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Common fixed point iterations for a class of multi-valued mappings in $CAT(0)$ spaces

Received: 21 July 2019 / Accepted: 20 July 2020 / Published online: 5 September 2020
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Abstract In this work, we propose an iterative scheme to approach common fixed point(s) of a finite family of generalized multi-valued nonexpansive mappings in a $CAT(0)$ space. We establish and prove convergence theorems for the algorithm. The results are new and interesting in the theory of $CAT(0)$ spaces and are the analogues of corresponding ones in uniformly convex Banach spaces and Hilbert spaces.

Mathematics Subject Classification 47H09 · 47H10

1 Introduction

The theory of fixed point for multi-valued mappings has become of great interest to many researchers, see, for instance, [3, 4, 8, 11, 13, 15] and references therein. This concept contributes significantly in convex approximation, fractals, optimal control, digital imaging, and economics.

The following statements are quoted from [26]: “Sastry and Babu [21] showed that under certain conditions, Mann and Ishikawa iterative algorithms for a multi-valued nonexpansive mapping with a fixed point x converge to a fixed point y of the mapping. Panyanak [18] extended their results to a uniformly convex Banach space. Song and Wang [23] improved the results of Panyanak [18]. Abbas et al. [1] proposed a one-step iterative scheme to find a common fixed point of two multi-valued nonexpansive mappings in a uniformly convex Banach space. Recently, Uddin et al. [26] discovered a few gaps in the scheme proposed in [1] and came up with a new one-step iterative scheme for fixed points of two multi-valued nonexpansive mappings in $CAT(0)$ space and removed the gaps found in [1]”.

Motivated by [1, 26], we recommend a new scheme to approximate common fixed point(s) of a finite family of generalized multi-valued nonexpansive mappings in $CAT(0)$ spaces.

2 Preliminary lemmas

We refer to [2] for the following definitions. “Suppose (M, d) is a metric space and $a, b \in M$.” A geodesic path from a to b is a mapping $s : [0, d(a, b)] \rightarrow X$, such that $s(0) = a$, $s(d(a, b)) = b$ and $d(s(r), s(r')) = |r - r'|$ for all $r, r' \in [0, d(a, b)]$. A geodesic segment is defined as the image of the geodesic path. If every pair of points in M is joined by a unique geodesic segment, then the space is known as unique geodesic metric

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space. A geodesic triangle is denoted by $\Delta(a, b, c)$ in a geodesic metric space (M, d) that consists of three points $a, b, c \in M$ (the vertices of the geodesic triangle Δ) and three geodesic segments among these points. A comparison triangle for this geodesic triangle $\Delta(a, b, c)$ in M is a triangle $\bar{\Delta}(\bar{a}, \bar{b}, \bar{c})$ in $\mathbb{R}^2$, such that $d_{\mathbb{R}^2}(\bar{i}, \bar{j}) = d(i, j)$ for $i, j \in \{a, b, c\}$.

Definition 2.1 [2] A geodesic space M is a $CAT(0)$ space if, for each pair $(\Delta, \bar{\Delta}) \in X \times \mathbb{R}^2$, the following inequality:

$$d_{\mathbb{R}^2}(\bar{a}, \bar{b}) \geq d(a, b)$$

holds for all $\bar{a}, \bar{b} \in \bar{\Delta}$ and $a, b \in \Delta$.

Definition 2.2 [2] The geodesic segment from a to b is denoted by $[a, b]$, that is:

$$[a, b] = \{\gamma a \oplus (1 - \gamma)b : \gamma \in [0, 1]\}.$$

For $\gamma \in [0, 1]$, we write $\gamma a \oplus (1 - \gamma)b$ for the unique point c on $[a, b]$, such that:

$$d(a, c) = (1 - \gamma)d(a, b) \text{ and } d(b, c) = \gamma d(a, b).$$

Let M be a $CAT(0)$ space. A subset C of M is convex if $[a, b] \subset C$ for all $a, b \in C$.

Definition 2.3 [13] For a fixed $x \in M$ and a bounded sequence $\{x_n\} \subset M$, we define:

$$r(x, \{x_n\}) = \limsup_{n \rightarrow \infty} d(x_n, x).$$

The asymptotic radius of $\{x_n\}$ is defined by:

$$r(\{x_n\}) = \inf_{x \in M} r(x, \{x_n\});$$

the asymptotic radius of $\{x_n\}$ with respect to $C \subset M$ is denoted by $r_C(\{x_n\})$ and is defined by:

$$r_C(\{x_n\}) = \inf_{x \in C} r(x, \{x_n\}).$$

The asymptotic center of $\{x_n\}$ is denoted by $A(\{x_n\})$ and is given by the set:

$$A(\{x_n\}) = \{\bar{x} \in X : r(\bar{x}, \{x_n\}) = r(\{x_n\})\}.$$

The asymptotic center of $\{x_n\}$ with respect to $C \subset M$ is the set:

$$A_C(\{x_n\}) = \{\bar{x} \in C : r(\bar{x}, \{x_n\}) = r(\{x_n\})\}.$$

Definition 2.4 [14] A sequence $\{x_n\}$ in M is said to be Δ -convergent to the point $x \in M$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. We call x the Δ -limit of $\{x_n\}$ and express this limit as $\Delta - \lim_n x_n = x$.

It has been observed in [12] that “if $\{x_n\} \subset M$ and $\{x_n\}$ is Δ -convergent to $x \in M$ and $\bar{x} (\neq x) \in M$, then:

$$\limsup_{n \rightarrow \infty} d(x_n, x) < \limsup_{n \rightarrow \infty} d(x_n, \bar{x}).”$$

As mentioned in [12], “this concludes that $CAT(0)$ space satisfies a condition known as the Opial’s property in Banach spaces”.

Let C be a nonempty subset of a metric space (M, d) . The set C is called proximal if for each $x \in M$, there exists $\bar{x} \in C$, such that $d(x, \bar{x}) = \text{dist}(x, C)$, where $\text{dist}(x, C) = \inf_{z \in C} d(x, z)$. We denote the family of all bounded proximal subsets of C by $\mathcal{P}(C)$.

Let $\mathcal{CB}(C)$ and $\mathcal{K}(C)$ be families of nonempty, closed, and bounded subsets of C and nonempty compact subsets of C , respectively. The Hausdorff metric H on $\mathcal{CB}(C)$ is defined as:

$$H(D, \bar{D}) = \max \left\{ \sup_{x \in \bar{D}} \text{dist}(x, D), \sup_{\bar{x} \in D} \text{dist}(\bar{x}, \bar{D}) \right\}, \text{ where } D, \bar{D} \in \mathcal{CB}(C).$$

An element $p \in C$ is a fixed point of $S : C \rightarrow \mathcal{CB}(C)$ if $p \in Sp$. The set of all fixed points of S is denoted by $F(S)$.

Definition 2.5 [13] A mapping $S : C \rightarrow \mathcal{CB}(C)$ is a multi-valued nonexpansive if $H(Sx, Sy) \leq d(x, y)$ for all $x, y \in C$.

Definition 2.6 [22] A mapping $S : C \rightarrow \mathcal{CB}(C)$ is a multi-valued quasi-nonexpansive if $H(Sx, Sp) \leq d(x, p)$ for all $x \in C, p \in F(S)$.

Definition 2.7 [21] The mapping $S : C \rightarrow \mathcal{CB}(C)$ is called generalized multi-valued nonexpansive if

$$H(Sx, Sy) \leq a_1 d(x, y) + a_2 [\text{dist}(x, Sx) + \text{dist}(y, Sy)] + a_3 [\text{dist}(x, Sy) + \text{dist}(y, Sx)] \quad (2.1)$$

for all $x, y \in C$ where $a_1 + 2a_2 + 2a_3 \leq 1$.

Example 2.8 Let $G : [0, 3] \rightarrow \mathcal{CB}([0, 3])$ be defined as under:

$$G(x) = \begin{cases} [0, \frac{x}{3}] & \text{if } x \neq 3 \\ \{1.2\} & \text{if } x = 3. \end{cases}$$

Then, G is a generalized nonexpansive mapping with $F(G) = \{0\}$.

Verification: We first show that G is nonexpansive ($a_1 = 1, a_2 = 0 = a_3$).

Case 1. If $x = y \neq 3$, then $H(Gx, Gy) = \|\frac{x}{3} - \frac{y}{3}\| \leq \|x - y\|$.

Case 2. If $x = 3 = y$, then $H(Gx, Gy) = 0 = \|x - y\|$.

Case 3. If $x \neq 3$ and $y = 3$, then $H(Gx, Gy) = 1.2 - \frac{x}{3} \leq 3 - x = \|x - y\|$. This relation holds if and only if $x \leq 2.7$. Hence, G is a nonexpansive mapping.

For generalized nonexpansivity of G , we further investigate two cases.

Case 4. If $x \in [0, 1.8)$ and $y \in [0, 3)$, then this case is similar to Case 1 as mentioned above.

Case 5. If $x \in [1.8, 3)$ and $y = 3$ with $a_1 = \frac{1}{8} = a_2$ and $a_3 = \frac{1}{4}$, then we compute:

$$H(Gx, Gy) = \left\| \frac{2x}{3} - 1.2 \right\| = \begin{cases} 0 & \text{if } x = 1.8 \\ < 0.8 & \text{if } x < 3. \end{cases} \quad (2.2)$$

The other estimates are as follows:

1. $\|x - y\| = \|x - 3\|$;
2. $\text{dist}(x, Gx) = \left\| \frac{2x}{3} \right\|$;
3. $\text{dist}(y, Gy) = \|3 - 1.2\|$;
4. $\text{dist}(x, Gy) = \|x - 1.2\|$;
5. $\text{dist}(y, Gx) = \|3 - [0, 1]\|$.

Now, observe that:

$$\begin{aligned} & a_1 \|x - y\| + a_2 (\text{dist}(x, Gx) + \text{dist}(y, Gy)) + a_3 (\text{dist}(x, Gy) + \text{dist}(y, Gx)) \\ &= \frac{1}{8} \|x - 3\| + \frac{1}{8} \left(\left\| \frac{2x}{3} \right\| + 1.8 \right) + \frac{1}{4} (\|x - 1.2\| + \|3 - [0, 1]\|) \\ &\geq \frac{1}{8} \|1.8 - 3\| + \frac{1}{8} \left(\left\| \frac{2(1.8)}{3} \right\| + 1.8 \right) + \frac{1}{4} (\|1.8 - 1.2\| + 2) \\ &= \frac{1.2}{8} + \frac{3}{8} + \frac{2.6}{4} \\ &= 1.175. \end{aligned} \quad (2.3)$$

It follows from (2.2) and (2.3) that:

$$H(Gx, Gy) \leq a_1 \|x - y\| + a_2 (\text{dist}(x, Gx) + \text{dist}(y, Gy)) + a_3 (\text{dist}(x, Gy) + \text{dist}(y, Gx)).$$

Hence, G is a generalized nonexpansive mapping.

If we choose $a_1 = 1$ and $a_2 = 0 = a_3$ in (2.1), it becomes multi-valued nonexpansive, but the converse does not hold, in general. Furthermore, we can verify that every generalized multi-valued nonexpansive mapping having at least one fixed point is multi-valued quasi-nonexpansive. For $x \in C$, $p \in F(S)$, we obtain:

$$\begin{aligned} H(Sx, Sp) &\leq a_1 d(x, p) + a_2 [\text{dist}(x, Sx) + \text{dist}(p, Sp)] \\ &\quad + a_3 [\text{dist}(x, Sp) + \text{dist}(p, Sx)] \\ &\leq a_1 d(x, p) + a_2 [d(x, p) + \text{dist}(p, Sx)] + a_3 [d(x, p) + \text{dist}(p, Sx)] \\ &\leq (a_1 + a_2 + a_3) d(x, p) + (a_2 + a_3) H(Sx, Sp). \end{aligned}$$

That is:

$$H(Sx, Sp) \leq \frac{a_1 + a_2 + a_3}{1 - (a_2 + a_3)} d(x, p) \leq d(x, p).$$

The following fixed point existence theorem has been proved by Lim [13].

Theorem 2.9 [13] *Let X be a uniformly convex Banach space and C be a nonempty bounded, closed, and convex subset of X . Let $T : C \rightarrow \mathcal{K}(C)$ be a multi-valued nonexpansive mapping. Then, T has a fixed point.*

Due to the importance of fixed point problems, iterative schemes for approximating fixed points of multi-valued mappings remain to be a flourishing subject in fixed point theory. Unfortunately, the existence result of common fixed points of a family of mappings is not known in many situations. Therefore, it is natural to consider approximation results for these classes of mappings, see, for example, [10, 19, 20, 24, 25]. Many iterative schemes have been introduced for different classes of mappings with a nonempty set of common fixed points. To define new iterative scheme, the following lemma will be needed.

Lemma 2.10 [16] *Let $D, \overline{D} \in \mathcal{CB}(C)$ and $x \in D$. For every $\varepsilon > 0$, there exists $y \in \overline{D}$, such that $d(x, y) \leq H(D, \overline{D}) + \varepsilon$.*

Let (M, d) be a $CAT(0)$ space. For $x_i \in M$ and $\alpha_i \in (0, 1)$, $i \in \{1, 2, \dots, n\}$ and $\sum_{i=1}^n \alpha_i = 1$, we can define, by induction, that:

$$\bigoplus_{i=1}^n \alpha_i x_i = (1 - \alpha_n) \left(\frac{\alpha_1}{1 - \alpha_n} x_1 \oplus \frac{\alpha_2}{1 - \alpha_n} x_2 \oplus \dots \oplus \frac{\alpha_{n-1}}{1 - \alpha_n} x_{n-1} \right) \oplus \alpha_n x_n.$$

Now, we write up our new iterative scheme to approximate common fixed point of a finite family of multi-valued generalized nonexpansive mappings in a $CAT(0)$ space as follows:

Let $S_1, S_2, \dots, S_m : C \rightarrow \mathcal{CB}(C)$ be multi-valued generalized nonexpansive mappings, such that $\mathcal{F} = \bigcap_{i \in J} F(S_i) \neq \emptyset$, where $J = \{1, 2, 3, \dots, m\}$ (we assume it throughout the paper).

In view of Lemma 2.10, we define the scheme $\{x_n\}$ in C as follows:

$$x_1 \in C, \quad x_{n+1} = \bigoplus_{i=0}^m \alpha_{n,i} y_{n,i}, \quad (2.4)$$

where $y_{n,0} = x_n$, $y_{n,i} \in S_i x_n$, such that:

$$d(y_{n,i}, p) \leq H(S_i x_n, S_i p) + \varepsilon_{n,i} \quad \text{for } i \in J \text{ and } p \in \mathcal{F}$$

with $\lim_{n \rightarrow \infty} \varepsilon_{n,i} = 0$ and the sequences $\{\alpha_{n,0}\}, \{\alpha_{n,1}\}, \dots, \{\alpha_{n,m}\}$ are in $(0, 1)$ with $\sum_{i=0}^m \alpha_{n,i} = 1$.

We recall some needed propositions and lemmas.

Proposition 2.11 [7] *If $\{x_n\}$ is a bounded sequence in a complete $CAT(0)$ space M and if C is nonempty closed convex subset of M , then there exists a unique point $u \in C$, such that:*

$$r(u, \{x_n\}) = \inf_{x \in C} r(x, \{x_n\}).$$

The above fact immediately yields the following proposition.

Proposition 2.12 [17] *Let $\{x_n\}$, C and M be as in Proposition 2.11. Then, $A(\{x_n\})$ and $A_C(\{x_n\})$ are singleton.*

Lemma 2.13 [6,26] *Let M be a $CAT(0)$ space and $\{x, x_1, x_2, \dots, x_n\} \subset M$. If $a_i \in (0, 1)$ for $i = 1, 2, 3, \dots, n$ with $\sum_{i=1}^n a_i = 1$, then for each $i, j \in \{1, 2, \dots, n\}$, such that $i < j$, we have:*

- (a) $d\left(\bigoplus_{i=1}^n a_i x_i, x\right) \leq \sum_{i=1}^n a_i d(x_i, x),$
- (b) $d\left(\bigoplus_{i=1}^n a_i x_i, x\right)^2 \leq \sum_{i=1}^n a_i d(x_i, x)^2 - a_i a_j d(x_i, x_j)^2.$

Lemma 2.14 [12] *Every bounded sequence in a complete $CAT(0)$ space M admits a Δ -convergent subsequence.*

Lemma 2.15 [5] *Let C be a closed and convex subset of a complete $CAT(0)$ space M . If $\{x_n\}$ is a bounded sequence in C , then the asymptotic center of $\{x_n\}$ lies in C .*

In this paper, we prove some convergence theorems for a finite family of multi-valued generalized nonexpansive mappings using scheme (2.4). Our results generalize the corresponding results in [1,26] and references cited therein.

3 Convergence theorems

We start with the following lemma.

Lemma 3.1 *Let C be a nonempty closed and convex subset of a complete $CAT(0)$ space M . Let $S_1, S_2, \dots, S_m : C \rightarrow \mathcal{CB}(C)$ be multi-valued generalized nonexpansive mappings, such that $S_i p = \{p\}$ for all $p \in \mathcal{F}$. If $\{x_n\}$ is a sequence in C given by (2.4), then $\lim_{n \rightarrow \infty} d(x_n, p)$ exists for each $p \in \mathcal{F}$.*

Proof Let $p \in \mathcal{F}$. Applying Lemma 2.13 (a) to the scheme (2.4), we have:

$$\begin{aligned} d(x_{n+1}, p) &= d\left(\bigoplus_{i=0}^m \alpha_{n,i} y_{n,i}, p\right) \\ &\leq \sum_{i=0}^m \alpha_{n,i} d(y_{n,i}, p) \\ &= \alpha_{n,0} d(x_n, p) + \sum_{i=1}^m \alpha_{n,i} \text{dist}(y_{n,i}, S_i p) \\ &\leq \alpha_{n,0} d(x_n, p) + \sum_{i=1}^m \alpha_{n,i} H(S_i x_n, S_i p) \\ &\leq d(x_n, p) \sum_{i=0}^m \alpha_{n,i} \\ &= d(x_n, p). \end{aligned}$$

That is, $\{d(x_n, p)\}$ is decreasing and bounded below. Hence, $\lim_{n \rightarrow \infty} d(x_n, p)$ exists. $\square$

Lemma 3.2 *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space M . Let $S_1, S_2, \dots, S_m : C \rightarrow \mathcal{CB}(C)$ be multi-valued generalized nonexpansive mappings, such that $S_i p = \{p\}$ for all $p \in \mathcal{F}$. If $\{x_n\}$ is a sequence in C given by (2.4), where $\{\alpha_{n,0}\}, \{\alpha_{n,1}\}, \dots, \{\alpha_{n,m}\}$ are sequences in $[\delta, 1 - \delta]$ for some $\delta \in (0, \frac{1}{2})$, such that $\sum_{i=0}^m \alpha_{n,i} = 1$, then $\lim_{n \rightarrow \infty} \text{dist}(x_n, S_j x_n) = 0$ for each $j \in J$.*

Proof Let $p \in \mathcal{F}$. By the scheme (2.4) and Lemma 2.13 (b):

$$\begin{aligned} d(x_{n+1}, p)^2 &= d\left(\bigoplus_{i=0}^m \alpha_{n,i} y_{n,i}, p\right)^2 \\ &\leq \sum_{i=0}^m \alpha_{n,i} d(y_{n,i}, p)^2 - \alpha_{n,1} \alpha_{n,j} d(x_n, y_{n,j})^2 \end{aligned}$$

$$\begin{aligned}
&\leq \alpha_{n,0}d(x_n, p)^2 + \sum_{i=1}^m \alpha_{n,i}H(S_i x_n, p)^2 - \alpha_{n,1}\alpha_{n,j}d(x_n, y_{n,j})^2 \\
&\leq \sum_{i=0}^m \alpha_{n,i}d(x_n, p)^2 - \alpha_{n,1}\alpha_{n,j}d(x_n, y_{n,j})^2 \\
&= d(x_n, p)^2 - \alpha_{n,1}\alpha_{n,j}d(x_n, y_{n,j})^2.
\end{aligned}$$

That is:

$$\delta^2 d(x_n, y_{n,j})^2 \leq d(x_n, p)^2 - d(x_{n+1}, p)^2.$$

For any positive integer $N \geq 1$, the above inequality provides that:

$$\delta^2 \sum_{n=1}^N d(x_n, y_{n,j})^2 \leq d(x_1, p)^2 - d(x_{N+1}, p)^2 \leq d(x_1, p)^2 < \infty.$$

If $N \rightarrow \infty$, we have $\delta^2 \sum_{n=1}^{\infty} d(x_n, y_{n,j})^2 < \infty$.

Hence, for each $j \in \{1, 2, \dots, m\}$, we deduce that:

$$\lim_{n \rightarrow \infty} d(x_n, y_{n,j}) = 0.$$

Since $\text{dist}(x_n, T_j x_n) \leq d(x_n, y_{n,j})$, it follows that:

$$\lim_{n \rightarrow \infty} \text{dist}(x_n, T_j x_n) = 0.$$

□

Now, we prove the demiclosed principle for multi-valued generalized nonexpansive mappings.

Theorem 3.3 *Let C be a nonempty closed and convex subset of a complete CAT(0) space M . Let $S : C \rightarrow \mathcal{K}(C)$ be a multi-valued generalized nonexpansive mapping and $\{x_n\} \subset C$. If $\Delta - \lim_n x_n = x \in C$ and $\lim_{n \rightarrow \infty} \text{dist}(x_n, Sx_n) = 0$, then x is a fixed point of S .*

Proof For $n \geq 1$, choose $y_n \in Sx$, such that $d(x_n, y_n) = \text{dist}(x_n, Sx)$ as $Sx \in \mathcal{K}(C)$. By the compactness of Sx , there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$, such that $\lim_{i \rightarrow \infty} y_{n_i} = y \in Sx$.

Now, we have:

$$\begin{aligned}
d(x_{n_i}, y_{n_i}) &= \text{dist}(x_{n_i}, Sx) \\
&\leq \text{dist}(x_{n_i}, Sx_{n_i}) + H(Sx_{n_i}, Sx) \\
&\leq \text{dist}(x_{n_i}, Sx_{n_i}) + a_1 d(x, x_{n_i}) + a_2 [\text{dist}(x_{n_i}, Sx_{n_i}) + \text{dist}(x, Sx)] \\
&\quad + a_3 [\text{dist}(x_{n_i}, Sx) + \text{dist}(x, Sx_{n_i})] \\
&= (1 + a_2) \text{dist}(x_{n_i}, Sx_{n_i}) + a_1 d(x, x_{n_i}) \\
&\quad + a_2 [d(x_{n_i}, x) + \text{dist}(x_{n_i}, Sx)] \\
&\quad + a_3 [\text{dist}(x_{n_i}, Sx) + d(x_{n_i}, x) + \text{dist}(x_{n_i}, Sx_{n_i})].
\end{aligned}$$

Thus:

$$\begin{aligned}
d(x_{n_i}, y_{n_i}) &= \text{dist}(x_{n_i}, Sx) \\
&\leq \frac{1 + a_2 + a_3}{1 - (a_2 + a_3)} \text{dist}(x_{n_i}, Sx_{n_i}) + \frac{a_1 + a_2 + a_3}{1 - (a_2 + a_3)} d(x_{n_i}, x) \\
&\leq \frac{1 + a_2 + a_3}{1 - (a_2 + a_3)} \text{dist}(x_{n_i}, Sx_{n_i}) + d(x_{n_i}, x).
\end{aligned}$$

Hence:

$$\begin{aligned}
d(x_{n_i}, y) &\leq d(x_{n_i}, y_{n_i}) + d(y_{n_i}, y) \\
&\leq \frac{1 + a_2 + a_3}{1 - (a_2 + a_3)} \text{dist}(x_{n_i}, Sx_{n_i}) + d(x_{n_i}, x) + d(y_{n_i}, y).
\end{aligned}$$

Taking the superior limit on both sides of the above inequality, we obtain:

$$\limsup_{i \rightarrow \infty} d(x_{n_i}, y) \leq \limsup_{i \rightarrow \infty} d(x_{n_i}, x) = r(x, \{x_{n_i}\}).$$

By the uniqueness of the asymptotic center, we have $x = y \in Sx$. $\square$

Theorem 3.4 *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space M . Let $S_1, S_2, \dots, S_m : C \rightarrow \mathcal{K}(C)$ be multi-valued generalized nonexpansive mappings, such that $S_i p = \{p\}$ for all $p \in \mathcal{F}$. If $\{x_n\}$ is a sequence in C given by (2.4), where $\{\alpha_{n,0}\}, \{\alpha_{n,1}\}, \dots, \{\alpha_{n,m}\}$ are sequences in $[\delta, 1 - \delta]$ for some $\delta \in (0, \frac{1}{2})$, such that $\sum_{i=0}^m \alpha_{n,i} = 1$, then $\Delta - \lim_n x_n = x \in \mathcal{F}$.*

Proof It follows from Lemma 3.1 that $\{x_n\}$ is bounded. By Proposition 2.11, $\{x_n\}$ has a unique asymptotic center; that is, $A_C(\{x_n\}) = \{x\}$ and $x \in C$. By Lemmas 2.14 and 2.15, there must be a subsequence $\{z_n\}$ of $\{x_n\}$, such that $\Delta - \lim_n z_n = z \in C$. Lemma 3.2 implies that $\lim_{n \rightarrow \infty} \text{dist}(z_n, S_j z_n) = 0$ for each $j \in J$. In view of Theorem 3.3, we have $S_j z = z$ for each $j \in J$. That is, $z \in \mathcal{F}$.

If $z \neq x$, by the uniqueness of the asymptotic center, we have:

$$\begin{aligned} \limsup_{n \rightarrow \infty} d(z_n, z) &< \limsup_{n \rightarrow \infty} d(z_n, x) \leq \limsup_{n \rightarrow \infty} d(x_n, x) \\ &< \limsup_{n \rightarrow \infty} d(x_n, z) = \limsup_{n \rightarrow \infty} d(z_n, z), \end{aligned}$$

which is a contradiction and, hence, $z = x \in \mathcal{F}$.

Let $\{v_n\}$ be any subsequence of $\{x_n\}$, such that $A_C(\{v_n\}) = \{v\}$ and $v \in C$. Suppose $z \neq v$; we have:

$$\begin{aligned} \limsup_{n \rightarrow \infty} d(z_n, z) &< \limsup_{n \rightarrow \infty} d(z_n, v) \\ &\leq \limsup_{n \rightarrow \infty} d(v_n, v) \\ &< \limsup_{n \rightarrow \infty} d(v_n, z) \\ &= \limsup_{n \rightarrow \infty} d(x_n, z) = \limsup_{n \rightarrow \infty} d(z_n, z), \end{aligned}$$

which is a contradiction. Hence, $z = v = x \in \mathcal{F}$. $\square$

Theorem 3.5 *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space M . Let $S_1, S_2, \dots, S_m : C \rightarrow \mathcal{CB}(C)$ be multi-valued generalized nonexpansive mappings, such that $S_i p = \{p\}$ for all $p \in \mathcal{F}$. If $\{x_n\}$ is a sequence given by (2.4), where $\{\alpha_{n,0}\}, \{\alpha_{n,1}\}, \dots, \{\alpha_{n,m}\}$ are sequences in $[\delta, 1 - \delta]$ for some $\delta \in (0, \frac{1}{2})$, such that $\sum_{i=0}^m \alpha_{n,i} = 1$, then $\{x_n\}$ converges to a point $p \in \mathcal{F}$ if and only if:*

$$\liminf_{n \rightarrow \infty} \text{dist}(x_n, \mathcal{F}) = 0 \text{ or } \limsup_{n \rightarrow \infty} \text{dist}(x_n, \mathcal{F}) = 0. \quad (3.1)$$

Proof It is easy to see that if $\{x_n\}$ converges to a point $p \in \mathcal{F}$, then (3.1) holds.

Conversely, suppose that:

$$\liminf_{n \rightarrow \infty} \text{dist}(x_n, \mathcal{F}) = 0 \text{ or } \limsup_{n \rightarrow \infty} \text{dist}(x_n, \mathcal{F}) = 0.$$

Lemma 3.1 implies that:

$$d(x_{n+1}, p) \leq d(x_n, p) \quad \text{for any } p \in \mathcal{F},$$

so that $\text{dist}(x_{n+1}, \mathcal{F}) \leq \text{dist}(x_n, \mathcal{F})$. Thus, $\lim_{n \rightarrow \infty} \text{dist}(x_n, \mathcal{F})$ exists. Since:

$$\liminf_{n \rightarrow \infty} \text{dist}(x_n, \mathcal{F}) = 0 \text{ or } \limsup_{n \rightarrow \infty} \text{dist}(x_n, \mathcal{F}) = 0,$$

we have $\lim_{n \rightarrow \infty} \text{dist}(x_n, \mathcal{F}) = 0$. Let $\varepsilon > 0$. We can find $n_0 \geq 1$, such that:

$$\text{dist}(x_n, \mathcal{F}) < \frac{\varepsilon}{3}, \text{ for all } n \geq n_0.$$

Therefore, there exists $p \in \mathcal{F}$, such that:

$$d(x_{n_0}, p) < \frac{\varepsilon}{2}.$$

For $m \geq n_0, n \geq n_0$, we have that:

$$d(x_{m+n}, x_n) \leq d(x_{m+n}, p) + d(x_n, p) < 2d(x_{n_0}, p) < \varepsilon.$$

Thus, $\{x_n\}$ is a Cauchy sequence in the closed subset C of X . This implies that $\{x_n\}$ converges to $x \in C$. For each $j \in \{1, 2, \dots, m\}$:

$$\begin{aligned} \text{dist}(x, S_j x) &\leq d(x, x_n) + \text{dist}(x_n, S_j x_n) + H(S_j x_n, S_j x) \\ &\leq d(x, x_n) + \text{dist}(x_n, S_j x_n) + a_2 [\text{dist}(x_n, S_j x_n) + \text{dist}(x, S_j x)] \\ &\quad + a_1 d(x_n, x) + a_3 [\text{dist}(x_n, S_j x) + \text{dist}(x, S_j x_n)] \\ &\leq \frac{a_1 + 2a_3 + 1}{1 - (a_2 + a_3)} d(x, x_n) + \frac{a_2 + a_3 + 1}{1 - (a_2 + a_3)} \text{dist}(x_n, S_j x_n) \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$. Hence, $S_j x = x$ for each $j \in J$. That is, $x \in \mathcal{F}$. $\square$

Definition 3.6 A finite family of multi-valued mappings $S_1, S_2, \dots, S_m : C \rightarrow \mathcal{CB}(C)$ satisfies condition (AV) if there exists a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$, such that $f(0) = 0$, $f(t) > 0$ for $t > 0$, and

$$\frac{1}{m} \sum_{i=1}^m \text{dist}(x, S_i x) \geq f(\text{dist}(x, \mathcal{F})), \quad x \in C.$$

The following theorem states the strong convergence of the algorithm.

Theorem 3.7 Let C be a nonempty closed and convex subset of a complete CAT(0) space M . Let $S_1, S_2, \dots, S_m : C \rightarrow \mathcal{CB}(C)$ be multi-valued generalized nonexpansive mappings satisfying condition (AV) and $S_i p = \{p\}$ for all $p \in \mathcal{F}$. If $\{x_n\}$ is a sequence given by (2.4) and $\{\alpha_{n,0}\}, \{\alpha_{n,1}\}, \dots, \{\alpha_{n,m}\}$ are sequences in $[\delta, 1 - \delta]$ for some $\delta \in (0, \frac{1}{2})$ such that $\sum_{i=0}^m \alpha_{n,i} = 1$, then $\{x_n\}$ converges to a point $p \in \mathcal{F}$.

Proof As a consequence of Lemma 3.1, $\lim_{n \rightarrow \infty} \text{dist}(x_n, \mathcal{F})$ exists.

By condition (B):

$$\lim_{n \rightarrow \infty} f(\text{dist}(x_n, \mathcal{F})) \leq \frac{1}{m} \sum_{i=1}^m \lim_{n \rightarrow \infty} \text{dist}(x_n, S_i x) = 0.$$

That is, $\lim_{n \rightarrow \infty} f(\text{dist}(x_n, \mathcal{F})) = 0$. Since f is a nondecreasing and $f(0) = 0$, $\lim_{n \rightarrow \infty} \text{dist}(x_n, \mathcal{F}) = 0$. The conclusion follows from Theorem 3.5. $\square$

4 Common fixed points on proximal subsets

Let $S_1, S_2, \dots, S_m : C \rightarrow \mathcal{P}(C)$ be multi-valued generalized nonexpansive mappings. For $x_1 \in C$ and $i \in J$. As $S_i x_1$ is a proximal subset of C , there exists $y_{1,i} \in S_i x_1$, such that $d(q, y_{1,i}) = \text{dist}(q, S_i x_1)$ whenever q is a fixed point of S_i . Hence, our scheme 2.4 reads as follows:

$$x_{n+1} = \bigoplus_{i=0}^m \alpha_{n,i} y_{n,i}, \quad (4.1)$$

where $y_{n,0} = x_n$. For $i \in J$, $y_{n,i} \in S_i x_n$ with $d(q, y_{n,i}) = \text{dist}(q, S_i x_n)$ for $p \in \mathcal{F}$. Moreover, $\{\alpha_{n,0}\}, \{\alpha_{n,1}\}, \dots, \{\alpha_{n,m}\}$ are sequences in $(0, 1)$, such that $\sum_{i=0}^m \alpha_{n,i} = 1$.

Using the scheme (4.1), we can prove the following results whose proofs carry on similar details to those in Sect. 2. In the following results, assume that C is a nonempty, convex, and closed subset of the space M .

Lemma 4.1 Let $S_1, S_2, \dots, S_m : C \rightarrow \mathcal{P}(C)$ be multi-valued generalized nonexpansive mappings. If $\{x_n\}$ is a sequence given by (4.1), then $\lim_{n \rightarrow \infty} d(x_n, q)$ exists for each $q \in \mathcal{F}$.

Lemma 4.2 Let $S_1, S_2, \dots, S_m : C \rightarrow \mathcal{P}(C)$ be multi-valued generalized nonexpansive mappings. If $\{x_n\}$ is a sequence given by (4.1), where $\{\alpha_{n,0}\}, \{\alpha_{n,1}\}, \dots, \{\alpha_{n,m}\}$ are sequences in $[\delta, 1 - \delta]$ for some $\delta \in (0, \frac{1}{2})$, such that $\sum_{i=1}^{m+1} \alpha_{n,i} = 1$, then $\lim_{n \rightarrow \infty} \text{dist}(x_n, S_j x_n) = 0$ for $j \in J$.

Theorem 4.3 Let $S_1, S_2, \dots, S_m : C \rightarrow \mathcal{P}(C)$ be multi-valued generalized nonexpansive mappings. If $\{x_n\}$ is a sequence given by (4.1) where $\{\alpha_{n,0}\}, \{\alpha_{n,1}\}, \dots, \{\alpha_{n,m}\}$ are sequences in $[\delta, 1 - \delta]$ for some $\delta \in (0, \frac{1}{2})$ with $\sum_{i=1}^{m+1} \alpha_{n,i} = 1$, then $\{x_n\}$ converges to a point $q \in \mathcal{F}$ if and only if:

$$\liminf_{n \rightarrow \infty} \text{dist}(x_n, \mathcal{F}) = 0 \quad \text{or} \quad \limsup_{n \rightarrow \infty} \text{dist}(x_n, \mathcal{F}) = 0.$$

Theorem 4.4 Let $S_1, S_2, \dots, S_m : C \rightarrow \mathcal{P}(C)$ be multi-valued generalized nonexpansive mappings satisfying condition (B). If $\{x_n\}$ is a sequence given by (4.1) where $\{\alpha_{n,0}\}, \{\alpha_{n,1}\}, \dots, \{\alpha_{n,m}\}$ are sequences in $[\delta, 1 - \delta]$ for some $\delta \in (0, \frac{1}{2})$, such that $\sum_{i=1}^{m+1} \alpha_{n,i} = 1$, then the sequence $\{x_n\}$ converges to a point $q \in \mathcal{F}$.

Remark 4.5 The above theorems are new in $CAT(0)$ spaces and reform the results of Abbas et al. [1], Khan and Fukhar-ud-din [9] and some others. The reason is that we consider a nonlinear space, a family of generalized multi-valued nonexpansive mappings and a one-step iterative scheme (instead of multi-step).

Acknowledgements The first author is grateful to King Fahd University of Petroleum and Minerals (KFUPM), Saudi Arabia for supporting this research.

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