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On a DGL-map between derivations of Sullivan minimal models

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Abstract For a map $f : X \rightarrow Y$, there is the relative model $M(Y) = (\Lambda V, d) \rightarrow (\Lambda V \otimes \Lambda W, D) \simeq M(X)$ by Sullivan model theory (Félix et al., Rational homotopy theory, graduate texts in mathematics, Springer, Berlin, 2007). Let $\text{Baut}_1 X$ be the Dold–Lashof classifying space of orientable fibrations with fiber X (Dold and Lashof, Ill J Math 3:285–305, 1959). Its DGL (differential graded Lie algebra)-model is given by the derivations $\text{Der} M(X)$ of the Sullivan minimal model $M(X)$ of X . Then we consider the condition that the restriction $b_f : \text{Der}(\Lambda V \otimes \Lambda W, D) \rightarrow \text{Der}(\Lambda V, d)$ is a DGL-map and the related topics.

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1 Introduction

Let X (and also Y) be a connected and simply connected finite CW complex with $\dim \pi_*(X)_{\mathbb{Q}} < \infty$ ($G_{\mathbb{Q}} = G \otimes \mathbb{Q}$) and $\text{Baut}_1 X$ be the Dold–Lashof classifying space of orientable fibrations [5]. Here $\text{aut}_1 X = \text{map}(X, X; id_X)$ is the identity component of the space $\text{aut} X$ of self-equivalences of X . Then any orientable fibration ξ with fibre X over a base space B is the pull-back of a universal fibration $X \rightarrow E_{\infty}^X \rightarrow \text{Baut}_1 X$ by a map $h : B \rightarrow \text{Baut}_1 X$ and equivalence classes of ξ are classified by their homotopy classes [2, 5, 23]. The Sullivan minimal model $M(X)$ [24] determines the rational homotopy type of X , the homotopy type of the rationalization X_0 [14] of X . Notice that $(\text{Baut}_1 X)_0 \simeq \text{Baut}_1(X_0)$ [17]. The differential graded Lie algebra (DGL) $\text{Der} M(X)$, the negative derivations of $M(X)$ (see §2), gives rise to the DGL model for $\text{Baut}_1 X$ due to Sullivan [24] (cf. [10, 25]), i.e., the spatial realization $||\text{Der} M(X)||$ is $(\text{Baut}_1 X)_0$. Therefore, we obtain a map $(\text{Baut}_1 X)_0 \rightarrow (\text{Baut}_1 Y)_0$ if there is a DGL-map $\text{Der} M(X) \rightarrow \text{Der} M(Y)$. However, it does *not* exist in general.

Let $f : X \rightarrow Y$ be a map whose homotopy fibration $\xi_f : F_f \rightarrow X \rightarrow Y$ is given by the relative model (Koszul–Sullivan extension)

$$M(Y) = (\Lambda V, d) \xrightarrow{i} (\Lambda V \otimes \Lambda W, D) \simeq M(X)$$

for a certain differential D with $D|_{\Lambda V} = d$, where $M(F_f) \cong (\Lambda W, \overline{D})$ for the homotopy fiber F_f of f [7]. In this paper, we propose

Question 1.1 When is the restriction map given by $b_f(\sigma) = \text{proj}_V \circ \sigma \circ i$

$$b_f : \text{Der}(\Lambda V \otimes \Lambda W, D) \rightarrow \text{Der}(\Lambda V, d)$$

a DGL-map ?

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Here $\text{proj}_V : \Lambda V \otimes \Lambda W \rightarrow \Lambda V$ is the algebra map with $\text{proj}_V(w) = 0$ for $w \in W$ and $\text{proj}_V|_{\Lambda V} = id_{\Lambda V}$.

Definition 1.2 We say that a $\mathbb{Q}$ -w.t. map $f : X \rightarrow Y$ strictly induces the map

$$a_f : (\text{Baut}_1 X)_0 \rightarrow (\text{Baut}_1 Y)_0$$

if its DGL model is given by the DGL-map $b_f : \text{Der}(\Lambda V \otimes \Lambda W, D) \rightarrow \text{Der}(\Lambda V, d)$ with $\|b_f\| = a_f$.

Let $\min \pi_*(S)_{\mathbb{Q}} := \min\{i > 0 \mid \pi_i(S)_{\mathbb{Q}} \neq 0\}$ and $\max \pi_*(S)_{\mathbb{Q}} := \max\{i \geq 0 \mid \pi_i(S)_{\mathbb{Q}} \neq 0\}$ for a space S . In particular, $\min \pi_*(S)_{\mathbb{Q}} = \infty$ when S is the one point space.

Definition 1.3 A fibration $\xi_f : F_f \rightarrow X \xrightarrow{f} Y$ or a map $f : X \rightarrow Y$ with homotopy fiber F_f is said to be $\pi_{\mathbb{Q}}$ -separable if $\min \pi_*(F_f)_{\mathbb{Q}} \geq \max \pi_*(Y)_{\mathbb{Q}}$.

We say a map is *rationaly weakly trivial* (abbr., $\mathbb{Q}$ -w.t.) if ξ_f is rationally weakly trivial, i.e., $\pi_*(X)_{\mathbb{Q}} = \pi_*(F_f)_{\mathbb{Q}} \oplus \pi_*(Y)_{\mathbb{Q}}$. Then $(\Lambda V \otimes \Lambda W, D)$ is just the minimal model $M(X)$ of X . If a map $f : X \rightarrow Y$ is $\pi_{\mathbb{Q}}$ -separable, it is $\mathbb{Q}$ -w.t. The condition to be $\pi_{\mathbb{Q}}$ -separable is equivalent to the condition that $\min W = \min\{i > 0 \mid W^i \neq 0\} \geq \max V = \max\{i > 0 \mid V^i \neq 0\}$ in the relative minimal model $M(Y) = (\Lambda V, d) \rightarrow (\Lambda V \otimes \Lambda W, D)$ of ξ_f .

Proposition 1.4 For a $\mathbb{Q}$ -w.t. map $f : X \rightarrow Y$ with relative model $M(Y) = (\Lambda V, d) \rightarrow (\Lambda V \otimes \Lambda W, D)$, the restriction $b_f : \text{Der}(\Lambda V \otimes \Lambda W, D) \rightarrow \text{Der}(\Lambda V, d)$ is a DGL-map if and only if f is $\pi_{\mathbb{Q}}$ -separable.

That means

Theorem 1.5 A $\mathbb{Q}$ -w.t. map $f : X \rightarrow Y$ strictly induces $a_f : (\text{Baut}_1 X)_0 \rightarrow (\text{Baut}_1 Y)_0$ if and only if f is $\pi_{\mathbb{Q}}$ -separable.

In §2, we give the proofs under some preparations of models of [7] and [25]. In this paper, we consider only $\mathbb{Q}$ -w.t. maps. For example, we do not consider the inclusion map $i_X : X \rightarrow X \times Y$, which is not $\mathbb{Q}$ -w.t. However i_X induces the monoid map $\psi : \text{aut}_1 X \rightarrow \text{aut}_1(X \times Y)$ by $\psi(g) = g \times 1_Y$ and, therefore, there exists the induced map $B\psi : \text{Baut}_1 X \rightarrow \text{Baut}_1(X \times Y)$ without rationalization. The DGL model is given by the natural inclusion $\text{Der} M(X) \rightarrow \text{Der}(M(X) \otimes M(Y))$, which is a DGL-map.

In §3, we give some conditions that the strictly induced map $a_f : (\text{Baut}_1 X)_0 \rightarrow (\text{Baut}_1 Y)_0$ admits a section. Some results are obtained by Proposition 3.1 induced from [25, VI.1.(3) Proposition] that the DGL-model of the homotopy fibration $\chi_f : F_{a_f} \rightarrow (\text{Baut}_1 X)_0 \xrightarrow{a_f} (\text{Baut}_1 Y)_0$ is given by

$$\text{Der}(\Lambda W, \Lambda V \otimes \Lambda W) \rightarrow \text{Der}(\Lambda V \otimes \Lambda W) \xrightarrow{b_f} \text{Der}(\Lambda V).$$

Let $\text{aut}_1 f$ be the identity component of the space of all fibre-homotopy self-equivalences of f , i.e., $\{g : X \rightarrow X \mid f \circ g = f\}$ and $\text{Baut}_1 f$ be the classifying space of this topological monoid. It is known that $\text{Baut}_1 f \simeq \widehat{\text{map}}(Y, \text{Baut}_1(F_f); h)$, where $h : Y \rightarrow \text{Baut}_1(F_f)$ is the classifying map of the fibration $F_f \rightarrow X \xrightarrow{f} Y$ and $\widehat{\text{map}}$ denotes the universal cover of the function space [3]. Notice that

$$\text{Der}(\Lambda W, \Lambda V \otimes \Lambda W) = \text{Der}_{\Lambda V}(\Lambda V \otimes \Lambda W),$$

where $\text{Der}_{\Lambda V}(\Lambda V \otimes \Lambda W)$ is the sub DGL of $\text{Der}(\Lambda V \otimes \Lambda W)$ sending the elements of ΛV to zero and it is a DGL model of $\text{Baut}_1 f$ when Y and F_f are finite [4, Theorem 1], [8]. Thus, we note

Theorem 1.6 If the homotopy fiber F_f is finite for a $\pi_{\mathbb{Q}}$ -separable map f , the fiber of χ_f has the rational homotopy type of $\text{Baut}_1 f$.

A space X is said to be *elliptic* if the dimensions of the rational cohomology algebra and homotopy group are both finite [7]. An elliptic space X is said to be *pure* if $dM(X)^{\text{even}} = 0$ and $dM(X)^{\text{odd}} \subset M(X)^{\text{rmeven}}$. A pure space is said to be an F_0 -space (or *positively elliptic*) if $\dim \pi_{\text{even}}(X) \otimes \mathbb{Q} = \dim \pi_{\text{odd}}(X) \otimes \mathbb{Q}$ and $H^{\text{odd}}(X; \mathbb{Q}) = 0$. Then it is equivalent to $H^*(X; \mathbb{Q}) \cong \mathbb{Q}[x_1, \dots, x_n]/(f_1, \dots, f_n)$, in which $|x_i|$, the degree of x_i , is even and $f_1, \dots, f_n$ forms a regular sequence in the $\mathbb{Q}$ -polynomial algebra $\mathbb{Q}[x_1, \dots, x_n]$, where $M(X) = (\mathbb{Q}[x_1, \dots, x_n] \otimes \Lambda(y_1, \dots, y_n), d)$ with $dx_i = 0$ and $dy_i = f_i$. In 1976, S. Halperin [12] conjectured that the Serre spectral sequences of all fibrations $X \rightarrow E \rightarrow B$ of simply connected CW complexes collapse at the E_2 -terms for any F_0 -space X [7]. For compact connected Lie groups G and H where H is a subgroup of G , when $\text{rank } G = \text{rank } H$, the homogeneous space G/H satisfies the Halperin conjecture [21]. Also the Halperin conjecture is true when $n \leq 3$ [16]. Finally, we note some relations with the Halperin conjecture [7, §39] due to Meier [18] as

Theorem 1.7 *Let Y be an F_0 -space. Then the fibration χ_f is fibre-trivial for any $\pi_{\mathbb{Q}}$ -separable map $f : X \rightarrow Y$ if and only if Y satisfies the Halperin’s conjecture.*

In §4, we observe the cellular obstruction for the lifting $\tilde{h}$ for a map $h : B \rightarrow (\text{Baut}_1 Y)_0$ for a simply connected CW complex B of finite type:

$$\begin{array}{ccc} & & (\text{Baut}_1 X)_0 \\ & \nearrow \tilde{h} & \downarrow a_f \\ B & \xrightarrow{h} & (\text{Baut}_1 Y)_0 \end{array}$$

for a $\pi_{\mathbb{Q}}$ -separable map $f : X \rightarrow Y$. Of course, it is sufficient to define as $\tilde{h} = s \circ h$ if a_f admits a section s . Specifically, for a $\pi_{\mathbb{Q}}$ -separable map $f : X \rightarrow Y$, let

$$\begin{array}{ccc} B & \xrightarrow{h_X} & (\text{Baut}_1 X)_0 \\ i \downarrow & & \downarrow a_f \\ B \cup_{\alpha} e^N & \xrightarrow{h_Y} & (\text{Baut}_1 Y)_0 \end{array}$$

be a commutative diagram. Then, from Proposition 1.6, we define an obstruction class by derivations in Theorem 4.1 so that

Theorem 1.8 *Let $f : X \rightarrow Y$ be a $\pi_{\mathbb{Q}}$ -separable map with Y and F_f finite. There is a lift h such that*

$$\begin{array}{ccc} B & \xrightarrow{h_X} & (\text{Baut}_1 X)_0 \\ i \downarrow & \nearrow h & \downarrow a_f \\ B \cup_{\alpha} e^N & \xrightarrow{h_Y} & (\text{Baut}_1 Y)_0 \end{array}$$

is commutative if and only if $\mathcal{O}_{\alpha}(h_X, h_Y) = 0$ in $\pi_{N-1}(\text{Baut}_1 f)_{\mathbb{Q}}$.

In §5, we consider an application to lifting actions. Let G be a topological group and acts on a CW complex Y . Recall the problem of lifting (up to homotopy) of Gottlieb [11]:

Problem 1.9 *When is a fibration $F_f \rightarrow X \xrightarrow{f} Y$ fibre homotopy equivalent to a G -fibration? i.e., when is there a fibration $f' : X' \rightarrow Y$ such that f' is fibre homotopy equivalent to f and there is a G -action on X' such that f' is equivariant?*

Suppose that G is a compact connected Lie group. Since $H^*(BG; \mathbb{Q})$ is evenly graded, the obstruction classes of Theorem 1.8 are contained in $\pi_{\text{odd}}(\text{Baut}_1 f)_{\mathbb{Q}}$ when $B = BG$. If $\pi_{\text{odd}}(\text{Baut}_1 f)_{\mathbb{Q}} = 0$, they vanish and there exists a lift $h : BG \rightarrow (\text{Baut}_1 X)_0$. Then from Theorem 2.6 in the case that $B = B' = BG$ and $g = (id_{BG})_0$, we obtain using Theorem 5.1 of D. H. Gottlieb.

Theorem 1.10 *Let $f : X \rightarrow Y$ be a $\pi_{\mathbb{Q}}$ -separable map with Y and F_f finite. Suppose that a compact Lie group G acts on Y . If $\pi_{\text{odd}}(\text{Baut}_1 f)_{\mathbb{Q}} = 0$, the action on Y is rationally lifted to X , i.e., f is rationally fibre homotopy equivalent to a G -equivariant map $f' : X' \rightarrow Y$ for a G -space X' .*

Due to Theorem 1.7 and the result of Shiga and Tezuka [21], we have

Corollary 1.11 *Let $f : X \rightarrow Y$ be a $\pi_{\mathbb{Q}}$ -separable map such that Y is a homogeneous space G/H with $\text{rank } G = \text{rank } H$. Then any group action on Y is rationally lifted to X . In particular, the natural G -action on Y is rationally lifted to X .*

Furthermore, we apply the obstruction argument to a rational homotopical invariant: let $r_0(X)$ be the *rational toral rank* of a simply connected complex X of $\dim H^*(X; \mathbb{Q}) < \infty$, i.e., the largest integer r such that an r -torus $T^r = S^1 \times \cdots \times S^1$ (r -factors) can act continuously on a CW-complex X' in the rational homotopy type of X with all its isotropy subgroups finite (almost free action) [1, 9, 13]. It is very difficult to calculate $r_0(\)$ in general. From the definition, we have the inequality $r_0(X \times Y) \geq r_0(X) + r_0(Y)$. Notice that it may sometimes be a strict inequality since there is an example that $r_0(X \times S^{12}) > 0$ even though $r_0(X) = r_0(S^{12}) = 0$ [15, Example 3.3]. For a map $f : X \rightarrow Y$, we see $r_0(Y) \leq r_0(X)$ when $X = F \times Y$ for any space F and f is the projection $F \times Y \rightarrow Y$. In general, when does a map $f : X \rightarrow Y$ induce $r_0(Y) \leq r_0(X)$?

Corollary 1.12 *Let $f : X \rightarrow Y$ be a $\pi_{\mathbb{Q}}$ -separable map with Y and F_f finite. If $\pi_{\text{odd}}(\text{Baut}_1 f)_{\mathbb{Q}} = 0$, we have $r_0(Y) \leq r_0(X)$.*

2 Sullivan models and derivations

Let $M(X) = (\Lambda V, d)$ be the Sullivan minimal model of simply connected CW complex X of finite type [24]. It is a free $\mathbb{Q}$ -commutative differential graded algebra (DGA) with a $\mathbb{Q}$ -graded vector space $V = \bigoplus_{i \geq 1} V^i$ where $\dim V^i < \infty$ and a decomposable differential, i.e., $d(V^i) \subset (\Lambda^+ V \cdot \Lambda^+ V)^{i+1}$ and $d \circ d = 0$. Here $\Lambda^+ V$ is the ideal of ΛV generated by elements of positive degree. The degree of a homogeneous element x of a graded algebra is denoted as $|x|$. Then $xy = (-1)^{|x||y|}yx$ and $d(xy) = d(x)y + (-1)^{|x|}xd(y)$. Note that $M(X)$ determines the rational homotopy type of X , namely the spatial realization is given as $\|M(X)\| \simeq X_0$. In particular,

$$V^n \cong \text{Hom}(\pi_n(X), \mathbb{Q}) \text{ and } H^*(\Lambda V, d) \cong H^*(X; \mathbb{Q}).$$

Here the second is an isomorphism as graded algebras. Refer to [7] for detail.

Let $\text{Der}_i M(X)$ be the set of $\mathbb{Q}$ -derivations of $M(X)$ decreasing the degree by i with $\sigma(xy) = \sigma(x)y + (-1)^{i \cdot |x|}x\sigma(y)$ for $x, y \in M(X)$. The boundary operator $\partial : \text{Der}_i M(X) \rightarrow \text{Der}_{i-1} M(X)$ is defined by

$$\partial(\sigma) = d \circ \sigma - (-1)^i \sigma \circ d$$

for $\sigma \in \text{Der}_i M(X)$. We denote $\bigoplus_{i \geq 0} \text{Der}_i M(X)$ by $\text{Der} M(X)$ in which $\text{Der}_1 M(X)$ is ∂ -cycles. Then $\text{Der} M(X)$ is a (non-free) DGL by the Lie bracket

$$[\sigma, \tau] := \sigma \circ \tau - (-1)^{|\sigma||\tau|} \tau \circ \sigma.$$

Note that $H_*(\text{Der} M) = H_*(\text{Der} N)$ when free DGAs M and N are quasi-isomorphic [20]. Furthermore, recall the definition of Tanré [25, p. 25]: let (L, ∂) be a DGL of finite type with positive degree. Then $C^*(L, \partial) = (\Lambda s^{-1} \sharp L, D = d_1 + d_2)$ with

$$\langle d_1 s^{-1} z; sx \rangle = -\langle z; \partial x \rangle \text{ and } \langle d_2 s^{-1} z; sx_1, sx_2 \rangle = (-1)^{|x_1|} \langle z; [x_1, x_2] \rangle,$$

where $\langle s^{-1} z; sx \rangle = (-1)^{|z|} \langle z; x \rangle$ and $\sharp L$ is the dual space of L .

Theorem 2.1 [24, §11], [10, 25] *For a Sullivan model $M(X)$ of X , $\text{Der} M(X)$ is a DGL-model of $\text{Baut}_1 X$. In particular, there is an isomorphism of graded Lie algebras $H_*(\text{Der} M(X)) \cong \pi_*(\Omega \text{Baut}_1 X)_{\mathbb{Q}}$ where the right hand has the Samelson bracket. Furthermore, $C^*(\text{Der} M(X))$ is a DGA-model of $\text{Baut}_1 X$.*

Two fibrations ξ_{f_1} and ξ_{f_2} are fibre homotopy equivalent if there is a diagram:

$$\begin{array}{ccccc} F_{f_1} & \xrightarrow{i_1} & X_1 & \xrightarrow{f_1} & Y \\ \sim \uparrow \overline{\psi} & & \uparrow \exists \psi & & \parallel \\ F_{f_2} & \xrightarrow{i_2} & X_2 & \xrightarrow{f_2} & Y, \end{array}$$

where $\psi \circ i_2 \simeq i_1 \circ \overline{\psi}$ and $f_1 \circ \psi = f_2$. Then its Sullivan model is given as

$$\begin{array}{ccccc} \Lambda V, d & \longrightarrow & \Lambda V \otimes \Lambda W, D_1 & \longrightarrow & \Lambda W, \overline{D}_1 \\ \parallel & & \downarrow \cong \psi & & \downarrow \cong \overline{\psi} \\ \Lambda V, d & \longrightarrow & \Lambda V \otimes \Lambda W, D_2 & \longrightarrow & \Lambda W, \overline{D}_2, \end{array}$$

where the left square is DGA-commutative and the right square is DGA-homotopy commutative.

Lemma 2.2 *Suppose that two maps f_1 and f_2 strictly induce a_{f_1} and a_{f_2} , respectively. If ξ_{f_1} and ξ_{f_2} are fibre homotopy equivalent, there is a DGL-isomorphism $\phi : \text{Der}(\Lambda V \otimes \Lambda W, D_1) \cong \text{Der}(\Lambda V \otimes \Lambda W, D_2)$ such that $\phi(\sigma) = \psi \circ \sigma \circ \psi^{-1}$ and $b_{f_2} \circ \phi = b_{f_1}$. Thus, there is a homotopy equivalence map $\phi : (\text{Baut}_1 X_1)_0 \xrightarrow{\sim} (\text{Baut}_1 X_2)_0$ such that $a_{f_2} \circ \phi = a_{f_1}$, i.e., a_{f_1} and a_{f_2} are fibre homotopy equivalent as fibrations.*

Proof Recall the DGA-diagram of §1. Then $D_2 = \psi \circ D_1 \circ \psi^{-1}$. Therefore, there is a DGL-isomorphism ϕ given by $\phi(\sigma) = \psi \circ \sigma \circ \psi^{-1}$ and

$$\begin{array}{ccc} \text{Der}(\Lambda V \otimes \Lambda W, D_1), \partial_1 & \xrightarrow{b_{f_1}} & \text{Der}(\Lambda V) \\ \phi \downarrow & & \parallel \\ \text{Der}(\Lambda V \otimes \Lambda W, D_2), \partial_2 & \xrightarrow{b_{f_2}} & \text{Der}(\Lambda V) \end{array}$$

is DGL-commutative since $\psi|_{\Lambda V} = id_{\Lambda V}$. In particular, we can check $\partial_2 \circ \phi = \phi \circ \partial_1$ by

$$\begin{aligned} \partial_2 \phi(\sigma) &= D_2 \psi \sigma \psi^{-1} - (-1)^i (\psi \sigma \psi^{-1}) D_2 = \psi D_1 \sigma \psi^{-1} - (-1)^i \psi \sigma D_1 \psi^{-1} \\ &= \phi(D_1 \sigma - (-1)^i \sigma D_1) = \phi \partial_1(\sigma) \end{aligned}$$

for $\sigma \in \text{Der}_i(\Lambda V \otimes \Lambda W, D_1)$. Similarly, we have $\phi([\sigma, \tau]) = [\phi(\sigma), \phi(\tau)]$. $\square$

Convention. For a DGA-model $(\Lambda V, d)$ the symbol (v, f) means the *elementary derivation* that takes a generator v of V to an element f of ΛV and the other generators to 0. Note that $|(v, f)| = |v| - |f|$.

Proof of Proposition 1.4 Let $M(Y) = (\Lambda V, d) \rightarrow (\Lambda V \otimes \Lambda W, D)$ be the model of f . Note that $b_f(\sigma)$ is a derivation on ΛV for $\sigma \in \text{Der}(\Lambda V \otimes \Lambda W)$ since

$$\begin{aligned} b_f(\sigma)(a \cdot b) &= (\text{proj}_V \circ \sigma \circ i)(a \cdot b) = \text{proj}_V(\sigma(a) \cdot b + (-1)^{|\sigma||a|} a \cdot \sigma(b)) \\ &= (\text{proj}_V \circ \sigma \circ i)(a) \cdot b + (-1)^{|\sigma||a|} a \cdot (\text{proj}_V \circ \sigma \circ i)(b) = b_f(\sigma)(a) \cdot b + (-1)^{|\sigma||a|} a \cdot b_f(\sigma)(b) \end{aligned}$$

for $a, b \in \Lambda V$. Thus, b_f is well defined.

(if) When $\min W \geq \max V$, there is a decomposition of vector spaces

$$\text{Der}(\Lambda V \otimes \Lambda W) = \text{Der}(\Lambda V) \oplus \text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)$$

from degree arguments. Then there is a DGL-map $b_f : \text{Der}(\Lambda V \otimes \Lambda W, D) \rightarrow \text{Der}(\Lambda V, d)$ by $b_f(\sigma_1) = \sigma_1$ and $b_f(\sigma_2) = 0$ for $\sigma = \sigma_1 + \sigma_2$ with $\sigma_1 \in \text{Der}(\Lambda V)$ and $\sigma_2 \in \text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)$. Then b_f is a Lie algebra map since

$$\begin{aligned} b_f([\sigma, \tau]) &= b_f([\sigma_1 + \sigma_2, \tau_1 + \tau_2]) = b_f([\sigma_1, \tau_1]) \\ &= \text{proj}_V \circ \sigma_1 \circ \tau_1 \circ i + (-1)^{|\sigma_1||\tau_1|} \text{proj}_V \circ \tau_1 \circ \sigma_1 \circ i \\ &= (\text{proj}_V \circ \sigma_1 \circ i) \circ (\text{proj}_V \circ \tau_1 \circ i) + (-1)^{|\sigma_1||\tau_1|} (\text{proj}_V \circ \tau_1 \circ i) \circ (\text{proj}_V \circ \sigma_1 \circ i) \\ &= [b_f(\sigma_1), b_f(\tau_1)] = [b_f(\sigma), b_f(\tau)] \end{aligned}$$

for $\sigma, \tau \in \text{Der}(\Lambda V \otimes \Lambda W)$. Furthermore, it preserves the differential since

$$\begin{aligned} (b_f \circ \partial_X)(\sigma) &= b_f(\partial_X(\sigma_1 + \sigma_2)) = b_f(\partial_X(\sigma_1)) = b_f(\partial_Y(\sigma_1) + \tau) \\ &= b_f(\partial_Y(\sigma_1)) = \partial_Y(\sigma_1) = \partial_Y(b_f(\sigma_1)) = \partial_Y(b_f(\sigma_1 + \sigma_2)) = (\partial_Y \circ b_f)(\sigma) \end{aligned}$$

for $\sigma \in \text{Der}(\Lambda V \otimes \Lambda W)$ with some $\tau \in \text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)$.

(only if) Suppose that $\min W < \max V$. There are elements $w \in W$ and $v \in V$ with $|w| < |v|$. Then b_f is not a DGL-map since $b_f([(w, 1), (v, w)]) = b_f(v, 1) \neq 0$ but $[b_f(w, 1), b_f(v, w)] = [0, 0] = 0$ from the definition of b_f . $\square$

Example 2.3 Consider the case that f is not $\pi_{\mathbb{Q}}$ -separable (not $\mathbb{Q}$ -w.t.). Let $f : S^7 \rightarrow S^4$ be the Hopf map. Then the model is given by

$$M(S^4) = (\Lambda(x, y), d) \rightarrow (\Lambda(x, y, z), D) \simeq M(S^7)$$

with $|x| = 4$, $|y| = 7$, $|z| = 3$, $dx = Dx = 0$, $Dy = dy = x^2$, $Dz = x$, $Dy = x^2$. Then the bases of derivations are given as,

n	$\text{Der}_n(\Lambda(x, y, z), D)$
7	$(y, 1)$
4	$(x, 1) (y, z)$
3	$(y, x) (z, 1)$
1	(x, z)

n	$\text{Der}_n(\Lambda(x, y), d)$
7	$(y, 1)$
4	$(x, 1)$
3	(y, x)
1	

where $H_*(\text{Der}(\Lambda(x, y))) = \mathbb{Q}\{(y, 1)\}$. By degree reason, any DGL-map

$$\psi : (\text{Der}(\Lambda(x, y, z), D) \rightarrow (\text{Der}(\Lambda(x, y), d)$$

is given by $\psi(y, 1) = a_1(y, 1)$, $\psi(x, 1) = a_2(x, 1)$, $\psi(y, z) = a_3(x, 1)$, $\psi(y, x) = a_4(y, x)$, $\psi(z, 1) = a_5(y, x)$ and $\psi_f(x, z) = 0$ for some $a_i \in \mathbb{Q}$.

From $(x, 1) = [(z, 1), (x, z)]$ and $(y, z) = [(x, z), (y, x)]$ we have $a_2 = 0$ and $a_3 = 0$, respectively. Then from $2(y, 1) = [(z, 1), (y, z)] + [(x, 1), (y, x)]$, we obtain $a_1 = 0$. Thus, $\|\psi\|$ is homotopic to the constant map.

Let $f : X \rightarrow Y$ be a map with a section s , i.e., there is a map $s : Y \rightarrow X$ with $f \circ s \sim id_Y$. Then there is a map $\psi_f : \text{aut}_1 X \rightarrow \text{aut}_1 Y$ with $\psi_f(g) := f \circ g \circ s$ for $g \in \text{aut}_1 X$. In general, this does not preserve the monoid structures.

Theorem 2.4 *If a $\pi_{\mathbb{Q}}$ -separable map f admits a section, there is a commutative diagram:*

$$\begin{array}{ccc}
 \pi_n(\text{aut}_1 X)_{\mathbb{Q}} & \xrightarrow{\pi_n(\psi_f)_{\mathbb{Q}}} & \pi_n(\text{aut}_1 Y)_{\mathbb{Q}} \\
 \downarrow \cong & & \downarrow \cong \\
 H_n(\text{Der}(\Lambda V \otimes \Lambda W, D)) & \xrightarrow{H_n(b_f)} & H_n(\text{Der}(\Lambda V, d)) \\
 \downarrow \cong & & \downarrow \cong \\
 \pi_n(\Omega \text{Baut}_1 X)_{\mathbb{Q}} & \xrightarrow{\pi_n(\Omega a_f)_{\mathbb{Q}}} & \pi_n(\Omega \text{Baut}_1 Y)_{\mathbb{Q}}.
 \end{array}$$

Proof The map $\pi_n(\psi_f) : \pi_n(\text{aut}_1 X) \rightarrow \pi_n(\text{aut}_1 Y)$ is given by $\pi_n(\psi_f)([\sigma]) = [\tau] := [f \circ \sigma \circ (s \times 1_{S^n})]$ in the following homotopy commutative diagram:

$$\begin{array}{ccccc}
 X \times S^n & \xleftarrow{s \times id_{S^n}} & Y \times S^n & & \\
 \downarrow \sigma & \swarrow i_X & \downarrow i_Y & \swarrow \tau & \\
 X & \xleftarrow{g} & X & \xleftarrow{s} & Y \\
 \downarrow f & \searrow & \downarrow & \searrow & \\
 X & \xrightarrow{f} & Y & &
 \end{array}$$

from adjointness. That is the pointed homotopy classes of maps $S^n \rightarrow \text{aut}_1 X = \text{map}(X, X; id_X)$ are in bijection with the homotopy classes of those maps $X \times S^n \rightarrow X$ that composed with the inclusion $i_X : X \rightarrow X \times S^n$ yield the identity [20, p. 43–44]. Let $M(Y) = (\Lambda V, d) \xrightarrow{i} (\Lambda V \otimes \Lambda W, D) \simeq M(X)$ be the model of f . We identify the rationalized map $\sigma_0 : (X \times S^n)_0 \rightarrow X_0$ to an element of $\text{Der}_n(\Lambda V \otimes \Lambda W)$ [24, p. 313] (cf. [20, Proposition 11]). Then there is a chain map

$$c_f : \text{Der}(\Lambda V \otimes \Lambda W, D) \rightarrow \text{Der}(\Lambda V, d)$$

given by $c_f(\sigma_0) = \text{proj}_V \circ \sigma_0 \circ i$. It is well defined, i.e., $\partial_Y \circ c_f = c_f \circ \partial_X$, from $DW \subset \Lambda V \otimes \Lambda^+ W$ [26] since it admits a section. (However, c_f is not a DGL-map in general.) Notice that

$$\pi_n(\psi_f)_{\mathbb{Q}}([\sigma_0]) \equiv H_n(c_f)([\sigma_0]).$$

If f is a $\pi_{\mathbb{Q}}$ -separable map, c_f is the DGL-map b_f . □

The following is obvious from the definition of b_f and useful:

Claim 2.5 For any $\pi_{\mathbb{Q}}$ -separable map $f : X \rightarrow Y$, we have $b_f(C) = 0$ and $b_f|_{\text{Der}(\Lambda V)} = id_{\text{Der}(\Lambda V)}$ for $\text{Der}(\Lambda V \otimes \Lambda W) = C \oplus \text{Der}(\Lambda V)$.

Theorem 2.6 For a $\pi_{\mathbb{Q}}$ -separable map $f : X \rightarrow Y$, let

$$\begin{array}{ccc} B_0 & \xrightarrow{h} & (\text{Baut}_1 X)_0 \\ g \downarrow & & \downarrow a_f \\ B'_0 & \xrightarrow{h'} & (\text{Baut}_1 Y)_0 \end{array}$$

be a commutative diagram. Then there exists a map between total spaces $k : E \rightarrow E'$ in the diagram:

$$\begin{array}{ccccc} X_0 & \xrightarrow{i} & E & \xrightarrow{p} & B_0 \\ f_0 \downarrow & & \downarrow k & & \downarrow g \\ Y_0 & \xrightarrow{i'} & E' & \xrightarrow{p'} & B'_0, \end{array}$$

where $k \circ i \simeq i' \circ f_0$ and $g \circ p = p' \circ k$. Here $p : E \rightarrow B_0$ and $p' : E' \rightarrow B'_0$ are induced by the rationalized classifying maps h and h' , respectively.

Proof Let $X \rightarrow E_{\infty}^X \xrightarrow{p_{\infty}^X} \text{Baut}_1 X$ and $Y \rightarrow E_{\infty}^Y \xrightarrow{p_{\infty}^Y} \text{Baut}_1 Y$ be the universal fibrations of X and Y , respectively. Let $C^*(\text{Der}(\Lambda V)) \otimes \Lambda V$, D_Y be the DGA-model of E_{∞}^Y and $C^*(\text{Der}(\Lambda V \otimes \Lambda W)) \otimes \Lambda V \otimes \Lambda W$, D_X be the DGA-model of E_{∞}^X . For a $\pi_{\mathbb{Q}}$ -separable map $f : X \rightarrow Y$, there exists a DGA-inclusion map ψ such that

$$\begin{array}{ccccc} C^*(\text{Der}(\Lambda V)) & \longrightarrow & C^*(\text{Der}(\Lambda V)) \otimes \Lambda V, D_Y & \longrightarrow & \Lambda V, d \\ C^*(b_f) \downarrow & & \downarrow \psi & & \downarrow i \\ C^*(\text{Der}(\Lambda V \otimes \Lambda W)) & \longrightarrow & C^*(\text{Der}(\Lambda V \otimes \Lambda W)) \otimes \Lambda V \otimes \Lambda W, D_X & \longrightarrow & \Lambda V \otimes \Lambda W, D \end{array}$$

is commutative from the universality. Indeed, $C^*(\text{Der}(\Lambda V)) \otimes \Lambda V$, D_Y is a sub-DGA of $C^*(\text{Der}(\Lambda V \otimes \Lambda W)) \otimes \Lambda V \otimes \Lambda W$, D_X from Claim 2.5 and $\max V \leq \min W$. Thus, there is a map $\tilde{a}_f := |\psi| : (E_{\infty}^X)_0 \rightarrow (E_{\infty}^Y)_0$ such that $(p_{\infty}^Y)_0 \circ \tilde{a}_f = a_f \circ (p_{\infty}^X)_0$. Since p' is the pull-back of $(p_{\infty}^Y)_0$ by h' , there exists a map $k : E \rightarrow E'$ such that

$$\begin{array}{ccccc} E & \xrightarrow{\tilde{h}} & (E_{\infty}^X)_0 & \xrightarrow{\tilde{a}_f} & (E_{\infty}^Y)_0 \\ \downarrow p & \searrow k & \downarrow (p_{\infty}^X)_0 & & \downarrow (p_{\infty}^Y)_0 \\ E' & \xrightarrow{\tilde{h}'} & (\text{Baut}_1 X)_0 & \xrightarrow{a_f} & (\text{Baut}_1 Y)_0 \\ \downarrow p' & & \downarrow h & & \downarrow h' \\ B_0 & \xrightarrow{h} & (\text{Baut}_1 X)_0 & \xrightarrow{a_f} & (\text{Baut}_1 Y)_0 \\ \downarrow g & & & & \\ B'_0 & \xrightarrow{h'} & (\text{Baut}_1 Y)_0 & & \end{array}$$

is commutative from the universality since $h' \circ g \circ p = a_f \circ h \circ p = a_f \circ (p_{\infty}^X)_0 \circ \tilde{h} = (p_{\infty}^Y)_0 \circ \tilde{a}_f \circ \tilde{h}$. □

Example 2.7 Let $X = K(\mathbb{Q}, n) \times K(\mathbb{Q}, 2n)$ and $Y = K(\mathbb{Q}, n)$ for some even integer n . Then $M(X) = \Lambda(x, y), 0$ and $M(Y) = \Lambda(z), 0$ with $|x| = |z| = n$ and $|y| = 2n$. Let a map $f : X \rightarrow Y$ be given by $M(f) : \Lambda(z) \rightarrow \Lambda(x, y)$ with $M(f)(z) = x$. The homotopy fibration of any $\pi_{\mathbb{Q}}$ -separable map is given by $\Lambda(z), 0 \rightarrow \Lambda(z, y), 0 \cong \Lambda(x, y), 0$ from the degree reason. Therefore, the DGL-map $\psi : \text{Der}\Lambda(x, y) \rightarrow \text{Der}\Lambda(z)$ such that $\psi((y, x)) = \psi((x, 1)) = (z, 1)$ is not DGL-homotopic to b_f from Claim 2.5.

Let $h : S_0^{n+1} \rightarrow (\text{Baut}_1 X)_0$ and $h' : S_0^{n+1} \rightarrow (\text{Baut}_1 Y)_0$ be given by $L(h) : \mathbb{L}(u) \rightarrow \text{Der}(\Lambda(x, y))$ with $|u| = n$, $L(h)(u) = (y, x)$ and $L(h') : \mathbb{L}(u) \rightarrow \text{Der}(\Lambda(z))$ with $L(h')(u) = (z, 1)$, respectively. Then the commutative diagram

$$\begin{array}{ccc} S_0^{n+1} & \xrightarrow{h} & (\text{Baut}_1 X)_0 \\ \downarrow = & & \downarrow ||\psi|| \\ S_0^{n+1} & \xrightarrow{h'} & (\text{Baut}_1 Y)_0 \end{array}$$

does not induce a map between total spaces $f' : E \rightarrow E'$ such that

$$\begin{array}{ccccc} X_0 & \longrightarrow & E & \xrightarrow{p} & S_0^{n+1} \\ \downarrow f_0 & & \downarrow f' & & \downarrow = \\ Y_0 & \longrightarrow & E' & \xrightarrow{p'} & S_0^{n+1} \end{array}$$

is homotopy commutative. Indeed, there does not exist a DGA-map $h : \Lambda(v, z), D' \rightarrow \Lambda(v, x, y), D$ with $D'z = v$, $Dy = vx$ and $Dx = 0$ such that

$$\begin{array}{ccccc} \Lambda v, 0 & \longrightarrow & \Lambda(v, z), D' & \xrightarrow{p} & \Lambda(z), 0 \\ \downarrow = & & \downarrow h & & \downarrow M(f) \\ \Lambda v, 0 & \longrightarrow & \Lambda(v, x, y), D & \xrightarrow{p'} & \Lambda(x, y), 0, \end{array}$$

where $|v| = n + 1$ is homotopy commutative since h cannot be a DGA-map from $Dh(z) = 0$ but $hD'(z) = v$.

3 When does a_f admit a section?

Let $f : X \rightarrow Y$ be a $\pi_{\mathbb{Q}}$ -separable map with homotopy fiber F_f and $\text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)$ the sub-DGL of $\text{Der}(\Lambda V \otimes \Lambda W)$ restricted to derivations out of ΛW .

Proposition 3.1 Let F_{a_f} be the homotopy fiber of a_f . Then the DGL-model of the fibration $\chi_f : F_{a_f} \xrightarrow{j} (\text{Baut}_1 X)_0 \xrightarrow{a_f} (\text{Baut}_1 Y)_0$ is given by

$$\text{Der}(\Lambda W, \Lambda V \otimes \Lambda W) \xrightarrow{\text{incl.}} \text{Der}(\Lambda V \otimes \Lambda W) \xrightarrow{b_f} \text{Der}(\Lambda V).$$

Proof Since b_f is surjective and $\text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)$ is $\text{Ker } b_f$, it follows from [25, VI.1.(3) Proposition]. $\square$

Let $L(F) = \oplus_{i \geq 0} L(F)_i$ be the degree decomposition of a DGL-model of a space F .

Theorem 3.2 $L(F_{a_f})_n \cong \oplus_{i-j=n} \text{Der}_i(\Lambda W) \otimes H^j(\Lambda V)$.

Proof A chain-map $\rho : \text{Der}_i(\Lambda W) \otimes H^j(\Lambda V) \rightarrow \text{Der}_i(\Lambda W, \Lambda W \otimes (\Lambda V)^j)$ is given by $\rho(\sigma \otimes [f])(w) := (-1)^{|w||j|} \sigma(w) \cdot f$ induced by an inclusion $H^j(\Lambda V) \rightarrow (\Lambda V)^j$. It is quasi-isomorphic, i.e., there is a decomposition $\text{Der}(\Lambda W, \Lambda W \otimes \Lambda V) = (\text{Der}(\Lambda W) \otimes H^*(\Lambda V)) \oplus C$ for a complex C of derivations with $H_*(C) = 0$. $\square$

The rational homotopy exact sequence of the strictly induced fibration χ_f :

$$\begin{aligned} \cdots \xrightarrow{j_\#} \pi_{n+2}(\text{Baut}_1 X)_{\mathbb{Q}} \xrightarrow{a_f^\#} \pi_{n+2}(\text{Baut}_1 Y)_{\mathbb{Q}} \xrightarrow{\delta_f} \\ \pi_{n+1}(F_{a_f})_{\mathbb{Q}} \xrightarrow{j_\#} \pi_{n+1}(\text{Baut}_1 X)_{\mathbb{Q}} \xrightarrow{a_f^\#} \pi_{n+1}(\text{Baut}_1 Y)_{\mathbb{Q}} \xrightarrow{\delta_f} \cdots \end{aligned}$$

is equivalent to the homology exact sequence:

$$\begin{aligned} \cdots \rightarrow H_{n+1}(\text{Der}(\Lambda V \otimes \Lambda W)) \xrightarrow{b_{f*}} H_{n+1}(\text{Der}(\Lambda V)) \xrightarrow{\delta_f} \\ H_n(\text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)) \rightarrow H_n(\text{Der}(\Lambda V \otimes \Lambda W)) \xrightarrow{b_{f*}} H_n(\text{Der}(\Lambda V)) \xrightarrow{\delta_f} \cdots \end{aligned}$$

Then we have the following from an ordinary chain complex property:

Claim 3.3 *The connecting map δ_f is given by $\delta_f([\sigma]) = [\tau]$ when $\partial_X(\sigma) = \tau$ for a ∂_Y -cycle σ of $\text{Der}(\Lambda V)$ and a ∂_X -cycle τ of $\text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)$.*

Recall that the following implications hold for a general fibration $\chi : F \rightarrow E \xrightarrow{p} B$ of simply connected spaces:

$$\chi \text{ is fibre-trivial} \Rightarrow p \text{ admits a section} \Rightarrow \chi \text{ is weakly trivial} \Leftrightarrow \delta = 0. \quad (*)$$

Here $\delta : \pi_*(B) \rightarrow \pi_{*-1}(F)$ is the connecting map of the homotopy exact sequence for χ . The following may be a characteristic phenomenon in our fibration χ_f .

Proposition 3.4 *a_f admits a section if and only if $\delta_f = 0$.*

Proof (if) Let the DGA-model of the fibration $\chi_f : F_{a_f} \xrightarrow{j} (\text{Baut}_1 X)_0 \xrightarrow{a_f} (\text{Baut}_1 Y)_0$ be given as the commutative diagram:

$$\begin{array}{ccccc} \Lambda U, d & \longrightarrow & \Lambda U \otimes \Lambda Z, D_2 & \longrightarrow & \Lambda Z, \overline{D}_2 \\ \rho_2|_{\Lambda U} \downarrow \simeq & & \rho_2 \downarrow \simeq & & \parallel \\ C^*(\text{Der} \Lambda V) & \longrightarrow & C^*(\text{Der} \Lambda V) \otimes \Lambda Z, D_1 & \longrightarrow & \Lambda Z, \overline{D}_1 \\ \parallel & & \rho_1 \downarrow \simeq & & \downarrow \simeq \\ C^*(\text{Der} \Lambda V) & \xrightarrow{C^*(b_f)} & C^*(\text{Der}(\Lambda V \otimes \Lambda W)), D & \xrightarrow{C^*(incl.)} & C^*(\text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)), D', \end{array}$$

where $M(\text{Baut}_1 Y) \cong (\Lambda U, d)$ with $U^{n+1} = H_n(\text{Der}(\Lambda V))$ and $M(F_{a_f}) \cong (\Lambda Z, \overline{D}_2)$ with $Z^{n+1} = H_n(\text{Der}(\Lambda W, \Lambda V \otimes \Lambda W))$. Here

$$(\Lambda U, d) \rightarrow (\Lambda U \otimes \Lambda Z, D_2) \rightarrow (\Lambda Z, \overline{D}_2)$$

is the model of χ_f . From the assumption χ_f is weakly equivalent, i.e., $M(\text{Baut}_1 X) \cong (\Lambda U \otimes \Lambda Z, D_2)$ as a minimal model. Let $D = d_1 + d_2$ as in §2. Notice that a linear component of any bracket representation of $\sigma \in \text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)$ is not contained in $[\text{Der}(\Lambda V), \text{Der}(\Lambda V)]$, where $[\ , \]$ is the Lie bracket. That means

$$d_2(s^{-1}\sigma^*) \in I(C^*(\text{Der}(\Lambda W, \Lambda V \otimes \Lambda W))) \quad (**).$$

Here $I(S)$ is the ideal in $C^*(\text{Der}(\Lambda V \otimes \Lambda W))$ generated by a basis of a vector space S . Let σ be a non-exact ∂_X -cycle of $\text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)$. Then

$$H^*(C^*(\text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)), d'_1)/H^* \cdot H^* \cong Z \ni [s^{-1}\sigma^*]$$

for $D' = d'_1 + d'_2$ as in §2. Since $\rho_1(D_1([s^{-1}\sigma^*]))$ and $d_2(s^{-1}\sigma^*)$ are D -cohomologous in $C^*(\text{Der}(\Lambda V \otimes \Lambda W))$, we can put

$$D_1([s^{-1}\sigma^*]) \in I(Z)$$

from (**), i.e., $D_1(Z) \subset C^*(\text{Der} \Lambda V) \otimes \Lambda^+ Z$. By ρ_2 , $D_2(Z) \subset \Lambda U \otimes \Lambda^+ Z$. Then we have done from [26]. (only if) It holds from the above implications (*). $\square$

Theorem 3.5 *If a $\pi_{\mathbb{Q}}$ -separable map $f : X \rightarrow Y$ is rationally fibre-trivial (i.e., $X_0 \sim (F_f)_0 \times Y_0$), a_f admits a section.*

Proof From the assumption and Claim 3.3, we have $\delta_f = 0$. Then it holds from Proposition 3.4. $\square$

Refer [19, page 292] for related topics. Conversely, when Y is an odd-sphere,

Theorem 3.6 *If a $\pi_{\mathbb{Q}}$ -separable map $f : X \rightarrow Y = S^{2n+1}$ is not rationally fibre-trivial, a_f does not admit a section. Furthermore, $a_f \sim *$ (the constant map).*

Proof Let $M(S^{2n+1}) = (\Lambda v, 0)$. Since there exists an element $w \in W$ such that $Dw \in \Lambda v \otimes \Lambda^+ W$ from the assumption, $\partial_X(v, 1) = \pm(w, \partial Dw/\partial v) + \dots \neq 0$ in $\text{Der}(\Lambda W)$. From Claim 3.3 δ_f is injective since $\delta_f([(v, 1)]) = [\pm(w, \partial Dw/\partial v) + \dots] \neq 0$. Then the former holds from Proposition 3.4. Furthermore, from the homotopy exact sequence, we have $a_{f\sharp} = 0$. Thus, the latter holds. $\square$

Example 3.7 (1) Let $S^5 \times S^7 \rightarrow X \rightarrow Y = S^3$ be a non-(fibre-)trivial $\pi_{\mathbb{Q}}$ -separable fibration given by the model

$$(\Lambda(v_1), 0) \rightarrow (\Lambda(v_1, w_1, w_2), D) \rightarrow (\Lambda(w_1, w_2), 0)$$

with $|v_1| = 3$, $|w_1| = 5$, $|w_2| = 7$, $Dw_1 = 0$ and $Dw_2 = v_1 w_1$. Then a_f does not admit a section from Theorem 3.6. Indeed $\delta_f : H_3(\text{Der}(\Lambda v')) \rightarrow H_2(\text{Der}(\Lambda(w_1, w_2), \Lambda(v_1, w_1, w_2)))$ is non-trivial from $\delta_f([(v_1, 1)]) = [(w_2, w_1)] \neq 0$.

(2) Let $S^5 \times S^7 \rightarrow X' \rightarrow Y'$ be a non-(fibre-)trivial $\pi_{\mathbb{Q}}$ -separable fibration given by the model

$$(\Lambda(v_1, v_2, v_3), d_{Y'}) \rightarrow (\Lambda(v_1, v_2, v_3, w_1, w_2), D') \rightarrow (\Lambda(w_1, w_2), 0)$$

with $|v_1| = |v_2| = 3$, $|v_3| = 5$, $|w_1| = 7$, $|w_2| = 9$, $d_{Y'}(v_1) = d_{Y'}(v_2) = 0$, $d_{Y'}(v_3) = v_1 v_2$, $D'w_1 = 0$ and $D'w_2 = v_1 w_1$. Then a_f admits a section from Proposition 3.4 since $\delta_f([(v_3, 1)]) = 0$ for $H_*(\text{Der}(\Lambda(v_1, v_2, v_3))) = \mathbb{Q}\{[(v_3, 1)]\}$. However, χ_f is not trivial from $[(v_3, 1), (w_2, v_2 v_3)] = (w_2, v_2)$. Indeed, then

$$\mathcal{D}(s^{-1}(w_2, v_2)^*) = d_2(s^{-1}(w_2, v_2)^*) = s^{-1}(v_3, 1)^* \cdot s^{-1}(w_2, v_2 v_3)^*$$

for $(C^*(\text{Der}(\Lambda(v_1, v_2, v_3, w_1, w_2)), \mathcal{D}))$ with $\mathcal{D} = d_1 + d_2$. Refer the proof of Proposition 3.4.

Proof of Theorem 1.7 Let $M(Y) = (\Lambda V, d) = (\mathbb{Q}[x_1, \dots, x_n] \otimes \Lambda(y_1, \dots, y_n), d)$ with $dx_i = 0$ and $dy_i = f_i$ for $i = 1, \dots, n$.

(if) Let $M(Y) = (\Lambda V, d) \rightarrow (\Lambda V \otimes \Lambda W, D)$ be the model of f . From the regularity of $f_1, \dots, f_n$, $\text{Im } D \subset \mathbb{Q}[x_1, \dots, x_n] \otimes \Lambda W$. Thus,

$$\partial_X(x_i, h_i) = \sum_{j=1}^n (y_j, (\partial f_j / \partial x_i) \cdot h_i) + \theta \quad \text{and} \quad \partial_X(y_i, h_i) = 0 \quad (i = 1, \dots, n)$$

for any $h_i \in \Lambda V^{\text{even}} = \mathbb{Q}[x_1, \dots, x_n]$ with suitable degree and some $\theta \in \text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)$. Then we have $\delta_f = 0$ from Claim 3.3 since $H_{\text{even}}(\text{Der } M(Y)) = 0$ [18] from the assumption. Then a_f admits a section from Proposition 3.4. Furthermore, from Theorem 3.2, we can suppose that the Lie bracket decomposition of an element of $\text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)$ does not have an element of $\text{Der}(\Lambda V)$ as a factor since $\text{Der } H^*(Y; \mathbb{Q}) = 0$ [18] again. Thus, we have

$$D_2 = d \otimes 1 \pm 1 \otimes \overline{D}_2$$

for the Sullivan minimal model $(\Lambda U, d) \rightarrow (\Lambda U \otimes \Lambda Z, D_2) \rightarrow (\Lambda Z, \overline{D}_2)$ of χ_f (in the proof of Proposition 3.4).

(only if) Suppose that Y does not satisfies the Halperin conjecture, i.e., there is a non-zero element $[\sum_i (x_i, h_i) + \sum_j (y_j, g_j)] \in H_{2m}(\text{Der } M(Y))$ for $h_i \in \mathbb{Q}[x_1, \dots, x_n]$, $g_j \in \Lambda V$ and some m [18]. Let $(S^a \times S^b \simeq_0) F \rightarrow X \xrightarrow{f} Y$ be a fibration of the model:

$$(\Lambda V, d) \rightarrow (\Lambda V \otimes \Lambda(w_1, w_2, w_3), D) \rightarrow (\Lambda(w_1, w_2, w_3), d_F),$$

where $|w_1| = a$ is even and $|w_2| = b$ is odd with $b - a = |x_k| - 1$ for some k , $d_F w_1 = d_F w_2 = 0$, $d_F w_3 = w_1^2$, $Dw_1 = 0$ and $Dw_2 = x_k w_1$. When h_k is not d_Y -exact, the element $h_k w_1$ is not D -exact. Then

$$\delta_f \left(\left[\sum_i (x_i, h_i) + \sum_j (y_j, g_j) \right] \right) = [(w_2, h_k w_1)] \neq 0$$

for $\delta_f : H_{2m}(\text{Der } \Lambda V) \rightarrow H_{2m-1}(\text{Der}(\Lambda(w_1, w_2, w_3), \Lambda V \otimes \Lambda(w_1, w_2, w_3)))$ from Claim 3.3. In particular, χ_f is not fibre-trivial. $\square$

Example 3.8 Let Y be the homogeneous space $SU(6)/SU(3) \times SU(3)$. Then Y is a pure space but not an F_0 -space since $\text{rank } SU(6) = 5 > 4 = \text{rank } (SU(3) \times SU(3))$. Let $\xi : (S^{11} \times S^{23} \simeq_0) F \rightarrow X \xrightarrow{f} Y$ be a fibration whose relative model is given as

$$(\Lambda(x_1, x_2, y_1, y_2, y_3), d_Y) \rightarrow (\Lambda(x_1, x_2, y_1, y_2, y_3) \otimes \Lambda(w_1, w_2), D) \rightarrow (\Lambda(w_1, w_2), 0),$$

where $|x_1| = 4$, $|x_2| = 6$, $|y_1| = 7$, $|y_2| = 9$, $|y_3| = 11$, $|w_1| = 11$, $|w_2| = 23$, $d_Y y_1 = x_1^2$, $d_Y y_2 = x_1 x_2$, $d_Y y_3 = x_2^2$, $Dw_1 = 0$ and $Dw_2 = (x_1 y_2 - x_2 y_1) w_1$. Then $\partial_X((y_1, 1)) = (w_2, x_2 w_1)$, i.e., $\delta_f[(y_1, 1)] = [(w_2, x_2 w_1)] \neq 0$ from Claim 3.3. In particular, χ_f is not trivial. Refer [19, Example 1.14(2)] for the Sullivan minimal model of $\text{Baut}_1 Y$.

4 The obstruction class for a lifting

Let $L(B) = (L(B), \partial_B)$ be the Quillen model of a simply connected CW complex B of finite type. Then $L(B \cup_\alpha e^N)$ is given by $L(B) \coprod \mathbb{L}(u), \partial_\alpha$ where $|u| = N - 1$, $\partial_\alpha|_{L(B)} = \partial_B$ and $\partial_\alpha(u) \in L(B)$ [25, Proposition III.3.(6)].

Theorem 4.1 For a $\pi_{\mathbb{Q}}$ -separable map $f : X \rightarrow Y$, let

$$\begin{array}{ccc} B & \xrightarrow{h_X} & (\text{Baut}_1 X)_0 \\ i \downarrow & & \downarrow a_f \\ B \cup_\alpha e^N & \xrightarrow{h_Y} & (\text{Baut}_1 Y)_0 \end{array}$$

be a commutative diagram. Then there is a lift h such that

$$\begin{array}{ccc} B & \xrightarrow{h_X} & (\text{Baut}_1 X)_0 \\ i \downarrow & \nearrow h & \downarrow a_f \\ B \cup_\alpha e^N & \xrightarrow{h_Y} & (\text{Baut}_1 Y)_0 \end{array}$$

is commutative if and only if

$$\mathcal{O}_\alpha(h_X, h_Y) := [\tau(h_Y(u)) - h_X''(\partial_\alpha(u))] = 0$$

in $H_{N-2}(\text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)) = \pi_{N-1}(F_{a_f})_{\mathbb{Q}}$ for the DGL-commutative diagram

$$\begin{array}{ccc} L(B) & \xrightarrow{h_X} & \text{Der}(\Lambda V \otimes \Lambda W), \partial_X \\ i \downarrow & & \downarrow b_f \\ L(B) \coprod \mathbb{L}(u), \partial_\alpha & \xrightarrow{h_Y} & \text{Der}(\Lambda V), \partial_Y \end{array}$$

with

- $\partial_X \mid_{\text{Der}(\Lambda V, \Lambda V \otimes \Lambda W)} = \partial_Y + \tau$ and $\partial_X \mid_{\text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)} = \tau$ for some $\tau : \text{Der}_*(\Lambda V \otimes \Lambda W) \rightarrow \text{Der}_{*-1}(\Lambda W, \Lambda V \otimes \Lambda W)$ and
- $h_X = h'_X + h''_X$ where $h'_X(b) \in \text{Der}(\Lambda V)$ and $h''_X(b) \in \text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)$ for $b \in L(B)$.

Proof Since $b_f \circ h_X = h_Y \circ i$ and h_Y is a DGL-map,

$$h'_X \partial_\alpha(u) = h_Y \partial_\alpha(u) = \partial_Y h_Y(u) \quad (1)$$

in $\text{Der}(\Lambda V)$. Notice that the obstruction element $\partial_X(h_Y(u)) - h_X(\partial_\alpha(u))$ is a ∂_X -cycle in $\text{Der}(\Lambda V \otimes \Lambda W)$. Therefore, $\tau(h_Y(u)) - h''_X(\partial_\alpha(u))$ is a ∂_X -cycle in $\text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)$ from (1).

(if) Suppose that $\mathcal{O}_\alpha(h_X, h_Y) = 0$. Then there is an element $q \in \text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)$ such that

$$\partial_X(q) = \tau(h_Y(u)) - h''_X(\partial_\alpha(u)). \quad (2)$$

Let

$$h \mid_{L(B)} := h_X \quad \text{and} \quad h(u) := h_Y(u) - q.$$

Then h is a DGL-map since

$$\begin{aligned} \partial_X(h(u)) &= \partial_X(h_Y(u)) - \partial_X(q) \\ &= (h'_X \partial_\alpha(u) + \tau h_Y(u)) - (\tau h_Y(u) - h''_X \partial_\alpha(u)) \\ &= h'_X \partial_\alpha(u) + h''_X \partial_\alpha(u) = h_X(\partial_\alpha(u)) = h(\partial_\alpha(u)) \end{aligned}$$

from (1) and (2). Furthermore,

$$\begin{array}{ccc} L(B) & \xrightarrow{h_X} & \text{Der}(\Lambda V \otimes \Lambda W), \partial_X \\ \downarrow i & \nearrow h & \downarrow b_f \\ L(B) \coprod \mathbb{L}(u), \partial_\alpha & \xrightarrow{h_Y} & \text{Der}(\Lambda V), \partial_Y \end{array} \quad (*)$$

is commutative since $b_f(q) = 0$. Thus, the (if)-part holds from the special realization of (*).

(only if) Suppose that there exists a map h such that (*) is commutative. Since h is a DGL-map,

$$\partial_X(h(u)) = h(\partial_\alpha(u)) \quad (3)$$

in $\text{Der}(\Lambda V \otimes \Lambda W)$ and

$$h''_X \partial_\alpha(u) = \tau h(u) \quad (4)$$

from (1) and (3). Furthermore,

$$\tau h(u) \sim \tau h_Y(u) \quad (5)$$

in $\text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)$. Here $\sim$ means “homologous”. Indeed, (5) follows since

$$h(u) = h_Y(u) + x$$

for some element $x \in \text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)$ from $b_f \circ h = h_Y$ and then since

$$\tau h(u) = \tau(h_Y(u) + x) = \tau h_Y(u) + \partial_X(x).$$

Thus, we obtain that $\mathcal{O}_\alpha(h_X, h_Y) = [\tau(h_Y(u)) - h''_X(\partial_\alpha(u))] = 0$ from (4) and (5). $\square$

From Theorem 3.2, we have

Corollary 4.2 *If $\pi_{\geq N-1}(\text{Baut}_1 F_f)_{\mathbb{Q}} = 0$ for the homotopy fiber F_f of f , there exists a lift h for the pair (h_X, h_Y) of above.*

Example 4.3 Let $B = S^2 = \mathbb{C}P^1$. Let $S^3 \times S^5 \rightarrow X \xrightarrow{f} Y = S^3$ be the fibration given by the model

$$(\Lambda(v), 0) \rightarrow (\Lambda(v, w_1, w_2), D) \rightarrow (\Lambda(w_1, w_2), 0)$$

with $|v| = |w_1| = 3$, $|w_2| = 5$, $Dw_1 = 0$ and $Dw_2 = vw_1$. Let $L(\mathbb{C}P^2) = L(B \cup_\alpha e^4) = (\mathbb{L}(u_1, u_2), \partial)$ with $|u_1| = 1$, $|u_2| = 3$, $\partial u_1 = 0$ and $\partial u_2 = [u_1, u_1]$ [25]. Let

$$\begin{array}{ccc} S^2 & \xrightarrow{h_X} & (\text{Baut}_1 X)_0 \\ i \downarrow & & \downarrow a_f \\ S^2 \cup_\alpha e^4 & \xrightarrow{h_Y} & (\text{Baut}_1 Y)_0 \end{array}$$

be a commutative diagram given by the DGL-model

$$\begin{array}{ccc} (\mathbb{L}(u_1), 0) & \xrightarrow{h_X} & (\text{Der}(\Lambda(v, w_1, w_2)), \partial_X) \\ i \downarrow & & \downarrow b_f \\ (\mathbb{L}(u_1, u_2), \partial) & \xrightarrow{h_Y} & (\text{Der}(\Lambda(v)), 0) \end{array}$$

by $h_X(u_1) = h_Y(u_1) = 0$ and $h_Y(u_2) = (v, 1)$. Then $\mathcal{O}_\alpha(h_X, h_Y) \neq 0$ in $H_2(\text{Der}(\Lambda(w_1, w_2), \Lambda(v, w_1, w_2)))$ since

$$\tau h_Y(u_2) = \partial_X(v, 1) = (w_2, w_1) \approx 0 = h_X''([u_1, u_1]) = h_X''(\partial_\alpha(u_2)).$$

Thus $h_Y : \mathbb{C}P^2 \rightarrow (\text{Baut}_1 Y)_0$ cannot lift to $h : \mathbb{C}P^2 \rightarrow (\text{Baut}_1 X)_0$. Note that h_Y is extended to $\mathbb{C}P^\infty \rightarrow (\text{Baut}_1 Y)_0$. Since $BS^1 = \mathbb{C}P^\infty$, we obtain that any free S^1 -action on Y cannot lift to X .

5 An application to lifting actions

Let BG and EG be the classifying space and the universal space of a compact connected Lie group G of rank $G = r$, respectively. If G acts on a space Y by $\mu : G \times Y \rightarrow Y$, there is the Borel fibration

$$Y \xrightarrow{i} EG \times_G^\mu Y \rightarrow BG,$$

where the Borel space $EG \times_G^\mu Y$ (or simply $EG \times_G Y$) is the orbit space of the diagonal action $g(e, y) = (eg^{-1}, gy)$ on the product $EG \times Y$. It is rationally given by the KS extension (model)

$$(\mathbb{Q}[t_1, \dots, t_r], 0) \rightarrow (\mathbb{Q}[t_1, \dots, t_r] \otimes \Lambda V, D_\mu) \rightarrow (\Lambda V, d) = M(Y) \quad (*),$$

where $|t_i|$ are even for $i = 1, \dots, r$, $D_\mu(t_i) = 0$ and $D_\mu(v) \equiv d(v)$ modulo the ideal $(t_1, \dots, t_r)$ for $v \in V$.

Recall the lifting theorem of D. H. Gottlieb:

Theorem 5.1 [11, Theorem 1] *Let a topological group G acts on a space Y . A fibration $X \xrightarrow{f} Y$ is fibre homotopy equivalent to a G -fibration if and only if it is fibre homotopy equivalent to the pull-back of a fibration over $EG \times_G Y$ induced by the inclusion $i : Y \rightarrow EG \times_G Y$.*

Proof of Theorem 1.10. Let $h_Y : BG \rightarrow (\text{Baut}_1 Y)_0$ be the rationalization of the classifying map of the Borel fibration $Y \xrightarrow{i} EG \times_G^\mu Y \rightarrow BG$ of the action $\mu : G \times Y \rightarrow Y$. Let B^n be the n -skeleton of a CW complex B . From Theorem 1.8, there is a lift h_X^α such that

$$\begin{array}{ccc} (BG)^n & \xrightarrow{h_X^n} & (\text{Baut}_1 X)_0 \\ \downarrow & \nearrow h_X^\alpha & \downarrow a_f \\ (BG)^n \cup_\alpha \vee_i e_i^{n+1} & \xrightarrow{h_Y^\alpha} & (\text{Baut}_1 Y)_0 \end{array}$$

is commutative for all n and attaching α since $\mathcal{O}_\alpha(h_X^n, h_Y^\alpha) = 0$. Indeed, $\pi_{\text{odd}}(\text{Baut}_1 f)_\mathbb{Q} = 0$ and $L(BG)$ is oddly graded since $H^*(BG; \mathbb{Q})$ is evenly graded. Thus, we have the commutative diagram:

$$\begin{array}{ccc} BG & \xrightarrow{h_X} & (\text{Baut}_1 X)_0 \\ \parallel & & \downarrow a_f \\ BG & \xrightarrow{h_Y} & (\text{Baut}_1 Y)_0. \end{array}$$

From Theorem 2.6, there is a commutative diagram:

$$\begin{array}{ccc} E & \longrightarrow & BG_0 \\ \downarrow g & & \parallel \\ (EG \times_G Y)_0 & \longrightarrow & BG_0 \end{array}$$

for some space E . Let $g' : E' \rightarrow EG \times_G Y$ be the pull-back of g by the rationalization l_0 and $f' : X' \rightarrow Y$ be the pull-back of g' by i

$$\begin{array}{ccccccc} X' & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & BG_0 \\ \downarrow f' & & \downarrow g' & & \downarrow g & & \parallel \\ Y & \xrightarrow{i} & EG \times_G Y & \xrightarrow{l_0} & (EG \times_G Y)_0 & \longrightarrow & BG_0. \end{array}$$

Notice that the model is given by the DGA-commutative diagram:

$$\begin{array}{ccccccc} R & \longrightarrow & R \otimes \Delta V & \xrightarrow{=} & R \otimes \Delta V & \longrightarrow & \Delta V \\ \parallel & & \downarrow M(g) & & \downarrow M(g') & & \downarrow M(f') \\ R & \longrightarrow & R \otimes \Delta V \otimes \Delta W & \xrightarrow{=} & R \otimes \Delta V \otimes \Delta W & \longrightarrow & \Delta V \otimes \Delta W \end{array}$$

for $R := H^*(BG; \mathbb{Q}) = \mathbb{Q}[t_1, \dots, t_r]$. Notice that the third square is given by the push-out [7, Proposition 15.8]. Thus, from Theorem 5.1, we obtain the commutative diagram

$$\begin{array}{ccccccc} G \times X' & \xrightarrow{\exists} & X' & \xrightarrow{l_0} & X'_0 & \xrightarrow{\cong} & X_0 \\ \downarrow f' \times id_G & & \downarrow f' & & \downarrow f'_0 & \nearrow f_0 & \\ G \times Y & \xrightarrow{\mu} & Y & \xrightarrow{l_0} & Y_0 & & \end{array}$$

since $M(X') \cong \Delta V \otimes \Delta W = M(X)$. $\square$

If the r -torus T^r acts on a space Y , $|t_1| = \dots = |t_r| = 2$ in (*).

Proposition 5.2 [13, Proposition 4.2] Suppose that Y is a simply connected CW-complex with $\dim H^*(Y; \mathbb{Q}) < \infty$. Put $M(Y) = (\Delta V, d)$. Then $r_0(Y) \geq r$ if and only if there is a KS extension (*) satisfying $\dim H^*(\mathbb{Q}[t_1, \dots, t_r] \otimes \Delta V, D) < \infty$. Moreover, if $r_0(Y) \geq r$, then T^r acts freely on a finite complex Y' that has the same rational homotopy type as Y and $M(ET^r \times_{T^r} Y') \cong (\mathbb{Q}[t_1, \dots, t_r] \otimes \Delta V, D)$.

Proof of Corollary 1.12. Let $r_0(Y) = r$. Notice that $L(BT^r)$ is oddly generated since $H^*(BT^r; \mathbb{Q}) = \mathbb{Q}[t_1, \dots, t_r]$. Since $\pi_{\text{odd}}(\text{Baut}_1 f)_\mathbb{Q} = 0$, there exists a lift $(BT^r)_0 \rightarrow (\text{Baut}_1 X)_0$ from Theorem 1.8. Then we have the homotopy commutative diagram:

$$\begin{array}{ccccccc} (F_f)_0 & \longrightarrow & X_0 & \xrightarrow{f_0} & Y_0 & \xrightarrow{\cong} & Y'_0 \\ \downarrow \cong & & \downarrow & & \downarrow & & \downarrow \\ F_g & \longrightarrow & \tilde{E} & \xrightarrow{\exists g} & E & \xrightarrow{\cong} & (ET^r \times_{T^r} Y')_0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \bullet & \longrightarrow & (BT^r)_0 & = & (BT^r)_0 & = & (BT^r)_0 \end{array}$$

from Theorem 2.6. Here $\bullet$ is the one point space. We have $\dim H^*(\tilde{E}; \mathbb{Q}) < \infty$ since $\dim H^*(F_g; \mathbb{Q}) < \infty$ and $\dim H^*(E; \mathbb{Q}) = \dim H^*(ET^r \times_{T^r} Y; \mathbb{Q}) < \infty$ for the fibration $F_g \rightarrow \tilde{E} \rightarrow E$. Thus there is a free T^r -action on X' with $X'_0 \simeq X_0$ and $\tilde{E} \simeq (ET^r \times_{T^r} X')_0$ from Proposition 5.2. Thus, we have $r_0(X) \geq r$. $\square$

Example 5.3 Let $S^5 \rightarrow X \xrightarrow{f} Y$ be a rationally non-trivial fibration given by the model

$$(\Lambda V, d_Y) = (\Lambda(v_1, v_2, v_3, v_4, v_5), d_Y) \rightarrow (\Lambda(v_1, v_2, v_3, v_4, v_5, w), D)$$

with $|v_1| = |v_2| = 2$, $|v_3| = |v_4| = |v_5| = |w| = 5$, $d_Y(v_1) = d_Y(v_2) = 0$, $d_Y(v_3) = v_1^3$, $d_Y(v_4) = v_1^2 v_2$, $d_Y(v_5) = v_2^3$ and $D(w) = v_1 v_2^2$. Then

$$\pi_{\text{odd}}(\text{Baut}_1 f)_{\mathbb{Q}} \cong H_{\text{even}}(\text{Der}_{\Lambda V}(\Lambda V \otimes \Lambda(w))) = 0$$

since there is no element of odd-degree < 5 in ΛV . Therefore, $r_0(Y) \leq r_0(X)$. Indeed, we can directly check that $r_0(Y) = 1$ and $r_0(X) = 2$.

On the other hand, let $S^5 \rightarrow X \xrightarrow{f} Y = S^3 \times S^3$ be a rationally non-trivial fibration. Then the model is given by

$$(\Lambda(v_1, v_2), 0) \rightarrow (\Lambda(v_1, v_2, w), D)$$

with $|v_1| = |v_2| = 3$, $|w| = 5$, and $D(w) = v_1 v_2$. Then

$$\pi_3(\text{Baut}_1 f)_{\mathbb{Q}} \cong H_2(\text{Der}_{\Lambda V}(\Lambda V \otimes \Lambda(w))) = \mathbb{Q}\{(w, v_1)\} \oplus \mathbb{Q}\{(w, v_2)\} \neq 0$$

and $r_0(Y) = 2 > 1 = r_0(X)$.

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