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Fractional flow equation in fractured aquifer using dual permeability model with non-singular kernel

Received: 24 July 2019 / Accepted: 21 August 2020 / Published online: 7 September 2020
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Abstract In this paper, a finite fractured aquifer, bounded by a stream and impervious layers on the other sides, has been considered. Variation in the level of groundwater is analyzed in confined aquifer for the unsteady flow. The governing differential equation for piezometric head involves the Caputo–Fabrizio fractional derivative operator with respect to time and is based on dual-porosity model with the assumption that the flow from fracture to block is in pseudo steady state. The obtained solutions can be used to anticipate the fluctuations in the waterlevels of the confined aquifer and for the numerical validation of a model in an aquifer.

Mathematics Subject Classification 34A08 · 26A33

1 Introduction

An aquifer is a geological formation of underground layer of permeable rocks, rock fractures or unconsolidated materials which can contain and transmit the groundwater. Aquifer may occur at various depths in ground, close to surface are used to water supply and irrigation. Many desert areas having limestones hills or mountains can be exploited as groundwater resources.

The fractured rocks are the type of an aquifer which contains groundwater. The fractured aquifer having two overlapping medium in which one represent the fracture and other representing the blocks. These formations are the type of heterogeneous medium in which blocks have very low permeability and containing large amount of storage comparing to fractures which have high permeability. Groundwater is transferred between fractures to blocks, but in any two blocks, there is no flow, because of its low permeability [15, 29]. Therefore, fractures are important for groundwater flow, whereas blocks act as a source or sink to the fractures.

The dual-porosity model has been recognized as a powerful tool to simulate flow and transport phenomena in fractured aquifer [10, 12, 16, 19, 21–24]. In this approach, the porous medium consists of two continua, one associated with the fractured system and other with a less permeable pore system of matrix block. For saturated and unsaturated flows, the continuum approach is good unless the porous media has very high heterogeneity. Heterogeneous porous media give rise to non-uniform flow with widely different velocity distributions. Such flow often referred as preferential or bypass flow [27]. The continuum approach cannot provide realistic

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predictions for the flow in porous media which contains high heterogeneity due to fractures, rocks and fissures. The dual-porosity model should be used for the analysis of such preferential flows. Agarwal et al. [1–5] and Yadav et al. [30] have discussed analytical and numerical solutions of various fractionalized groundwater flow problems.

Many authors such as Atangana et al. [7,8] and Baleanu et al. [9,11,18,25] have applied the general fractional order derivative with non-singular kernels in their models and problems. Yang et al. [31–37] have applied Caputo–Fabrizio fractional derivative to investigate the mathematical models in mathematical physics, the general fractional order diffusion, the general fractional order rheological model, the anomalous heat transfer model, steady heat-conduction problem.

We assume that fractures and blocks are homogeneous and isotropic in the confined aquifer and the flow is fully saturated, unaffected by chemical dissolution and follows Darcy law. Here, in stream stage, constant rise or drawdown are not encountered as frequently as constant discharge does. The flow between two pores system is described using a space- and time-dependent exchange term in dual-porosity model. Using continuity equation and Darcy law, we obtain the governing differential equation for the flow from fracture to block in confined fractured aquifer as

$$T_1 \frac{\partial^2 h_1}{\partial x^2} = S_1 \frac{\partial h_1}{\partial t} + w_d, \quad (1)$$

$$T_2 \frac{\partial^2 h_2}{\partial x^2} = S_2 \frac{\partial h_2}{\partial t} - w_d, \quad (2)$$

respectively. Here, T_i ($i = 1, 2$) is the coefficient of transmissibility, S_i ($i = 1, 2$) is the coefficient of storage, h_i ($i = 1, 2$) is the mean piezometric head averaged over the thickness of the aquifer, w_d is the water transfer rate w.r.t. time from fractures to blocks in a finite aquifer, x is the distance along the flow direction and t is the time.

The quasi-steady transfer of water between fractures and pores of blocks (see, e.g. [12,17,20,26]) is expressed as

$$w_d = \delta T_2 (h_1 - h_2), \quad (3)$$

here δ denotes geometry of the fractured rocks. In fractured aquifer, the flow rate in blocks is less important than their storage capacity, hence, we neglected the term, $T_2 \left(\frac{\partial^2 h_2}{\partial x^2} \right)$ as given in [12,28,29], and from Eqs. (1), (2) and (3) we obtain

$$T_1 \frac{\partial^2 h_1}{\partial x^2} = S_1 \frac{\partial h_1}{\partial t} + S_2 \frac{\partial h_2}{\partial t}, \quad (4)$$

$$S_2 \frac{\partial h_2}{\partial t} = \delta T_2 (h_1 - h_2), \quad (5)$$

The Eqs. (4) and (5) are solved for transient flow because of a step change of stream stage, depicted in Fig. 1. The initial and boundary conditions for the problem are given as

$$h_1 = h_2 = h_0, \quad 0 \leq x \leq L, \quad t = 0, \quad (6)$$

$$h_1 = h_0 + s_0, \quad x = 0, \quad t > 0, \quad (7)$$

$$\frac{\partial h_1}{\partial x} = 0, \quad x = L, \quad t > 0, \quad (8)$$

where h_0 is the initial piezometric head in the fractures and blocks, s_0 is the step change in stream and L denotes length of the aquifer horizontally. In Fig. 1a, a naturally fractured aquifer composed of a rock matrix surrounded by an irregular system of vugs and natural fractures is shown. Here, actual flow system is replaced by an equivalent finite flow system as shown in Fig. 1b (for details refer [22]). For homogeneous dual-porosity model, the boundary condition is:

$$h_1 = h_0 + s_0, \quad x = 2L, \quad t > 0. \quad (9)$$

Now, we define dimensionless variables for the governing equations

$$u_1 = \frac{h_1 - h_0}{s_0} \quad (10)$$

Fig. 1 The flow in fractured aquifer **a** real flow and **b** equivalent flow

$$u_2 = \frac{h_2 - h_0}{s_0} \quad (11)$$

$$y = \frac{x}{2L} \quad (12)$$

$$\theta = \frac{T_1 t}{4S_1 L^2} \quad (13)$$

and substitute into Eqs. (4) and (5). We obtain a set of differential equations for the fracture flow and flow in the blocks as

$$\frac{\partial^2 u_1}{\partial y^2} = \frac{\partial u_1}{\partial \theta} + \eta \frac{\partial u_2}{\partial \theta}, \quad (14)$$

$$\frac{\partial u_2}{\partial \theta} = \zeta(u_1 - u_2), \quad (15)$$

respectively, $\eta = \frac{S_2}{S_1}$ and $\zeta = 4\delta \frac{T_2}{S_2} \frac{S_1}{T_1} L^2$ with initial and boundary conditions rewritten as

$$u_1(y, 0) = u_2(y, 0) = 0 \quad (16)$$

$$u_1(0, \theta) = u_1(1, \theta) = 1. \quad (17)$$

In this paper, we present analytic solution of the one dimensional flow in the fractured aquifer assuming that groundwater flows from fracture to block in pseudo steady-state condition under a step drawdown condition rather than flux condition because of greater mathematical simplicity [22]. Since we consider a finite fractured aquifer, bounded by a stream and impervious layers and variation in the level of groundwater is analyzed in confined aquifer for the unsteady flow. All these physical observations and facts cannot be described with more accuracy via the local derivative and the well-known derivative with fractional order. Caputo Fabrizio fractional derivative does not have singularity in its kernel, therefore, memory is described better by fractional order with non-singular kernels compared to fractional order with singular kernels. Therefore, we use Caputo Fabrizio fractional derivative to describe the movement of water within geological formations called aquifers due to their memory effect in Eqs. (14) and (15) for the time variable, respectively, as follows:

$$\frac{\partial^2 u_1}{\partial y^2} = \frac{\partial^\alpha u_1}{\partial \theta^\alpha} + \eta \frac{\partial^\beta u_2}{\partial \theta^\beta}, \quad (18)$$

$$\frac{\partial^\beta u_2}{\partial \theta^\beta} = \zeta(u_1 - u_2), \quad (19)$$

where $0 < \alpha \leq 1$, $0 < \beta \leq 1$ along with initial and boundary conditions (16) and (17).

2 Mathematical preliminaries

Let $f \in H'(a, b)$, $b > a$ then, definition of Caputo fractional derivative, given by Caputo and Fabrizio [14] as

$${}_a^C D_t^\alpha f(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t f'(\tau) \exp\left(-\alpha \frac{t-\tau}{1-\alpha}\right) d\tau, \quad a < t < b, \quad (20)$$

where $\alpha \in (0, 1]$, $M(\alpha)$ is a normalized function s.t. $M(0) = M(1) = 1$, and defined as

$$M(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}. \quad (21)$$

Caputo Fabrizio fractional derivative does not have singularity at $t = \tau$ in its kernel. Therefore, memory is described better by fractional order with non-singular kernels compared to fractional order with singular kernels.

Laplace transform of the Caputo–Fabrizio fractional derivative is given by Caputo and Fabrizio [14] as

$$\mathcal{L}\left[{}_0^C D_t^\alpha f(t); s\right] = M(\alpha) \frac{sF(s) - f(0)}{s + \alpha(1-s)}, \quad 0 < \alpha \leq 1, \quad (22)$$

if exists.

For a finite interval $0 < x < L$, Fourier sine transform is given as

$$F_s(f) = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2, \dots, \quad (23)$$

with the inverse Fourier sine transform

$$F^{-1}[F_s(f)] = \sum_{n=1}^{\infty} F_s(f) \sin\left(\frac{n\pi x}{L}\right). \quad (24)$$

3 Flow in fractured confined aquifer

The fractional model of flow in confined fractured strip aquifer is obtained by replacing time derivatives by Caputo–Fabrizio fractional derivatives ${}_0^C D_t^\alpha$ and ${}_0^C D_t^\beta$ in Eqs. (4) and (5), respectively,

$${}_0^C D_\theta^\alpha u_1(y, \theta) + \eta {}_0^C D_\theta^\beta u_2(y, \theta) = \frac{\partial^2 u_1(y, \theta)}{\partial y^2}, \quad (25)$$

$${}_0^C D_t^\beta u_2(y, \theta) = \zeta(u_1(y, \theta) - u_2(y, \theta)), \quad (26)$$

along with initial and boundary conditions

$$u_1(y, 0) = u_2(y, 0) = 0 \quad (27)$$

$$u_1(0, \theta) = u_1(1, \theta) = 1, \quad (28)$$

where $0 < \alpha \leq 1$, $0 < \beta \leq 1$, $y > 0$, $\theta > 0$ and ${}_0^C D_\theta^\alpha$, ${}_0^C D_\theta^\beta$ are the Caputo–Fabrizio fractional derivatives of order α and β respectively.

Applying finite Fourier sine transform on space variable y w.r.t. s , the Eqs. (25) and (26), respectively, becomes

$${}_0^C D_\theta^\alpha u_1^*(s, \theta) + \eta {}_0^C D_\theta^\beta u_2^*(s, \theta) = n\pi[u_1(0, \theta) - (-1)^n u_1(1, \theta)] - n^2 \pi^2 u_1^*(s, \theta), \quad n \in \mathbb{N} \quad (29)$$

$${}_0^C D_t^\beta u_2^*(s, \theta) = \zeta(u_1^*(s, \theta) - u_2^*(s, \theta)). \quad (30)$$

Applying boundary conditions in (28) on Eq. (29), we obtain

$${}_0^C D_\theta^\alpha u_1^*(s, \theta) + \eta {}_0^C D_\theta^\beta u_2^*(s, \theta) = n\pi[1 - (-1)^n] - n^2 \pi^2 u_1^*(s, \theta), \quad (31)$$

$${}_0^CF D_t^\beta u_2^*(s, \theta) = \zeta(u_1^*(s, \theta) - u_2^*(s, \theta)). \quad (32)$$

Now, applying Laplace transform on time variable θ w.r.t. p , the Eqs. (31) and (32) yields

$$\frac{p\bar{u}_1^*(s, p)}{p(1-\alpha) + \alpha} + \eta \frac{p\bar{u}_2^*(s, p)}{p(1-\beta) + \beta} = \frac{n\pi}{p} [1 - (-1)^n] - n^2\pi^2 \bar{u}_1^*(s, p), \quad (33)$$

$$\frac{p\bar{u}_2^*(s, p)}{p(1-\beta) + \beta} = \zeta(\bar{u}_1^*(s, p) - \bar{u}_2^*(s, p)). \quad (34)$$

Since initial conditions (27) becomes $u_1(s, 0) = u_2(s, 0) = 0$. On applying inverse Laplace transform to Eqs. (33) and (34) and solving, we obtain

$$u_1^*(s, \theta) = \frac{n\pi [1 - (-1)^n]}{\frac{1}{1-\alpha} \left[\frac{1}{\theta} - \frac{\alpha}{1-\alpha} e^{-\frac{\alpha\theta}{1-\alpha}} + n^2\pi^2 \right] + \eta \zeta \frac{\left[\frac{1}{\theta} - \frac{\beta}{1-\beta} e^{-\frac{\beta\theta}{1-\beta}} \right]}{\left[\frac{1}{\theta} - \frac{\beta}{1-\beta} e^{-\frac{\beta\theta}{1-\beta}} + \zeta \right]}}, \quad (35)$$

$$u_2^*(s, \theta) = \frac{\zeta(1-\beta)}{\left[\frac{\beta}{1-\beta} e^{-\frac{\beta\theta}{1-\beta}} + \zeta \right]} \times \frac{n\pi [1 - (-1)^n]}{\frac{1}{1-\alpha} \left[\frac{1}{\theta} - \frac{\alpha}{1-\alpha} e^{-\frac{\alpha\theta}{1-\alpha}} + n^2\pi^2 \right] + \eta \zeta \frac{\left[\frac{1}{\theta} - \frac{\beta}{1-\beta} e^{-\frac{\beta\theta}{1-\beta}} \right]}{\left[\frac{1}{\theta} - \frac{\beta}{1-\beta} e^{-\frac{\beta\theta}{1-\beta}} + \zeta \right]}}. \quad (36)$$

Further taking the inverse finite Fourier sine transform of the Eqs. (35) and (36), we obtain

$$u_1(y, \theta) = \sum_{n=1}^{\infty} \frac{n\pi [1 - (-1)^n] \sin n\pi y}{\frac{1}{1-\alpha} \left[\frac{1}{\theta} - \frac{\alpha}{1-\alpha} e^{-\frac{\alpha\theta}{1-\alpha}} + n^2\pi^2 \right] + \eta \zeta \frac{\left[\frac{1}{\theta} - \frac{\beta}{1-\beta} e^{-\frac{\beta\theta}{1-\beta}} \right]}{\left[\frac{1}{\theta} - \frac{\beta}{1-\beta} e^{-\frac{\beta\theta}{1-\beta}} + \zeta \right]}}, \quad (37)$$

$$u_2(y, \theta) = \frac{\zeta(1-\beta)}{\left[\frac{\beta}{1-\beta} e^{-\frac{\beta\theta}{1-\beta}} + \zeta \right]} \times \sum_{n=1}^{\infty} \frac{n\pi [1 - (-1)^n] \sin n\pi y}{\frac{1}{1-\alpha} \left[\frac{1}{\theta} - \frac{\alpha}{1-\alpha} e^{-\frac{\alpha\theta}{1-\alpha}} + n^2\pi^2 \right] + \eta \zeta \frac{\left[\frac{1}{\theta} - \frac{\beta}{1-\beta} e^{-\frac{\beta\theta}{1-\beta}} \right]}{\left[\frac{1}{\theta} - \frac{\beta}{1-\beta} e^{-\frac{\beta\theta}{1-\beta}} + \zeta \right]}}, \quad (38)$$

which is the analytic solution for Eqs. (25) and (26).

3.1 Flow rate

In fractured aquifer the groundwater flow rate q_0 of the stream may be calculated using Darcy law

$$q_0 = T_1 \frac{\partial h_1}{\partial x} \bigg|_{x=0}. \quad (39)$$

Since the conducting capabilities of blocks as compared to the fracture is negligible [20], so we obtain

$$q_d = \sum_{n=1}^{\infty} \frac{n^2\pi^2 [1 - (-1)^n]}{\frac{1}{1-\alpha} \left[\frac{1}{\theta} - \frac{\alpha}{1-\alpha} e^{-\frac{\alpha\theta}{1-\alpha}} + n^2\pi^2 \right] + \eta \zeta \frac{\left[\frac{1}{\theta} - \frac{\beta}{1-\beta} e^{-\frac{\beta\theta}{1-\beta}} \right]}{\left[\frac{1}{\theta} - \frac{\beta}{1-\beta} e^{-\frac{\beta\theta}{1-\beta}} + \zeta \right]}}, \quad (40)$$

where $q_d = \frac{2Lq_0}{T_1 s_0}$ is the dimensionless discharge.

4 Numerical simulation

In this section, the behavior of fracture drawdown w.r.t. various parameters has been observed and graphical representation of fractures drawdown w.r.t. time is presented. To obtain the effect of the fractional order derivative to the solution, we compare different parameters. Figures are plotted in Matlab. Here, we assume that in an observation well of finite depth many fractures intersect with it, so the drawdown in the well would be of fractures [13]. Drawdown in the block matrix is not useful hence no drawdown curve is plotted.

The behavior of dimensionless drawdown in Figs. 2, 3 and 4 and radial flow is qualitatively similar and have three stages. First, the increase in drawdown is in fracture storage only. Second, when flow is from fracture to block, a transitional stage develops, drawdown becomes slow and the shape of the curve depends on the parameters η and ζ for the fixed fractional order $\alpha = 0.90$, $\beta = 0.90$. The third stage represents the drawdown of homogeneous type in total storage. In Figs. 5 and 6, the dimensionless drawdown w.r.t. dimensionless time is depicted for different order of fractional derivatives. In Figs. 7 and 8, the dimensionless discharge w.r.t. dimensionless time is depicted for the fixed values of $\alpha = 0.90$, $\beta = 0.90$ and different values of η , ζ .

Fig. 2 Graphical representation of the dimensionless piezometric head w.r.t. dimensionless time for $\zeta = 10$, and $\eta = 2, 4, 6, 8$ at distance $y = 5.5$ and fractional order $\alpha = 0.90$, $\beta = 0.90$

Fig. 3 Graphical representation of the dimensionless piezometric head w.r.t. dimensionless time for $\eta = 10$, and $\zeta = 2, 4, 6, 8$ at distance $y = 5.5$ and fractional order $\alpha = 0.90$, $\beta = 0.90$

Fig. 4 Graphical representation of the dimensionless piezometric head w.r.t. dimensionless time at distance $y = 5.5$. for various η and ζ values such that $\eta\zeta = 50$ and fractional order $\alpha = 0.90$, $\beta = 0.90$

Fig. 5 Graphical representation of the dimensionless piezometric head w.r.t. dimensionless time at distance $y = 5.5$. for fixed $\beta = 0.90$, $\zeta = 10$, $\eta = 2$ and $\alpha = 0.90, 0.95, 0.99$

Fig. 6 Graphical representation of the dimensionless piezometric head w.r.t. dimensionless time at distance $y = 5.5$. for fixed $\beta = 0.90$, $\zeta = 10$, $\eta = 8$ and $\alpha = 0.90, 0.95, 0.99$

Fig. 7 Dimensionless discharge w.r.t. dimensionless time for $\zeta = 10$, $\eta = 2, 4, 6, 8$ and fractional order $\alpha = 0.90$, $\beta = 0.90$

Fig. 8 Dimensionless discharge w.r.t. dimensionless time for $\eta = 10$, $\zeta = 2, 4, 6, 8$ and fractional order $\alpha = 0.90$, $\beta = 0.90$

5 Conclusion

In this paper, the dual-porosity model is used to analyse the groundwater flow in finite fractured confined flow. Here, we assume that the aquifer is bounded by stream and impervious layer. We use Caputo Fabrizio fractional derivative with non-singular kernel to describe the movement of water within geological formations called aquifers due to their memory effect and apply Laplace and Fourier transforms to obtain the analytic solution for the finite fractured aquifer. This work helps researchers to investigate the properties and behavior of fractured aquifer and one can further extend the work for the other types of aquifers also. If the field data are not available for the testing of a numerical model, one can compare the results with the present solution.

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