively, $(\phi_t^+)_*$) decays tamely on U_1 (respectively, on U_2) of degree 0 and base m_1 (respectively, m_2), in other words:

(i) For every arbitrary positive constant ρ'_1 , there is a constant $m_1 = \left\lceil r \cdot \frac{a_L}{a_R} \right\rceil + \rho'_1 > 0$ and $C_1 > 0$ such that

$$\|(\exp t X_0^-)_* \cdot Y^1\|_r^{\Omega_1} \leq C_1 \cdot e^{-t \cdot \rho'_1 a_R} \cdot \|Y^1\|_{r+m_1}^{\Omega_1} \quad \forall Y^1 \in U_1.$$

(ii) For every arbitrary positive constant ρ'_2 , there is a constant $m_2 = \left\lceil r \cdot \frac{b_R}{b_L} \right\rceil + \rho'_2 > 0$ and $C_2 > 0$ such that

$$\|(\exp -t X_0^+)_* \cdot Y^2\|_r^{\Omega_2} \leq C_2 \cdot e^{-t \rho'_2 b_L} \cdot \|Y^2\|_{r+m_2}^{\Omega_2} \quad \forall Y^2 \in U_2.$$

Proof (i) Consider the following diffeomorphism:

$$\begin{aligned} (\phi_t^-)_* \cdot Y^1 &= (D\phi_t^- \cdot Y^1) \circ (\phi_t^-)^{-1} \\ (\phi_t^-)_* \cdot Y^1 &= \sum_{j=1}^k e^{t X_0^-} \left[u_j \left(e^{-t X_0^-} \cdot x \right) \right] \frac{\partial}{\partial x_j} \end{aligned}$$

with

$$Y^1 = \sum_{j=1}^k u_j(x) \frac{\partial}{\partial x_j} \in U_1.$$

For any integer $r \geq 0$ and any multi-index $\beta = (\beta_1, \dots, \beta_k) \in \mathbb{N}^k$ of module $|\beta| = \beta_1 + \dots + \beta_k$, we obtain for $x \in \mathbb{R}^k$:

$$D_x^\beta (\phi_t^-)_* \cdot Y^1(x) = (D_x^\beta e^{t X_0^-} \cdot u_1(e^{-t X_0^-} \cdot x), \dots, D_x^\beta e^{t X_0^-} \cdot u_k(e^{-t X_0^-} \cdot x)).$$

We will have

$$|(D_x^\beta(\phi_t^-)_* \cdot Y^1(x))| \leq e^{-a_R t} \sup_{j=1 \dots k} \|D_x^\beta u_j(e^{-tX_0^-} \cdot x)\|.$$

Since $\|\phi_{-t}^-(x)\| \leq \|x\|e^{a_L t}$, and posing $z = e^{-tX_0^-} x$, then

$$|(D_x^\beta(\phi_t^-)_* \cdot Y^1(x))| \leq e^{-a_R t} \cdot e^{t|\beta|a_L} \cdot \sup_{j=1 \dots k} \|D_z^\beta \cdot u_j(z)\|,$$

from which we obtain

$$\|(\phi_t^-)_* \cdot Y^1\|_r^{\Omega_1} \leq e^{-t(-ra_L + a_R)} \|Y^1\|_r^{\phi_{-t}^-(\Omega_1)}, \forall r \in \mathbb{N}.$$

Let ρ'_1 be an arbitrary positive constant Then there is $m_1 = \left\lceil r \cdot \frac{a_L}{a_R} \right\rceil + \rho'_1 > 0$, we will have

$$\|(\phi_t^-)_* \cdot Y^1\|_r^{\Omega_1} \leq e^{-t(a_R - ra_L)} \cdot \|Y^1\|_{r+m_1}^{\phi_{-t}^-(\Omega_1)} \cdot \sup_{x \in \mathbb{R}^k} \left(\frac{1}{1 + \|\phi_{-t}^-(x)\|^2} \right)^{\frac{m_1}{2}}.$$

From Lemma 3.1, $\|x\| e^{a_R t} \leq \|\phi_{-t}^-(x)\|$, there is a constant $C_1 > 0$ such that

$$\|(\phi_t^-)_* \cdot Y^1\|_r^{\Omega_1} \leq C_1 \cdot e^{-t \cdot \rho'_1 a_R} \cdot \|Y^1\|_{r+m_1}^{N_1^+}.$$

And as the integral $\int_0^\infty e^{-t \rho'_1 a_R} dt$ is convergent, $(\phi_t^-)_*$ decays tamely on U_1 .

(ii) We have, respectively,

For any vector field $Y^2 \in U_2$, with $Y^2 = \sum_{j=1}^l v_j(y) \frac{\partial}{\partial y_j}$, we consider

$$(\phi_{-t}^+)_* \cdot Y^2 = \sum_{j=1}^l e^{-tX_0^+} \left[v_j(e^{tX_0^+} \cdot y) \right] \frac{\partial}{\partial y_j}.$$

For any integer $r \geq 0$ and any multi-index $\beta' = (\beta'_1, \dots, \beta'_l) \in \mathbb{N}^l$ of module $|\beta'| = \beta'_1 + \dots + \beta'_l$, we obtain

$$D_y^{\beta'}(\phi_{-t}^+)_* \cdot Y^1(y) = (D_y^{\beta'} e^{-tX_0^+} \cdot v_1(e^{tX_0^+} \cdot y), \dots, D_y^{\beta'} e^{-tX_0^+} \cdot v_l(e^{tX_0^+} \cdot y)).$$

We will have

$$|(D_y^{\beta'}(\phi_{-t}^+)_* \cdot Y^2(y))| \leq e^{-b_L t} \sup_{j=1 \dots l} \|D_y^{\beta'} v_j(e^{tX_0^+} \cdot y)\|.$$

Since $\|\phi_t^+(y)\| \leq \|y\|e^{b_R t}$, and posing $z' = e^{tX_0^+} \cdot y$,

$$|(D_y^{\beta'}(\phi_{-t}^+)_* \cdot Y^2(y))| \leq e^{-b_L t} \cdot e^{t|\beta'|b_R} \cdot \sup_{j=1 \dots l} \|D_{z'}^{\beta'} \cdot v_j(z')\|,$$

from which we obtain

$$\|(\phi_{-t}^+)_* \cdot Y^2\|_r^{\Omega_2} \leq e^{-t(-rb_R + b_L)} \|Y^2\|_r^{\phi_t^+(\Omega_2)}.$$

Let ρ'_2 be an arbitrary positive constant Then there is $m_2 = \left\lceil r \cdot \frac{b_R}{b_L} \right\rceil + \rho'_2 > 0$, we will have

$$\|(\phi_{-t}^+)_* \cdot Y^2\|_r^{\Omega_2} \leq e^{-t(b_L - rb_R)} \cdot \|Y^2\|_{r+m_2}^{\phi_t^+(\Omega_2)} \cdot \sup_{y \in \mathbb{R}^l} \left(\frac{1}{1 + \|\phi_t^+(y)\|^2} \right)^{\frac{m_2}{2}}.$$

From Lemma 3.1, $\|y\| e^{b_L t} \leq \|\phi_t^+(y)\|$, there is a constant $C_2 > 0$ such that

$$\|(\phi_{-t}^+)_* \cdot Y^2\|_r^{\Omega_2} \leq C_2 \cdot e^{-t \rho'_2 b_L} \cdot \|Y^2\|_{r+m_2}^{N_2^+}.$$

And as the integral $\int_0^\infty e^{-t(\rho'_2 b_L)} dt$ is convergent, $(\phi_{-t}^+)_*$ operates smoothly on U_2 . □

(b) Wave operators

Let

$$\begin{aligned} \phi_t^- &= \exp t X_0^- \quad \text{and} \quad \psi_t^- = \exp t Y_0^- \\ (\text{resp } \phi_t^+ &= \exp t X_0^+ \quad \text{and} \quad \psi_t^+ = \exp t Y_0^+) \end{aligned}$$

and the diffeomorphism $f_t^-(x) = (\phi_{-t}^- \circ \psi_t^-)(x) \in G$. According to expression (1) (in Sect. 3.1) of $\psi_t^-(x)$, the diffeomorphism f_t^- becomes

$$f_t^-(x) = x + \int_0^t \phi_{-s}^-(Z_0^- \circ \psi_s^-)(x) ds$$

(respectively,

$$f_{-t}^+(y) = y - \int_0^t \phi_s^+(Z_0^+ \circ \psi_{-s}^+)(y) ds).$$

The wave operator is defined by

$$f^- = \lim_{t \rightarrow +\infty} \phi_{-t}^- \circ \psi_t^- \quad (\text{resp } f^+ = \lim_{t \rightarrow +\infty} \phi_{-t}^+ \circ \psi_t^+).$$

Lemma 3.6 (1) f_t^- , $D^r f_t^-$ (respectively, f_{-t}^+ , $D^r f_{-t}^+$), f_{-t}^- and $D^r f_{-t}^-$ (respectively, f_t^+ and $D^r f_t^+$) are infinite globally uniformly bounded $\forall t > 0, \forall r \in \mathbb{N}$.

(2) For all compact $K_1 \subseteq \Omega_1$ (respectively, $K_2 \subseteq \Omega_2$), there is $\varepsilon > 0$ such that $\forall t \in \mathbb{R}^+$:

$$\|f_{-t}^- - id\|_r^{K_1} \leq \varepsilon \quad \text{and} \quad \|f_t^+ - id\|_r^{K_2} \leq \varepsilon.$$

Proof (1) Let us show that f_t^- and $D^r f_t^-$ (respectively, f_{-t}^+ and $D^r f_{-t}^+$) are infinite globally uniformly bounded $\forall t > 0, \forall r \in \mathbb{N}$.

Let

$$f_t^-(x) = x + \int_0^t \phi_{-s}^-(Z_0^- \circ \psi_s^-)(x) ds,$$

then for all $r \in \mathbb{N}; \forall v \in \mathbb{R}^n$ such as $\|v\| = 1$, we have

$$D_x^r (f_t^- - id)(x) v^r = \int_0^t e^{-A^- s} D_x^r (Z_0^- \circ \psi_s^-)(x) \cdot v^r ds.$$

As

$$D_x^r (Z_0^- \circ \psi_s^-)(x) \cdot v^r = \sum_{\gamma=1}^r D_\xi^\gamma Z_0^-(\xi) \sum_{l_1+\dots+l_\gamma=r} D^{l_1} \psi_s^-(x) \cdot v^{l_1} \dots D^{l_\gamma} \psi_s^-(x) \cdot v^{l_\gamma},$$

where $\xi = \psi_s^-(x)$.

According to Lemma 3.2, there exist constants $M_i > 0$ such that $\|\psi_s^-(x)\|^{K_1} \leq M_1 e^{-sa_R}$ with K_1 a compact included in Ω_1 and $\forall k_1 > 1$; we will then have

$$\|D_\xi^\gamma Z_0^-(\xi)\| \leq M_\gamma \|\xi\|^{k_1} = M_\gamma \|\psi_s^-(x)\|^{k_1},$$

from which,

$$\begin{aligned} \|D_x^r (f_t^- - id)(x)\|^{K_1} &\leq M_2 \int_0^t e^{sa_L} \|\psi_s^-(x)\|^{k_1} ds \\ &\leq M_3 \int_0^t e^{sa_L} \cdot e^{-sa_R k_1} ds \\ &\leq M_3 \int_0^t e^{-s[k_1 a_R - a_L]} ds. \end{aligned}$$

By hypothesis (I) in Sect. 2.3, $\exists \rho_1 > 1/k_1 a_R - a_L \geq \rho_1 > 1$, then

$$\|D_x^r (f_t^- - id) (x)\|^{K_1} \leq M_3 \int_0^t e^{-s\rho_1} ds \leq \frac{M_3}{\rho_1},$$

from which,

$$\|(f_t^- - id) (x)\|_r^{K_1} \leq \frac{M_3}{\rho_1}; \quad \forall t > 0.$$

It follows that f_t^- and Df_t^- are infinite-uniformly bounded $\forall t > 0; \forall r \in \mathbb{N}$.

And by a similar reasoning, with K_2 a compact included in Ω_2 and with the hypothesis (II) in Sect. 2.3 $\exists \rho_2 > 1/k_2 b_L - b_R \geq \rho_2 > 1$, we can demonstrate that

$$\|(f_{-t}^+ - id) (y)\|_r^{K_2} \leq \frac{M_4}{\rho_2},$$

from which f_{-t}^+ is infinite-globally bounded $\forall t \geq 0$, i.e.

$$\|D_y^r (f_{-t}^+ - id) (y)\|^{K_2} \leq M_4 \int_0^t e^{-s(k_2 b_L - b_R)} ds \leq M_4 \int_0^t e^{-s\rho_2} ds = M_4 \frac{1 - e^{-t\rho_2}}{\rho_2} \leq \frac{M_4}{\rho_2},$$

where $k_2 b_L - b_R \geq \rho_2 > 1; k_2 > 1$.

Let us show that f_{-t}^- and $D^r f_{-t}^-$ (respectively, f_t^+ and $D^r f_t^+$) are infinite globally uniformly bounded $\forall t > 0, \forall l' \in \mathbb{N}$.

Let

$$f_{-t}^- (x) = x - \int_0^t \phi_s^- (Z_0^- \circ \psi_{-s}^-) (x) ds,$$

then, for all $r \in \mathbb{N}; \forall v \in \mathbb{R}^n$ such as $\|v\| = 1$, we have

$$D_x^r (f_{-t}^- - id) (x) v^r = - \int_0^t e^{A^-s} D_x^r (Z_0^- \circ \psi_{-s}^-) (x) \cdot v^r ds.$$

As

$$D_x^r (Z_0^- \circ \psi_{-s}^-) (x) \cdot v^r = \sum_{\gamma=1}^r D_{\xi'}^\gamma Z_0^- (\xi') \sum_{l_1+\dots+l_\gamma=r} D^{l_1} \psi_{-s}^- (x) \cdot v^{l_1} \dots D^{l_\gamma} \psi_{-s}^- (x) \cdot v^{l_\gamma}, \quad \text{where } \xi' = \psi_{-s}^- (x).$$

According to Lemma 3.2, $\|\psi_{-s}^- (x)\|^{K_1} \leq M e^{sa_L}$, we deduce that there is a constant $M' > 0$ such that

$$\begin{aligned} \|D_x^r (f_{-t}^- - id) (x)\|^{K_1} &\leq M' \int_0^t e^{-sa_R} \|Z_0^-\|_{r+m'_1}^{\psi_{-s}^-(K_1)} \sup_{x \in K_1} \left(\frac{1}{1 + \|\psi_{-s}^- (x)\|^2} \right)^{m'_1/2} e^{rsa_L} ds \\ &\leq M' \int_0^t e^{-s[(1+m'_1)a_R - (k_1+r)a_L]} ds. \end{aligned}$$

We choose

$$m'_1 = (r + k_1) \cdot \left[\frac{a_L}{a_R} + 1 \right] + \rho > 0$$

with ρ any positive constant.

$$\|D_x^r (f_{-t}^- - id) (x)\|^{K_1} \leq M' \int_0^t e^{-s(\rho+1)a_R} ds = M' \frac{1 - e^{-t(\rho+1)a_R}}{(\rho+1)a_R}$$

As K_1 is arbitrary, we can find $\varepsilon_1 > 0$ and $\rho > 0$ such that

$$\|D_x^r (f_{-t}^- - id) (x)\|^{K_1} \leq \varepsilon_1; \quad \forall t > 0. \quad (3.1)$$

It follows that f_{-t}^- and $D^r f_{-t}^-$ are infinite-uniformly bounded $\forall t > 0; \forall r \in \mathbb{N}$.
And by a similar reasoning, there is a constant $M'' > 0$ such that

$$\|D_y^r (f_t^+ - id)(y)\|^{K_2} \leq M'' \int_0^t e^{-s((m'_2+1)b_L - (r+k_2)b_R)} ds.$$

By choosing

$$m'_2 = (r + k_2) \cdot \left[\frac{b_R}{b_L} + 1 \right] + \rho' > 0$$

with ρ' an arbitrary positive constant, we deduce that

$$\|D_y^r (f_{-t}^+ - id)(y)\|^{K_2} \leq M'' \int_0^t e^{-s(\rho'+1)b_R} ds.$$

As K_2 is arbitrary, we can find $\varepsilon_2 > 0$ and $\rho' > 0$ such that

$$\|D_y^r (f_t^+ - id)(y)\|^{K_2} \leq \varepsilon_2; \quad (3.2)$$

from which f_t^+ and $D^r f_t^+$ are infinite-globally bounded $\forall t \geq 0$. $\square$

2) From Eqs. (3.1) and (3.2), $\forall t > 0; \exists \varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$:

$$\|f_{-t}^- - id\|_r^{K_1} \leq \varepsilon; \quad \text{and} \quad \|f_t^+ - id\|_r^{K_2} \leq \varepsilon.$$

c) Estimation of $(\exp t Y_0^-)_*$ and $(\exp t Y_0^-)_*$

Let $B_\varepsilon = B(0, \varepsilon)$ be the ball centered at the origin with radius a certain $\varepsilon > 0$.

Lemma 3.7 For all $Y^i \in U_i$ and $\forall t \geq 0$, there exists C', C'' positive constants such that

$$\|(\exp t Y_0^-)_* \cdot Y^1\|_r^{\Omega_1} \leq C' \cdot e^{-t\rho'_1 \cdot a_R} \cdot \|Y^1\|_{r+m_1}^{\Omega_1^\varepsilon}$$

and, respectively,

$$\|(\exp -t Y_0^+)_* \cdot Y^2\|_r^{\Omega_2} \leq C'' \cdot e^{-t\rho'_2 \cdot b_L} \cdot \|Y^2\|_{r+m_2}^{\Omega_2^\varepsilon}.$$

Proof Let $Y^1 \in U_1$ and $\forall t \geq 0$, we have

$$\begin{aligned} (\exp t Y_0^-)_* \cdot Y^1 &= (\psi_t^-)_* Y^1 = (\phi_t^- \circ f_t^-)_* \cdot Y^1 \\ &= (D(\phi_t^- \circ f_t^-) \cdot Y^1) \circ (\phi_t^- \circ f_t^-)^{-1} \\ &= D\phi_t^-(f_t^- \circ (\phi_t^-)^{-1} \circ \phi_{-t}^-) \cdot Y^1((f_t^-)^{-1} \circ \phi_{-t}^-) \cdot Df_t^-(f_t^-)^{-1} \circ \phi_{-t}^- \\ &= D\phi_t^-(\phi_{-t}^-) \cdot Y^1(\psi_{-t}^-) \cdot Df_t^-(\psi_{-t}^-) \\ &= D\phi_t^-(\phi_{-t}^-) \cdot Y^1(\phi_{-t}^- \circ f_{-t}^-) \cdot Df_t^-(\phi_{-t}^- \circ f_{-t}^-). \end{aligned}$$

Let K_1 be a compact of Ω_1 , we then deduce the following estimate:

$$\|(\exp t Y_0^-)_* \cdot Y^1\|^{K_1} \leq \|D\phi_t^-(\phi_{-t}^-) \cdot Y^1(\phi_{-t}^- \circ f_{-t}^-)\|^{K_1} \cdot \|Df_t^-(\phi_{-t}^- \circ f_{-t}^-)\|^{K_1}.$$

According to Lemma 3.6, Df_t^- and f_{-t}^- are uniformly bounded, then there is a constant $C > 0$ such that

$$\|(\exp t Y_0^-)_* \cdot Y^1\|^{K_1} \leq C \|(\phi_t^-)_* \cdot Y^1\|_r^{f_{-t}^-(K_1)} \quad \forall r \geq 0 \quad (*).$$

As

$$\left| \|f_{-t}^-\|^{K_1} - \|id\|^{K_1} \right| \leq \|f_{-t}^- - id\|^{K_1} \leq \varepsilon,$$

then

$$\|id\|^{K_1} - \varepsilon \leq \|f_{-t}^-\|^{K_1} \leq \|id\|^{K_1} + \varepsilon;$$

hence,

$$K_1 - B_\varepsilon \subseteq f_{-t}^-(K_1) \subseteq K_1 + B_\varepsilon \quad (**).$$

We will then have, from equations (*) and (**).

$$\|(\exp t Y_0^-)_* \cdot Y^1\|_r^{K_1} \leq C' e^{-a_R t \rho'_1} \|Y^1\|_{r+m_1}^{\Phi_{-t}^-(f_{-t}^-(K_1))} \leq C' e^{-a_R t \rho'_1} \|Y^1\|_{r+m_1}^{\Phi_{-t}^-(K_1+B_\varepsilon)},$$

from which,

$$\Phi_{-t}^-(f_{-t}^-(K_1)) \subseteq \Omega_1^\varepsilon.$$

We will then have

$$\|(\exp t Y_0^-)_* \cdot Y^1\|_r^{K_1} \leq C' e^{-a_R t \rho'_1} \|Y^1\|_{r+m_1}^{\Omega_1^\varepsilon}.$$

As K_1 is arbitrary on Ω_1 , so

$$\|(\exp t Y_0^-)_* \cdot Y^1\|_r^{\Omega_1} \leq C' e^{-a_R t \rho'_1} \|Y^1\|_{r+m_1}^{\Omega_1^\varepsilon}.$$

We proceed in the same way to demonstrate

$$\|(\exp -t Y_0^+)_* \cdot Y^2\|_r^{\Omega_2} \leq C'' \cdot e^{-t \rho'_2 \cdot b_L} \cdot \|Y^2\|_{r+m_2}^{\Omega_2^\varepsilon}$$

□

4 Admissible algebra with hyperbolic structure

4.1 $(\exp t X_0)_*$ decays tamely

Lemma 4.1 (i) $(\phi_t)_*$ is invariant on U ; $\forall t \in \mathbb{R}$.

(ii) $(\phi_t)_*$ decays tamely on U_1 and $(\phi_{-t})_*$ decays tamely on U_2 for all $t \geq 0$. That is to say, $\forall t \geq 0$, we have

$$\|(\exp t X_0)_* \cdot Y\|_r^{\Omega_1} \leq e^{-\omega t} \cdot \|Y\|_{r+m_1}^{\Omega_1} \quad \forall \omega \in \mathbb{R}^{*+}, \forall Y \in U$$

and

$$\|(\exp -t X_0)_* \cdot Y\|_r^{\Omega_2} \leq e^{-\omega' t} \cdot \|Y\|_{r+m_2}^{\Omega_2} \quad \forall \omega' \in \mathbb{R}^{*+}, \forall Y \in U.$$

Proof Let $Y \in U = U_1 \oplus U_2$; then $\exists! Y^i \in U_i (i = 1, 2)$ such that $Y = Y^1 + Y^2$; and let $t \in \mathbb{R}, \forall (x, y) \in \mathbb{R}^n = \Omega_1 \cup \Omega_2$, we have

$$(\phi_t)_* Y(x, y) = ((\phi_t^-)_* Y^1(x), (\phi_t^+)_* Y^2(y)).$$

(i) We will show that $(\phi_t)_* Y \in U, \forall t \in \mathbb{R}$.

- We project $(\phi_t)_* Y$ on U_1 , So

$$(\phi_t)_* Y(x, y) = ((\phi_t^-)_* Y^1(x), 0),$$

from which $(\phi_t)_* Y \in U$.

- Similarly, we project $(\phi_t)_* Y$ on U_2 , So

$$(\phi_t)_* Y(x, y) = (0, (\phi_t^+)_* Y^2(x)),$$

from which $(\phi_t)_* Y \in U$.

We deduce that $(\phi_t)_*$ is invariant on $U, \forall t \in \mathbb{R}$.

(ii) Let us first show that $(\phi_t)_*$ decays tamely on U_1 ; $\forall t > 0$:

$$\begin{aligned} \|(\phi_t)_* \cdot Y\|_r^{\Omega_1} &= \|(\phi_t^-)_* \cdot Y^1\|_r^{\Omega_1} \quad \text{and for all } \rho'_1 > 0, \text{ we have} \\ \|(\phi_t)_* \cdot Y\|_r^{\Omega_1} &\leq C_1 \cdot e^{-t\rho'_1 a_R} \cdot \|Y^1\|_{r+m_1}^{\Omega_1} \quad (\text{see Lemma 3.5}) \\ &\leq C_1 \cdot e^{-\omega t} \|Y\|_{r+m_1}^{\Omega_1} \quad \text{with } \omega = \rho'_1 \cdot a_R. \end{aligned}$$

Then we show that $(\phi_{-t})_*$ decays tamely on U_2 ; $\forall t > 0$:

$$\begin{aligned} \|(\phi_{-t})_* \cdot Y\|_r^{\Omega_2} &= \|(\phi_{-t}^+)_* \cdot Y^2\|_r^{\Omega_2} \quad \text{and for all } \rho'_2 > 0, \text{ we have} \\ \|(\phi_{-t})_* \cdot Y\|_r^{\Omega_2} &\leq C_2 \cdot e^{-t\rho'_2 b_L} \cdot \|Y^2\|_{r+m_2}^{\Omega_2} \quad (\text{see Lemma 3.5}) \\ &\leq C_2 \cdot e^{-\omega' t} \|Y\|_{r+m_2}^{\Omega_2} \quad \text{with } \omega' = \rho'_2 \cdot b_L. \end{aligned}$$

□

4.2 $(\text{expt} Y_0)_*$ decays tamely

Lemma 4.2 (i) $(\psi_t)_*$ is invariant on U , $\forall t \in \mathbb{R}$.

(ii) The diffeomorphism $(\psi_t)_*$ decays tamely on U_1^ε and $(\psi_{-t})_*$ decays tamely on U_2^ε , that is to say, $\forall t \geq 0$, there exist $\omega \in \mathbb{R}^{*+}$ and $\omega' \in \mathbb{R}^{*+}$ such that

$$\|(\psi_t)_* \cdot Y\|_r^{\Omega_1} \leq e^{-\omega t} \cdot \|Y\|_{r+m_1}^{\Omega_1^\varepsilon},$$

respectively,

$$\|(\psi_{-t})_* \cdot Y\|_r^{\Omega_2} \leq e^{-\omega' t} \cdot \|Y\|_{r+m_2}^{\Omega_2^\varepsilon}.$$

Proof Let $Y \in U = U_1 \oplus U_2$. Then $\exists! Y^i \in U_i$ ($i = 1, 2$) such that $Y = Y^1 + Y^2$.

Let $t \in \mathbb{R}$, $\forall (x, y) \in \mathbb{R}^n = \Omega_1 \cup \Omega_2$

$$(\psi_t)_* Y(x, y) = ((\psi_t^-)_* Y^1(x), (\psi_t^+)_* Y^2(y)).$$

(i) We will prove that $(\psi_t)_* Y \in U$, $\forall t \geq 0$.

- We project on $U_1 \Rightarrow Y(x, y) = (Y^1(x), 0)$.

Therefore,

$$(\psi_t)_* Y(x, y) = ((\psi_t^-)_* Y^1(x), 0),$$

from which $(\psi_t)_* Y \in U$

- Similarly, if we project on $U_2 \Rightarrow Y(x, y) = (0, Y^2(x))$,

then

$$(\psi_t)_* Y(x, y) = (0, (\psi_t^+)_* Y^2(x)),$$

from which $(\psi_t)_* Y \in U$. We deduce that $(\psi_t)_*$ is invariant on U , $\forall t \in \mathbb{R}$.

(ii) We note that

$$\begin{aligned} \|(\psi_t)_* \cdot Y\|_r^{\Omega_1} &= \|(\psi_t^-)_* \cdot Y^1\|_r^{\Omega_1} \\ &\leq C'_1 \cdot e^{-t\rho'_1 a_R} \cdot \|Y^1\|_{r+m_1}^{\Omega_1^\varepsilon} \quad (\text{cf Lemma 3.7}) \\ &\leq C'_1 \cdot e^{-\omega t} \|Y\|_{r+m_1}^{\Omega_1^\varepsilon} \quad \text{with } \omega = \rho'_1 \cdot a_R. \end{aligned}$$

This means that $(\psi_t)_*$ decays tamely on U_1^ε .

- Similarly,

$$\begin{aligned} \|(\psi_{-t})_* \cdot Y\|_r^{\Omega_2} &= \|(\psi_{-t}^+)_* \cdot Y^2\|_r^{\Omega_2} \\ &\leq C''_2 \cdot e^{-t\rho'_2 b_L} \cdot \|Y^2\|_{r+m_2}^{\Omega_2^\varepsilon} \quad (\text{cf Lemma 3.7}) \\ &\leq C''_2 \cdot e^{-\omega' t} \|Y\|_{r+m_2}^{\Omega_2^\varepsilon} \quad \text{with } \omega' = \rho'_2 \cdot b_L. \end{aligned}$$

We deduce that $(\psi_{-t})_*$ decays tamely on U_2^ε .

□

Theorem 4.3 *The admissible algebra U admits a hyperbolic structure for the flow ψ_{t*} .*

Proof According to Lemma 4.2, we deduce that U has a hyperbolic structure for ψ_t . $\square$

Part II: applications

We will show, as an application, that the ideal of finite codimension extends over all the hyperbolic-type admissible algebra, and this using the following lemma:

Fundamental lemma: ([8]) Let V be a finite codimension subspace of a $\mathbb{R}$ -vector space E and an endomorphism ψ on E such that

1. $\psi(V) \subset V$;
2. $\psi + b.id$ is surjective on V for all $b \in \mathbb{R}$;
3. for all numbers $b, c \in \mathbb{R}$ such as $b^2 - 4c < 0$, the operator $\psi^2 + b\psi + c.id$ is surjective on V .

Then $V = E$.

5 Surjectivity of the operator $ad_{Y_0} + bI$

In this section, we study the surjectivity of some linear operators. Note by

$\varphi = (ad_{-Y_0^-}, ad_{Y_0^+})$, where $ad_{-Y_0^-} = \varphi_1$ and $ad_{Y_0^+} = \varphi_2$ two adjoint endomorphisms and id the identity application.

Lemma 5.1 *For all $b \in \mathbb{R}$, the operator $\varphi_1 + b.id_{\mathbb{R}^k}$ (respectively, $\varphi_2 + b.id_{\mathbb{R}^l}$) is surjective on U_1^ε (respectively, on U_2^ε).*

Proof Let $Y^i \in U_i$ ($i = 1, 2$) such that

$$\begin{cases} Y^1 = \sum_{i=1}^k f_i(x) \frac{\partial}{\partial x_i} \in U_1 \\ Y^2 = \sum_{j=1}^l g_j(y) \frac{\partial}{\partial y_j} \in U_2. \end{cases}$$

By putting

$$\begin{cases} W_1 = \int_0^{+\infty} R(t)(\psi_t^-)_* Y^1 dt \in U_1^\varepsilon \\ W_2 = \int_0^{+\infty} R(t)(\psi_t^+)_* Y^2 dt \in U_2^\varepsilon \end{cases}$$

- **First stage:** Let us prove that W_i is a solution of the equation:

$$\begin{aligned} (\varphi_i + b.id)(W_i) &= Y^i, (i = 1, 2), \forall b \in \mathbb{R} \\ ad_{-Y_0^-}(W_1) &= \lim_{s \rightarrow 0} \frac{d}{ds} (\psi_s^-)_* \left(\int_0^{+\infty} R(t)(\psi_t^-)_* Y^1 dt \right) \\ &= \lim_{s \rightarrow 0} \int_0^{+\infty} R(t) \frac{d}{ds} (\psi_{(t+s)}^-)_* Y^1 dt \\ &= \int_0^{+\infty} R(\tau) \frac{d}{d\tau} (\psi_\tau^-)_* Y^1 d\tau, \quad \text{where } \tau = t + s \\ &= R(\tau)(\psi_\tau^-)_* \cdot Y^1|_0^{+\infty} - \int_0^{+\infty} (\psi_\tau^-)_* Y^1 \frac{d}{d\tau} R(\tau) d\tau \\ ad_{-Y_0^-}(W_1) + bW_1 &= R(\tau)(\psi_\tau^-)_* Y^1|_0^{+\infty} + \int_0^{+\infty} (bR(\tau) - R'(\tau))(\psi_\tau^-)_* Y^1 d\tau. \end{aligned}$$

We put

$$\begin{cases} R'(\tau) - bR(\tau) = 0 \\ R(0) = -1 \end{cases} \quad \text{then} \quad R(\tau) = -e^{b\tau}$$

and

$$\begin{aligned} \lim_{\tau \rightarrow +\infty} \| R(\tau)(\psi_{\tau}^{-})_* \cdot Y^1 \|_r^{\Omega_1} &\leq C' \lim_{\tau \rightarrow +\infty} e^{b\tau} \cdot e^{-\tau\rho'_1 a_R} \cdot \| Y^1 \|_{r+m_1}^{\Omega_1^{\varepsilon}} \\ &\leq C' \lim_{\tau \rightarrow +\infty} e^{-\tau[-b+\rho'_1 a_R]} \cdot \| Y^1 \|_{r+m_1}^{\Omega_1^{\varepsilon}}. \end{aligned}$$

As ρ'_1 is arbitrary, then

$$\forall b \in \mathbb{R}, \exists \rho'_{1,b} > 0 / -b + \rho'_{1,b} \cdot a_R > 0,$$

then

$$\lim_{\tau \rightarrow +\infty} \| R(\tau)(\psi_{\tau}^{-})_* \cdot Y^1 \|_r = 0.$$

It results

$$ad_{-Y_0^-}(W_1) + bW_1 = R(\tau)(\psi_{\tau}^{-})_* Y^1|_0^{+\infty} = Y^1.$$

Similarly,

$$\begin{aligned} ad_{Y_0^+}(W_2) &= \lim_{s \rightarrow 0} \frac{d}{ds} (\psi_{-s}^+)_* \left(\int_0^{+\infty} R(t)(\psi_{-t}^+)_* Y^2 dt \right) \\ &= \lim_{s \rightarrow 0} \int_0^{+\infty} R(t) \frac{d}{ds} ((\psi_{-s-t}^+)_* Y^2) dt \\ &= \int_0^{+\infty} R(\tau) \frac{d}{d\tau} (\psi_{-\tau}^+)_* Y^2 d\tau, \quad \text{where } \tau = s + t \\ &= R(\tau)(\psi_{-\tau}^+)_* \cdot Y^2|_0^{+\infty} - \int_0^{+\infty} ((\psi_{-\tau}^+)_* \cdot Y^2) \frac{d}{d\tau} R(\tau) d\tau. \end{aligned}$$

By adding bW_2 in both members, we will have

$$ad_{Y_0^+}(W_2) + bW_2 = R(\tau)(\psi_{-\tau}^+)_* Y^2|_0^{+\infty} + \int_0^{+\infty} (-R'(\tau) + bR(\tau))(\psi_{-\tau}^+)_* \cdot Y^2 d\tau.$$

We put

$$\begin{cases} -R'(\tau) + bR(\tau) = 0 \\ R(0) = -1. \end{cases} \quad \text{then } R(\tau) = -e^{b\tau}$$

We have

$$\begin{aligned} \lim_{\tau \rightarrow +\infty} \| R(\tau)(\psi_{-\tau}^+)_* \cdot Y^2 \|_r^{\Omega_2} &\leq C'' \lim_{\tau \rightarrow +\infty} e^{b\tau} \cdot e^{-\tau\rho'_2 b_L} \cdot \| Y^2 \|_{r+m_2}^{\Omega_2^{\varepsilon}} \\ &\leq C'' \lim_{\tau \rightarrow +\infty} e^{-\tau[-b+\rho'_2 b_L]} \cdot \| Y^2 \|_{r+m_2}^{\Omega_2^{\varepsilon}}. \end{aligned}$$

As ρ'_2 is arbitrary, then

$$\forall b \in \mathbb{R}, \exists \rho'_{2,b} > 0 / -b + \rho'_{2,b} \cdot b_L > 0,$$

from which

$$\lim_{\tau \rightarrow +\infty} \| R(\tau)(\psi_{-\tau}^+)_* \cdot Y^2 \|_r = 0.$$

Therefore,

$$ad_{Y_0^+}(W_2) + bW_2 = R(\tau)(\psi_{-\tau}^+)_* Y^2|_0^{+\infty} = Y^2.$$

- Second stage: Let us prove that W_1 is of class C^∞ on any compact. Let K_1 be a compact of Ω_1 , and as

$$W_1 = \int_0^{+\infty} R(t)(\psi_t^-)_* Y^1 dt,$$

according to Lemma 3.7, we will have

$$\| R(t)(\exp t Y_0^-)_* \cdot Y^1 \|_{r}^{K_1} \leq C' e^{-t(-b+\rho'_1 \cdot a_R)} \| Y^1 \|_{r+m_1}^{\Omega_1^\varepsilon}.$$

As ρ'_1 is arbitrary, then

$$\forall b \in \mathbb{R}, \exists \rho'_{1,b} > 0 / -b + \rho'_{1,b} \cdot a_R > 0, \\ \int_0^{+\infty} e^{-t(-b+\rho'_{1,b} \cdot a_R)} dt \quad \text{converges } \forall b \in \mathbb{R}.$$

It follows that W_1 converges uniformly, $\forall x \in \Omega_1$, where ultimately W_1 is of class C^∞ on any compact of Ω_1 . The same reasoning is valid to prove that W_2 is of class C^∞ on any compact of Ω_2 . $\square$

Lemma 5.2 For all $b \in \mathbb{R}$, $\varphi + b.id$ is surjective on U .

Proof Let $Y \in U$, seeking a $W \in U$ such that $\forall (x, y) \in \mathbb{R}^n$; $Y(x, y) = (\varphi + b.id)W(x, y)$, $\forall b \in \mathbb{R}$. As $Y \in U = U_1 \oplus U_2$, $\exists ! Y^i \in U_i$ such that

$$Y(x, y) = (Y^1(x), Y^2(y)) = (\varphi_1 W_1(x) + b W_1(x), \varphi_2 W_2(y) + b W_2(y)).$$

We project on U_i :

- On U_1 , we have $Y(x, y) = (Y^1(x), 0) = (\varphi_1 W_1(x) + b W_1(x), 0)$, and according to Lemma 5.1 $\exists \varepsilon > 0$ and W_1 , such that $W_1(x) = -\int_0^{+\infty} e^{bt} (\psi_t^-)_* Y^1(x) dt \in \Omega_1^\varepsilon$.
- On U_2 , we have $Y(x, y) = (0, Y^2(y)) = (0, \varphi_2 W_2(y) + b W_2(y))$, and according to Lemma 5.1 $\exists \varepsilon > 0$ and W_2 , such that $W_2(y) = -\int_0^{+\infty} e^{bt} (\psi_{-t}^+)_* Y^2(y) dt \in \Omega_2^\varepsilon$.

It follows that

$$W_i \in U_i^\varepsilon$$

.

We put $W(x, y) = (W_1(x), W_2(y)) \in \mathbb{R}^k \times \mathbb{R}^l = \mathbb{R}^n$. As $W = W_1 + W_2 \in U_1^\varepsilon + U_2^\varepsilon = U$, it follows that $\varphi + b.id$ is surjective on U , for all $b \in \mathbb{R}$. $\square$

6 Surjectivity of the operator $(ad_{Y_0})^2 + b.ad_{Y_0} + cI$

Lemma 6.1 For all numbers $b, c \in \mathbb{R}$ such that $b^2 - 4c < 0$, the operator $\varphi_1^2 + b\varphi_1 + c.id_{\mathbb{R}^k}$, (respectively, $\varphi_2^2 + b\varphi_2 + c.id_{\mathbb{R}^l}$) is surjective on U_1^ε (respectively, on U_2^ε).

Proof Let $Y^i \in U_i$ such that

$$Y^1 = \sum_{i=1}^k f_i(x) \frac{\partial}{\partial x_i} \in U_1 \\ Y^2 = \sum_{i=1}^l g_i(y) \frac{\partial}{\partial y_i} \in U_2,$$

such that $(\varphi_1^2 + b\varphi_1 + cid_{\mathbb{R}^k})W_1 = [-Y_0^-, [-Y_0^-, W_1]] + b[-Y_0^-, W_1] + cW_1$ and $(\varphi_2^2 + b\varphi_2 + cid_{\mathbb{R}^l})W_2 = [-Y_0^+, [-Y_0^+, W_2]] + b[-Y_0^+, W_2] + cW_2$.

$$\begin{cases} W_1 = \int_0^\infty R(t)(\psi_t^-)_* Y^1 dt \in U_1^\varepsilon \\ W_2 = \int_0^\infty R(t)(\psi_t^+)_* Y^2 dt \in U_2^\varepsilon. \end{cases}$$

With

$$R(t) = \frac{2\exp(\frac{b}{2}t)}{\sqrt{4c-b^2}} \cdot \sin(\sqrt{4c-b^2}/2)t.$$

- First stage: Let us show that W_i is a solution of the equation $(\varphi_i^2 + b\varphi_i + cid)W_i = Y^i$.

If we put $Z_1 = [-Y_0^-, W_1] = ad_{-Y_0^-}(W_1)$, then

$$\begin{aligned} Z_1 &= \lim_{s \rightarrow 0} \frac{d}{ds} (\psi_s^-)_* \left(\int_0^{+\infty} R(t)(\psi_t^-)_* Y^1 dt \right) \\ &= R(\tau)(\psi_\tau^-)_* \cdot Y^1|_0^{+\infty} - \int_0^{+\infty} (\psi_\tau^-)_* Y^1 \frac{d}{d\tau} R(\tau) d\tau \\ Z_1 &= R(\tau)(\psi_\tau^-)_* Y^1|_0^{+\infty} - \int_0^{+\infty} R'(\tau)(\psi_\tau^-)_* Y^1 d\tau. \end{aligned}$$

As $R(t) = \frac{2\exp(\frac{b}{2}t)}{\sqrt{4c-b^2}} \cdot \sin(\sqrt{4c-b^2}/2)t$ and

$$\begin{aligned} \| R(\tau)(\psi_\tau^-)_* \cdot Y^1 \|_{r}^{\Omega_1} &\leq C' e^{\frac{b}{2}\tau} \cdot e^{-\tau\rho'_1 a_R} \cdot \| Y^1 \|_{r+m_1}^{\Omega_1^\varepsilon} \\ &\leq C' e^{-\tau[-b/2+\rho'_1 a_R]} \cdot \| Y^1 \|_{r+m_1}^{\Omega_1^\varepsilon} \end{aligned}$$

ρ'_1 being arbitrary; therefore, $\forall b \in \mathbb{R}, \exists \rho'_{1,b} > 0 / -b/2 + \rho'_{1,b} \cdot a_R > 0$, where ultimately

$$\lim_{\tau \rightarrow +\infty} \| R(\tau)(\psi_\tau^-)_* \cdot Y^1 \|_r = 0,$$

then the first member vanishes and Z_1 becomes

$$Z_1 = - \int_0^{+\infty} R'(\tau)(\psi_\tau^-)_* Y^1 d\tau.$$

We will have

$$\begin{aligned} \varphi_1^2(W_1) &= [-Y_0^-, [-Y_0^-, W_1]] = [-Y_0^-, Z_1] \\ &= - \lim_{s \rightarrow 0} \frac{d}{ds} (\psi_s^-)_* \int_0^{+\infty} R'(\tau)(\psi_\tau^-)_* Y^1 d\tau \\ &= - \lim_{s \rightarrow 0} \int_0^{+\infty} R'(\tau) \frac{d}{ds} (\psi_{s+\tau}^-)_* Y^1 d\tau \\ &= - [R'(t)(\psi_t^-)_* Y^1]_0^{+\infty} + \int_0^{+\infty} R''(t)(\psi_t^-)_* Y^1 dt \quad \text{where } t = s + \tau \\ &= Y^1 + \int_0^{+\infty} R''(t)(\psi_t^-)_* Y^1 dt. \end{aligned}$$

Consequently,

$$\begin{aligned}(\varphi_1^2 + b\varphi_1 + c.id_{\mathbb{R}^k})W_1 &= [-Y_0^-, [-Y_0^-, W_1]] + b[-Y_0^-, W_1] + cW_1 \\&= Y^1 + \int_0^{+\infty} (R''(t) - bR'(t) + cR(t))(\psi_t^-)_* Y dt \\&= Y^1,\end{aligned}$$

where $R(t) = \frac{2\exp(\frac{b}{2}t)}{\sqrt{4c - b^2}} \cdot \sin(\sqrt{4c - b^2}/2)t$ is solution of the Cauchy problem:

$$\begin{cases} R''(t) - bR'(t) + cR(t) = 0 \\ R(0) = 0 \\ R'(0) = 1. \end{cases}$$

The operator $\varphi_1^2 + b\varphi_1 + c.id_{\mathbb{R}^k}$ is, therefore, surjective on U_1^ε . The proof of the surjectivity of the operator $\varphi_2^2 + b\varphi_2 + c.id_{\mathbb{R}^l}$ on U_2^ε is made in the same way.

- Second stage: According to the previous lemma, W_i , ($i = 1, 2$) are of class C^∞ on all compact. $\square$

Lemma 6.2 For all $b, c \in \mathbb{R}$, $b^2 - 4.c < 0$ the operator $\varphi^2 + b.\varphi + c.id$ is surjective in U .

Proof Let $Y \in U$, seeking a $W \in U$ such that

$$\forall (x, y) \in \mathbb{R}^n, Y(x, y) = (\varphi^2 + b.\varphi + c.id)W(x, y), \text{ for all } b, c \in \mathbb{R}, b^2 - 4.c < 0.$$

As $Y \in U = U_1 \oplus U_2$, $\exists! Y^i \in U_i$ such that

$$Y(x, y) = (Y^1(x), Y^2(y)) = \left(\varphi_1^2 W_1(x) + b.\varphi_1 W_1(x) + c.W_1(x), \varphi_2^2 W_2(y) + b.\varphi_2 W_2(y) + c.W_2(y) \right).$$

We project on U_i :

- On U_1 , we have $Y(x, y) = (Y^1(x), 0) = \left(\varphi_1^2 W_1(x) + b.\varphi_1 W_1(x) + c.W_1(x), 0 \right)$, and according to Lemma 6.1, $\exists \varepsilon > 0$ and W_1 , such that $W_1(x) = \int_0^{+\infty} R(t)(\psi_t^-)_* Y^1(x) dt \in \Omega_1^\varepsilon$ with

$$R(t) = \frac{2\exp(\frac{b}{2}t)}{\sqrt{4c - b^2}} \cdot \sin(\sqrt{4c - b^2}/2)t$$

- On U_2 , we have $Y(x, y) = (0, Y^2(y)) = \left(0, \varphi_2^2 W_2(y) + b.\varphi_2 W_2(y) + c.W_2(y) \right)$, and according to Lemma 6.1, $\exists \varepsilon > 0$ and W_2 such that $W_2(y) = \int_0^{+\infty} R(t)(\psi_{-t}^+)_* Y^2(y) dt \in \Omega_2^\varepsilon \subset \Omega_2$ with

$$R(t) = \frac{2\exp(\frac{b}{2}t)}{\sqrt{4c - b^2}} \cdot \sin(\sqrt{4c - b^2}/2)t.$$

It follows that

$$W_i \in U_i^\varepsilon.$$

We put $W(x, y) = (W_1(x), W_2(y)) \in \mathbb{R}^k \times \mathbb{R}^l = \mathbb{R}^n$. Since $W = W_1 + W_2 \in U_1^\varepsilon + U_2^\varepsilon = U$. Consequently, $\varphi^2 + b.\varphi + c.id$ is surjective on U , for all $b, c \in \mathbb{R}$, $b^2 - 4.c < 0$. $\square$

7 Ideals of finite codimension in hyperbolic-type algebra

Theorem Let U be the admissible algebra having a hyperbolic structure for the flow:

$(\psi_t)_* = (\exp t Y_0)_* = (\exp t (X_0 + Z_0))_*$, if $\dim(E - U)$ is finite, and $\varphi = (ad_{-Y_0^-}, ad_{Y_0^+})$ an endomorphism on U , such that

- (i) $\varphi(U) \subset U$,
- (ii) $\varphi + b.id_{\mathbb{R}^n}$ is surjective on U ; $\forall b \in \mathbb{R}$,
- (iii) $\varphi^2 + b\varphi + c.id_{\mathbb{R}^n}$ is surjective on U ; $\forall b, c \in \mathbb{R}/b^2 - 4c < 0$ Then $U = E$.

Proof The hypotheses of the fundamental lemma is satisfied thanks to Lemmas 4.2, 5.2 and 6.2, and, if in addition we have $\dim(E - U)$ finite, then we deduce

$$U = E.$$

□

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