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Boundary value problems for Caputo fractional differential equations with nonlocal and fractional integral boundary conditions

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Abstract In this paper, we study the existence and uniqueness of solutions for fractional differential equations with nonlocal and fractional integral boundary conditions. New existence and uniqueness results are established using the Banach contraction principle. Other existence results are obtained using O'Regan fixed point theorem and Burton and Kirk fixed point. In addition, an example is given to demonstrate the application of our main results.

Mathematics Subject Classification 34A08 · 26A33

1 Introduction

Fractional differential equations have gained much importance and attention due to the fact that they have been proved to be valuable tools in the modeling of many phenomena in engineering and sciences such as physics, mechanics, economics, biology, etc. (see, for example, [16, 19, 23, 24] and references therein). Actually, the concepts of fractional derivatives are not the only generalization of the ordinary derivatives, but also it has been found that they can efficiently and properly describe the behavior of many physical systems (real-life phenomena) more accurately than integer-order derivatives.

Different kind of fixed point theorems are widely used as fundamental tools to prove the existence and uniqueness of solutions for various classes of fractional differential equations; for example, we refer the reader to [1–4, 9, 12, 14, 15, 17, 18, 21, 25, 27] and the references cited therein. Moreover, some mathematicians considered Caputo fractional differential equations with a nonlinear term depending on the Caputo derivative (see Benchohra and Soudi [5], Benchohra and Lazreg [6], Benchohra et al. [7], El-Sayed and Bin-Taher [10, 11], Guezane-Lakoud and Khaldi [12], Guezane-Lakoud and Bensebaa [13], Houas and Benbachir [14], Nieto et al. [20], and Yan et al. [27]).

Motivated by the above papers, in this paper, we establish various existence and uniqueness results of solutions for a boundary value problem of nonlinear fractional differential equations of order $\alpha \in (2, 3]$ with nonlocal and fractional integral boundary conditions given by:

$$\begin{cases} {}^c D_{0+}^{\alpha} u(t) = f(t, u(t), {}^c D_{0+}^{\beta} u(t)), t \in J := [0, 1] \\ u(0) = g(u), \\ u'(0) = a I_{0+}^{\sigma_1} u(\eta_1), \\ {}^c D_{0+}^{\beta_1} u(1) = b I_{0+}^{\sigma_2} u(\eta_2), \end{cases} \quad \begin{matrix} 0 < \eta_1 < 1 \\ 0 < \eta_2 < 1, \end{matrix} \quad (1)$$

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where ${}^c D_{0+}^\kappa$ is the Caputo fractional derivative of order $\kappa \in \{\alpha, \beta, \beta_1\}$, such that $2 < \alpha \leq 3$, and $0 < \beta, \beta_1 \leq 1$, and $f : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ and $g : C([0, 1], \mathbb{R}) \rightarrow \mathbb{R}$ are continuous functions, $I_{0+}^{\sigma_i}$ is the Riemann–Liouville fractional integral of order $\sigma_i > 0$, $i = 1, 2$, η_1, η_2, a, b are suitably chosen real constants, such that:

$$\Delta = \left[\left(\frac{a\eta_1^{\sigma_1+1}}{\Gamma(\sigma_1+2)} - 1 \right) \left(\frac{b\eta_2^{\sigma_2+2}}{\Gamma(\sigma_2+3)} - \frac{1}{\Gamma(3-\beta_1)} \right) - \left(\frac{b\eta_2^{\sigma_2+1}}{\Gamma(\sigma_2+2)} - \frac{1}{\Gamma(2-\beta_1)} \right) \left(\frac{a\eta_1^{\sigma_1+2}}{\Gamma(\sigma_1+3)} \right) \right] \neq 0. \quad (2)$$

The paper is organized as follows. In Sect. 2, we collect some background material and basic results about fractional calculus that will be used later. In Sect. 3, we establish the existence of solutions for the boundary value problem (1) using Banach contraction principle, O'Regan fixed point theorem, and Burton and Kirk fixed point. In the fourth section, we give an example to demonstrate the application of our main results. Finally, we add a conclusion to the paper in the last section.

2 Preliminaries

In this section, we will recall some notions and definition which we will need in the sequel (details can be found, e.g., in [16, 23, 24]).

Definition 2.1 ([16]) The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $u \in L^1([0, 1])$ is defined by:

$$I_{0+}^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds, \quad (t > 0, \alpha > 0).$$

where $\Gamma(\cdot)$ is the (Euler's) Gamma function $\Gamma(\alpha) = \int_0^{+\infty} e^{-t} t^{\alpha-1} dt$, $\alpha > 0$. Moreover, for $\alpha = 0$, we set $I_{0+}^0 u := u$.

Definition 2.2 ([16, 23]) The Caputo fractional derivative of order α of a function $u \in AC^n([0, 1])$ is represented by:

$${}^c D_{0+}^\alpha u(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s) ds, & \text{if } \alpha \notin \mathbb{N}, \\ u^{(n)}(t), & \text{if } \alpha \in \mathbb{N}, \end{cases}$$

where $u^{(n)}(t) = \frac{d^n u(t)}{dt^n}$, $\alpha > 0$, $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of the real number α .

Lemma 2.3 Let $\alpha, \beta > 0$, $n = [\alpha] + 1$, and then, the following relation holds:

$${}^c D_{0+}^\alpha t^\beta = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, & (\beta \in \mathbb{N} \text{ and } \beta \geq n \text{ or } \beta \notin \mathbb{N} \text{ and } \beta > n-1), \\ 0, & \beta \in \{0, \dots, n-1\}. \end{cases}$$

Lemma 2.4 Let $\alpha > \beta > 0$, and $u \in L^1([0, 1])$. Then, we have:

- (1) $I_{0+}^\alpha I_{0+}^\beta u(t) = I_{0+}^{\alpha+\beta} u(t)$,
- (2) ${}^c D_{0+}^\alpha I_{0+}^\alpha u(t) = u(t)$,
- (3) ${}^c D_{0+}^\beta I_{0+}^\alpha u(t) = I_{0+}^{\alpha-\beta} u(t)$.



Lemma 2.5 *Let $\alpha > 0$. Then, the differential equation*

$$({}^cD_{0+}^\alpha u)(t) = 0,$$

has solutions

$$u(t) = \sum_{j=0}^{n-1} c_j t^j, \quad c_j \in \mathbb{R}, j = 0 \dots n - 1,$$

where $n - 1 < \alpha < n$

Lemma 2.6 *Let $\alpha > 0$. Then:*

$$I_{0+}^\alpha ({}^cD_{0+}^\alpha u(t)) = u(t) + \sum_{j=0}^{n-1} c_j t^j,$$

for some $c_j \in \mathbb{R}, j = 0, 1, 2, \dots, m - 1$, where $n = [\alpha] + 1$.

Lemma 2.7 *Let $\alpha > 0, u \in L^1([0, 1], \mathbb{R})$. Then, for all $t \in [0, 1]$, we have:*

$$I_{0+}^{\alpha+1} u(t) \leq \|I_{0+}^\alpha u\|_{L^1}.$$

Proof Let $u \in L^1([0, 1], \mathbb{R})$, and from Lemma 2.4, we have:

$$I_{0+}^{\alpha+1} u(t) = I_{0+}^1 I_{0+}^\alpha u(t) = \int_0^t I_{0+}^\alpha u(s) ds \leq \int_0^1 |I_{0+}^\alpha u(s)| ds = \|I_{0+}^\alpha u\|_{L^1}.$$

□

Lemma 2.8 *The fractional integral $I_{0+}^\alpha, \alpha > 0$ is bounded in $L^1([0, 1], \mathbb{R})$ with*

$$\|I_{0+}^\alpha u\|_{L^1} \leq \frac{\|u\|_{L^1}}{\Gamma(\alpha + 1)}.$$

Proof Let $u \in L^1([0, 1], \mathbb{R})$, and then:

$$\begin{aligned} \|I_{0+}^\alpha u\|_{L^1} &= \int_0^1 |I_{0+}^\alpha u(t)| dt \leq \frac{1}{\Gamma(\alpha)} \int_0^1 \int_0^t (t - s)^{\alpha-1} |u(s)| ds dt \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^1 |u(s)| ds \int_s^1 (t - s)^{\alpha-1} dt \\ &\leq \frac{1}{\Gamma(\alpha + 1)} \int_0^1 (t - s)^\alpha |u(s)| ds \\ &\leq \frac{\|u\|_{L^1}}{\Gamma(\alpha + 1)}. \end{aligned}$$

□

Let us now introduce the space $E = \{u : u \in C([0, 1], \mathbb{R}) : {}^cD_{0+}^\beta u \in C([0, 1], \mathbb{R})\}$ equipped with the norm:

$$\|u\|_E = \|u\|_\infty + \|{}^cD_{0+}^\beta u\|_\infty = \sup_{t \in J} |u(t)| + \sup_{t \in J} |{}^cD_{0+}^\beta u(t)|.$$

Clearly, $(E, \|\cdot\|_E)$ is a Banach space [26].

3 Main results

For the existence of solutions for the problem (1), we need the following lemma:

Lemma 3.1 For a given $h \in C(J, \mathbb{R})$, the unique solution of the linear fractional boundary value problem:

$${}^c D_{0+}^{\alpha} u(t) = h(t), \quad 2 < \alpha \leq 3, \quad (3)$$

supplemented with boundary conditions

$$u(0) = g(u), \quad u'(0) = a I_{0+}^{\sigma_1} u(\eta_1), \quad {}^c D_{0+}^{\beta_1} u(1) = b I_{0+}^{\sigma_2} u(\eta_2), \quad (4)$$

is given by:

$$u(t) = I_{0+}^{\alpha} h(t) + (v_6 t^2 - v_1 t) I_{0+}^{\alpha+\sigma_1} h(\eta_1) + (v_2 t - v_5 t^2) I_{0+}^{\alpha+\sigma_2} h(\eta_2) \\ + (v_8 t^2 - v_4 t) I_{0+}^{\alpha-\beta_1} h(1) + (1 - v_3 t - v_7 t^2) g(u), \quad (5)$$

where

$$v_1 = \frac{a}{\Delta} \left(\frac{b \eta_2^{\sigma_2+2}}{\Gamma(\sigma_2+3)} - \frac{1}{\Gamma(3-\beta_1)} \right), \quad v_2 = \frac{ab}{\Delta} \left(\frac{\eta_1^{\sigma_1+2}}{\Gamma(\sigma_1+3)} \right), \\ v_3 = \frac{\eta_1^{\sigma_1} v_1}{\Gamma(\sigma_1+1)} - \frac{\eta_2^{\sigma_2} v_2}{\Gamma(\sigma_2+1)}, \quad v_4 = \frac{v_2}{b}, \quad v_5 = \frac{b}{2\Delta} \left(\frac{a \eta_1^{\sigma_1+1}}{\Gamma(\sigma_1+2)} - 1 \right), \\ v_6 = \frac{a}{2\Delta} \left(\frac{b \eta_2^{\sigma_2+1}}{\Gamma(\sigma_2+2)} - \frac{1}{\Gamma(2-\beta_1)} \right), \quad v_7 = \frac{\eta_2^{\sigma_2} v_5}{\Gamma(\sigma_2+1)} - \frac{\eta_1^{\sigma_1} v_6}{\Gamma(\sigma_1+1)}, \\ v_8 = \frac{v_5}{b}, \quad (6)$$

and Δ is defined by (2).

Proof Applying the Riemann–Liouville fractional integral operator of order α to both sides of (3) and using Lemma 2.6, we have:

$$u(t) = I_{0+}^{\alpha} h(t) - c_0 - c_1 t - c_2 t^2, \quad c_0, c_1, c_2 \in \mathbb{R}. \quad (7)$$

Applying the boundary conditions (4) in (7), we find that:

$$c_0 = -g(u).$$

Using the boundary conditions of (4) in (7), we find a system of equations:

$$\left(\frac{a \eta_1^{\sigma_1+1}}{\Gamma(\sigma_1+2)} - 1 \right) c_1 + 2 \left(\frac{a \eta_1^{\sigma_1+2}}{\Gamma(\sigma_1+3)} \right) c_2 = a I_{0+}^{\alpha+\sigma_1} h(\eta_1) + \left(\frac{a \eta_1^{\sigma_1}}{\Gamma(\sigma_1+1)} \right) g(u). \quad (8)$$

$$\left(\frac{b \eta_2^{\sigma_2+1}}{\Gamma(\sigma_2+2)} - \frac{1}{\Gamma(2-\beta_1)} \right) c_1 + 2 \left(\frac{b \eta_2^{\sigma_2+2}}{\Gamma(\sigma_2+3)} - \frac{1}{\Gamma(3-\beta_1)} \right) c_2 = b I_{0+}^{\alpha+\sigma_2} h(\eta_2) \\ + \left(\frac{b \eta_2^{\sigma_2}}{\Gamma(\sigma_2+1)} \right) g(u) - I_{0+}^{\alpha-\beta_1} h(1). \quad (9)$$

Solving the system of equations (8) and (9) together with notations (6), we find that:

$$c_1 = v_1 I_{0+}^{\alpha+\sigma_1} h(\eta_1) - v_2 I_{0+}^{\alpha+\sigma_2} h(\eta_2) + v_3 g(u) + v_4 I_{0+}^{\alpha-\beta_1} h(1), \\ c_2 = v_5 I_{0+}^{\alpha+\sigma_2} h(\eta_2) - v_6 I_{0+}^{\alpha+\sigma_1} h(\eta_1) + v_7 g(u) - v_8 I_{0+}^{\alpha-\beta_1} h(1).$$

Substituting the values of c_0, c_1, c_2 in (7), we obtain the solution (5). This completes the proof. \square



For computational convenience, we introduce the notations:

$$I_{0+}^{\alpha} f(s, u(s), {}^cD_{0+}^{\beta} u(s))(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s), {}^cD_{0+}^{\beta} u(s)) ds, \quad \forall t \in J.$$

Define the integral operator $T : E \rightarrow E$ by:

$$\begin{aligned} T(u)(t) &= I_{0+}^{\alpha} f(s, u(s), {}^cD_{0+}^{\beta} u(s))(t) \\ &\quad + (v_6 t^2 - v_1 t) I_{0+}^{\alpha+\sigma_1} f(s, u(s), {}^cD_{0+}^{\beta} u(s))(\eta_1) \\ &\quad + (v_2 t - v_5 t^2) I_{0+}^{\alpha+\sigma_2} f(s, u(s), {}^cD_{0+}^{\beta} u(s))(\eta_2) \\ &\quad + (v_8 t^2 - v_4 t) I_{0+}^{\alpha-\beta_1} f(s, u(s), {}^cD_{0+}^{\beta} u(s))(1) \\ &\quad + (1 - v_3 t - v_7 t^2) g(u). \end{aligned} \tag{10}$$

Next, we introduce two operators $T_{1,2} : E \rightarrow E$ as follows:

$$\begin{aligned} T_1(u)(t) &= I_{0+}^{\alpha} f(s, u(s), {}^cD_{0+}^{\beta} u(s))(t) \\ &\quad + (v_6 t^2 - v_1 t) I_{0+}^{\alpha+\sigma_1} f(s, u(s), {}^cD_{0+}^{\beta} u(s))(\eta_1) \\ &\quad + (v_2 t - v_5 t^2) I_{0+}^{\alpha+\sigma_2} f(s, u(s), {}^cD_{0+}^{\beta} u(s))(\eta_2) \\ &\quad + (v_8 t^2 - v_4 t) I_{0+}^{\alpha-\beta_1} f(s, u(s), {}^cD_{0+}^{\beta} u(s))(1), \end{aligned} \tag{11}$$

and

$$T_2 u(t) = (1 - v_3 t - v_7 t^2) g(u). \tag{12}$$

Clearly:

$$T u(t) = T_1 u(t) + T_2 u(t), \quad t \in J. \tag{13}$$

Remark 3.2 In our work, we assume that the restriction $\Delta \neq 0$ is satisfied, because in the case when $\Delta = 0$, the problem (1) will be under the resonant conditions, and hence, the previously mentioned theorems are not applicable, and then, the problem can be managed using the method of Mawhin continuation theorem.

Lemma 3.3 Let $f \in C([0, 1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. Then, $u \in E$ is a solution of the fractional boundary value problem (1) if and only if $T u(t) = u(t), \forall t \in [0, 1]$.

Proof Let u be a solution of (1). Then, using the same method as used in Lemma 3.1, we can prove that:

$$\begin{aligned} u(t) &= T(u)(t) = I_{0+}^{\alpha} f(s, u(s), {}^cD_{0+}^{\beta} u(s))(t) \\ &\quad + (v_6 t^2 - v_1 t) I_{0+}^{\alpha+\sigma_1} f(s, u(s), {}^cD_{0+}^{\beta} u(s))(\eta_1) \\ &\quad + (v_2 t - v_5 t^2) I_{0+}^{\alpha+\sigma_2} f(s, u(s), {}^cD_{0+}^{\beta} u(s))(\eta_2) \\ &\quad + (v_8 t^2 - v_4 t) I_{0+}^{\alpha-\beta_1} f(s, u(s), {}^cD_{0+}^{\beta} u(s))(1) \\ &\quad + (1 - v_3 t - v_7 t^2) g(u). \end{aligned}$$

Conversely, u satisfies:

$$\begin{aligned} u(t) &= I_{0+}^{\alpha} f(s, u(s), {}^cD_{0+}^{\beta} u(s))(t) \\ &\quad + (v_6 t^2 - v_1 t) I_{0+}^{\alpha+\sigma_1} f(s, u(s), {}^cD_{0+}^{\beta} u(s))(\eta_1) \\ &\quad + (v_2 t - v_5 t^2) I_{0+}^{\alpha+\sigma_2} f(s, u(s), {}^cD_{0+}^{\beta} u(s))(\eta_2) \\ &\quad + (v_8 t^2 - v_4 t) I_{0+}^{\alpha-\beta_1} f(s, u(s), {}^cD_{0+}^{\beta} u(s))(1) \\ &\quad + (1 - v_3 t - v_7 t^2) g(u), \end{aligned}$$

and denotes the right-hand side of the equation by $v(t)$. Then, by Lemma 2.4, we obtain:

$$\begin{aligned} {}^cD_{0+}^\alpha v(t) &= {}^cD_{0+}^\alpha I_{0+}^\alpha f(s, u(s), {}^cD_{0+}^\beta u(s))(t) \\ &\quad + {}^cD_{0+}^\alpha (v_6 t^2 - v_1 t) I_{0+}^{\alpha+\sigma_1} f(s, u(s), {}^cD_{0+}^\beta u(s))(\eta_1) \\ &\quad + {}^cD_{0+}^\alpha (v_2 t - v_5 t^2) I_{0+}^{\alpha+\sigma_2} f(s, u(s), {}^cD_{0+}^\beta u(s))(\eta_2) \\ &\quad + {}^cD_{0+}^\alpha (v_8 t^2 - v_4 t) I_{0+}^{\alpha-\beta_1} f(s, u(s), {}^cD_{0+}^\beta u(s))(1) \\ &\quad + {}^cD_{0+}^\alpha (1 - v_3 t - v_7 t^2) g(u) \\ &= f(t, u(t), {}^cD_{0+}^\beta u(t)). \end{aligned}$$

Hence, $v(t)$ is a solution of the fractional differential Eq. (1). Also, it is easy to verify by direct computation that v satisfies conditions (4). This completes the proof. \square

To simplify the computations in the main results, we set some notations:

$$\begin{aligned} M_1 &= \frac{1}{\Gamma(\alpha)} + \frac{|v_6| + |v_1|}{\Gamma(\alpha + \sigma_1)} + \frac{|v_2| + |v_5|}{\Gamma(\alpha + \sigma_2)} + \frac{|v_8| + |v_4|}{\Gamma(\alpha - \beta_1)}, \\ M_2 &= \frac{1}{\Gamma(\alpha - \beta)} + \frac{2|v_6| + (2 - \beta)|v_1|}{\Gamma(3 - \beta)\Gamma(\alpha + \sigma_1)} + \frac{2|v_5| + (2 - \beta)|v_2|}{\Gamma(3 - \beta)\Gamma(\alpha + \sigma_2)} \\ &\quad + \frac{2|v_8| + (2 - \beta)|v_4|}{\Gamma(3 - \beta)\Gamma(\alpha - \beta_1)}, \\ N_1 &= (1 + |v_3| + |v_7|), \quad N_2 = \frac{2|v_7| + (2 - \beta)|v_3|}{\Gamma(3 - \beta)}. \end{aligned} \quad (14)$$

In the sequel, we need the following assumptions.

(H1) The function $f : J \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous.

(H2)

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq g(t) (|u_1 - u_2| + |v_1 - v_2|).$$

for all $u_1, u_2, v_1, v_2 \in \mathbb{R}$; $g \in L^1(J, \mathbb{R}_+)$ and $t \in J$.

(H3) There exist a positive constant $\omega < 1$ and a continuous function $\phi : [0, \infty) \rightarrow [0, \infty)$, such that $\phi(v) \leq \omega v$, $\omega < \frac{1}{N_1 + N_2}$ and $|g(u) - g(v)| \leq \phi(\|u - v\|)$, for all $u, v \in C(J, \mathbb{R})$.

(H4) $g(0) = 0$.

(H5) There exist continuous nondecreasing function $\psi : [0, \infty) \rightarrow (0, \infty)$ and function $p \in L^1([0, 1], \mathbb{R}^+)$, such that:

$$|f(t, u, v)| \leq p(t)\psi(|u| + |v|), \quad (15)$$

for each $(t, u, v) \in [0, 1] \times \mathbb{R} \times \mathbb{R}$.

(H16)

$$\sup_{r \in (0, \infty)} \frac{r}{(M_1 + M_2)\|p\|_{L^1}\psi(r)} > \frac{1}{1 - (N_1 + N_2)\omega}. \quad (16)$$

(H7) There exists non-negative function $p \in L^1(J, \mathbb{R}_+)$, such that:

$$|f(t, u, v)| \leq p(t)(1 + |u| + |v|), \quad (17)$$

for all $u, v \in \mathbb{R}$ and $t \in J$.

Our first results is based on the Banach contraction principle.

Theorem 3.4 Assume (H1), (H2), and (H3) hold. If

$$\gamma_1 = (M_1 + M_2)\|g\|_{L^1} + (N_1 + N_2)\omega < 1. \quad (18)$$

holds, then the problem (1) has a unique solution on J .



Proof Transform the problem (1) into a fixed point problem. Clearly, the fixed points of the operator T defined by (10) are solutions of the problem (1).

Let $u, v \in E$ and $t \in J$, and from the definition of T , we can write:

$$\begin{aligned} |(Tu)(t) - (Tv)(t)| &= I_{0+}^\alpha |f(s, u(s), {}^cD_{0+}^\beta u(s)) - f(s, v(s), {}^cD_{0+}^\beta v(s))|(t) \\ &+ |v_6 t^2 - v_1 t| I_{0+}^{\alpha+\sigma_1} |f(s, u(s), {}^cD_{0+}^\beta u(s)) - f(s, v(s), {}^cD_{0+}^\beta v(s))|(\eta_1) \\ &+ |v_2 t - v_5 t^2| I_{0+}^{\alpha+\sigma_2} |f(s, u(s), {}^cD_{0+}^\beta u(s)) - f(s, v(s), {}^cD_{0+}^\beta v(s))|(\eta_2) \\ &+ |v_8 t^2 - v_4 t| I_{0+}^{\alpha-\beta_1} |f(s, u(s), {}^cD_{0+}^\beta u(s)) - f(s, v(s), {}^cD_{0+}^\beta v(s))|(1) \\ &+ |1 - v_3 t - v_7 t^2| |g(u) - g(v)|. \end{aligned}$$

By (H2), we can find that:

$$\begin{aligned} |(Tu)(t) - (Tv)(t)| &\leq \left(\|u - v\|_\infty + \|{}^cD_{0+}^\beta u - {}^cD_{0+}^\beta v\|_\infty \right) [I_{0+}^\alpha g(s)(t) \\ &+ (|v_6| + |v_1|) I_{0+}^{\alpha+\sigma_1} g(s)(\eta_1) + (|v_2| + |v_5|) I_{0+}^{\alpha+\sigma_2} g(s)(\eta_2) \\ &+ (|v_8| + |v_4|) I_{0+}^{\alpha-\beta_1} g(s)(1)] \\ &+ (1 + |v_3| + |v_7|) |g(u) - g(v)|. \end{aligned}$$

According to the Lemmas 2.7, 2.8, and (H3), we have:

$$\begin{aligned} \|Tu - Tv\|_\infty &\leq \|u - v\|_E \left(\frac{1}{\Gamma(\alpha)} + \frac{|v_6| + |v_1|}{\Gamma(\alpha + \sigma_1)} + \frac{|v_2| + |v_5|}{\Gamma(\alpha + \sigma_2)} + \frac{|v_8| + |v_4|}{\Gamma(\alpha - \beta_1)} \right) \|g\|_{L^1} \\ &+ (1 + |v_3| + |v_7|) \omega \|u - v\|_\infty. \end{aligned}$$

Thus:

$$\|Tu - Tv\|_\infty \leq (M_1 \|g\|_{L^1} + N_1 \omega) \|u - v\|_E. \tag{19}$$

Observe that:

$$\begin{aligned} {}^cD_{0+}^\beta Tu(t) &= I_{0+}^{\alpha-\beta} f(s, u(s), {}^cD_{0+}^\beta u(s))(t) \\ &+ \frac{2v_6 t^{2-\beta} + (2-\beta)v_1 t^{1-\beta}}{\Gamma(3-\beta)} I_{0+}^{\alpha+\sigma_1} f(s, u(s), {}^cD_{0+}^\beta u(s))(\eta_1) \\ &+ \frac{2v_5 t^{2-\beta} + (2-\beta)v_2 t^{1-\beta}}{\Gamma(3-\beta)} I_{0+}^{\alpha+\sigma_2} f(s, u(s), {}^cD_{0+}^\beta u(s))(\eta_2) \\ &+ \frac{2v_8 t^{2-\beta} + (2-\beta)v_4 t^{1-\beta}}{\Gamma(3-\beta)} I_{0+}^{\alpha-\beta_1} f(s, u(s), {}^cD_{0+}^\beta u(s))(1) \\ &+ \frac{2v_7 t^{2-\beta} + (2-\beta)v_3 t^{1-\beta}}{\Gamma(3-\beta)} g(u), \end{aligned}$$

using similar techniques as to get (19), it yields:

$$\|{}^cD_{0+}^\beta Tu - {}^cD_{0+}^\beta Tv\|_\infty \leq (M_2 \|g\|_{L^1} + N_2 \omega) \|u - v\|_E. \tag{20}$$

Combining (19) and (20), we get:

$$\|Tu - Tv\|_E \leq ((M_1 + M_2) \|g\|_{L^1} + (N_1 + N_2) \omega) \|u - v\|_E.$$

Therefore, we have:

$$\|Tu - Tv\|_E \leq \gamma_1 \|u - v\|_E.$$

In view of the given condition $\gamma_1 = (M_1 + M_2) \|g\|_{L^1} + (N_1 + N_2) \omega < 1$, it follows that the mapping T is a contraction. Hence, by the Banach fixed point theorem, T has a unique fixed point which is a unique solution of Problem (1). This completes the proof. \square

Our next result relies on a fixed point theorem of O'Regan [22]

Lemma 3.5 Denote by U an open set in a closed, convex set C of a Banach space E . Assume $0 \in U$. Also assume that $T(\overline{U})$ is bounded and that $F : \overline{U} \rightarrow C$ is given by $T = T_1 + T_2$, in which $T_1 : \overline{U} \rightarrow E$ is continuous and completely continuous and $T_2 : \overline{U} \rightarrow E$ is a nonlinear contraction (i.e., there exists a non-negative nondecreasing function $\phi : [0, \infty) \rightarrow [0, \infty)$ satisfying $\phi(z) < z$ for $z > 0$, such that $\|T_2(u) - T_2(v)\| \leq \phi(\|u - v\|)$ for all $u, v \in \overline{U}$). Then, either

- (c1) T has a fixed point in \overline{U} , or
(c2) there exists $u \in \partial U$ and $\lambda \in (0, 1)$, such that $u = \lambda Tu$, where \overline{U} and ∂U , respectively, represent the closure and boundary of U .

Let

$$B_r = \{u \in E : \|u\|_E < r\},$$

and denote the maximum number by:

$$M_r = \max \{|f(t, u, v)| : (t, u, v) \in J \times [-r, r] \times [-r, r]\}.$$

Theorem 3.6 Suppose that (H1),(H3)–(H6) are satisfied. Then, the problem (1) has at least one solution on J .

Proof From (H6), there exists a number $r_0 > 0$, such that:

$$\frac{r_0}{(M_1 + M_2)\|p\|_{L^1}\psi(r_0)} > \frac{1}{1 - (N_1 + N_2)\omega}. \quad (21)$$

We shall prove that T_1 and T_2 defined by (11) and (12), respectively, satisfy the conditions of Lemma 3.5. We split the proof into several steps.

Step 1 : The operator T_1 is completely continuous. It is clear that T_1 is continuous, since f is continuous. Next, we shall prove that the operator T_1 maps bounded sets into bounded sets in E . For $u \in \overline{B}_{r_0}$ and for each $t \in J$, we have:

$$\begin{aligned} |(T_1 u)(t)| &\leq I_{0+}^\alpha |f(s, u(s), {}^c D_{0+}^\beta u(s))|(t) \\ &\quad + (|v_6| + |v_1|) I_{0+}^{\alpha+\sigma_1} |f(s, u(s), {}^c D_{0+}^\beta u(s))|(\eta_1) \\ &\quad + (|v_2| + |v_5|) I_{0+}^{\alpha+\sigma_2} |f(s, u(s), {}^c D_{0+}^\beta u(s))|(\eta_2) \\ &\quad + (|v_8| + |v_4|) I_{0+}^{\alpha-\beta_1} |f(s, u(s), {}^c D_{0+}^\beta u(s))|(1). \end{aligned}$$

In view of (15), we obtain:

$$\begin{aligned} |(T_1 u)(t)| &\leq \psi(\|u\|) \left\{ I_{0+}^\alpha p(s)(t) + (|v_6| + |v_1|) I_{0+}^{\alpha+\sigma_1} p(s)(\eta_1) \right. \\ &\quad \left. + (|v_2| + |v_5|) I_{0+}^{\alpha+\sigma_2} p(s)(\eta_2) + (|v_8| + |v_4|) I_{0+}^{\alpha-\beta_1} p(s)(1) \right\}. \end{aligned}$$

By Lemmas (2.7), (2.8), we obtain:

$$|(T_1 u)(t)| \leq M_1 \|p\|_{L^1} \psi_1(r_0).$$

Thus:

$$\|T_1 u\|_\infty \leq M_1 \|p\|_{L^1} \psi_1(r_0). \quad (22)$$

With the same arguments as in (22), we get:

$$\|{}^c D_{0+}^\beta T_1 u\|_\infty \leq M_2 \|p\|_{L^1} \psi_1(r_0). \quad (23)$$



Combining (22) with (23), we get:

$$\|T_1u\|_E \leq (M_1 + M_2)\|p\|_{L^1}\psi_1(r_0), \tag{24}$$

which shows that T_1 is uniform bounded.

Now, we will prove that $T_1(\overline{B}_{r_0})$ is equi-continuous. Let $t_1, t_2 \in [0, 1], t_1 < t_2, u \in \overline{B}_{r_0}$:

$$\begin{aligned} & |(T_1u)(t_2) - (T_1u)(t_1)| \\ & \leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] |f(s, u(s), {}^cD_{0+}^\beta u(s))| ds \\ & \quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} |f(s, u(s), {}^cD_{0+}^\beta u(s))| ds \\ & \quad + (|v_6|(t_2^2 - t_1^2) + |v_1|(t_2 - t_1)) I_{0+}^{\alpha+\sigma_1} |f(s, u(s), {}^cD_{0+}^\beta u(s))|(\eta_1) \\ & \quad + (|v_2|(t_2 - t_1) + |v_5|(t_2^2 - t_1^2)) I_{0+}^{\alpha+\sigma_2} |f(s, u(s), {}^cD_{0+}^\beta u(s))|(\eta_2) \\ & \quad + (|v_8|(t_2^2 - t_1^2) + |v_4|(t_2 - t_1)) I_{0+}^{\alpha-\beta_1} |f(s, u(s), {}^cD_{0+}^\beta u(s))|(1). \end{aligned}$$

Some computations give:

$$\begin{aligned} |(T_1u)(t_2) - (T_1u)(t_1)| \leq M_r & \left[\frac{1}{\Gamma(\alpha)} + \frac{2|v_6| + |v_1|}{\Gamma(\alpha + \sigma_1 + 1)} \eta_1^{\alpha+\sigma_1} \frac{2|v_2| + |v_5|}{\Gamma(\alpha + \sigma_2 + 1)} \eta_2^{\alpha+\sigma_2} \right. \\ & \left. + \frac{2|v_8| + |v_4|}{\Gamma(\alpha - \beta_1 + 1)} \right] (t_2 - t_1). \end{aligned}$$

In a like manner:

$$\begin{aligned} |{}^cD_{0+}^\beta (T_1u)(t_2) - {}^cD_{0+}^\beta (T_1u)(t_1)| \leq M_r & \left\{ \frac{t_2^{\alpha-\beta} - t_1^{\alpha-\beta}}{\Gamma(\alpha - \beta + 1)} \right. \\ & + \frac{2|v_6||t_2^{2-\beta} - t_1^{2-\beta}| + (2 - \beta)|v_1||t_2^{1-\beta} - t_1^{1-\beta}|}{\Gamma(3 - \beta)\Gamma(\alpha + \sigma_1 + 1)} \eta_1^{\alpha+\sigma_1} \\ & + \frac{2|v_5||t_2^{2-\beta} - t_1^{2-\beta}| + (2 - \beta)|v_2||t_2^{1-\beta} - t_1^{1-\beta}|}{\Gamma(3 - \beta)\Gamma(\alpha + \sigma_2 + 1)} \eta_2^{\alpha+\sigma_2} \\ & \left. + \frac{2|v_8||t_2^{2-\beta} - t_1^{2-\beta}| + (2 - \beta)|v_4||t_2^{1-\beta} - t_1^{1-\beta}|}{\Gamma(3 - \beta)\Gamma(\alpha - \beta_1 + 1)} \right\}. \end{aligned}$$

In consequence, we obtain:

$$\sup_{u \in \overline{B}_{r_0}} |(T_1u)(t_2) - (T_1u)(t_1)| + \sup_{u \in \overline{B}_{r_0}} |{}^cD_{0+}^\beta (T_1u)(t_2) - {}^cD_{0+}^\beta (T_1u)(t_1)| \rightarrow 0,$$

as $t_1 \rightarrow t_2$ independent of $u \in \overline{B}_{r_0}$. Therefore, the operator T_1 is equi-continuous and uniformly bounded. Hence, by the Arzelà–Ascoli theorem, it follows that $T_1(\overline{B}_{r_0})$ is relatively compact in E .

Step 2 : The operator $T_2 : \overline{B}_{r_0} \rightarrow E$ is contractive. Let $u, v \in$ and $t \in J$ by (H3), we can show that:

$$\|T_2(u) - T_2(v)\|_E \leq (N_1 + N_2)\omega\|u - v\|_E,$$

which, in view of (H3), implies that T_2 is a contraction.

Step 3: The set $T_2(\overline{B}_{r_0})$ is bounded. For $u \in \overline{B}_{r_0}$ and for each $t \in J$, and by (H3) and (H4), we obtain:

$$\|T_2u\|_E \leq (N_1 + N_2)\omega r_0.$$

Hence, $T_2(\overline{B}_{r_0})$ is bounded.

Step 4 : Finally, we show that the case (c2) in Lemma 3.5 does not occur. To this end, we suppose that (c2) holds. Let $u \in \partial B_{r_0}$, and then, $u = \lambda Tu$, for some $0 < \lambda < 1$. Therefore, we have $\|u\|_E = r_0$, and:

$$\begin{aligned}
 u(t) = \lambda \{ & I_{0+}^\alpha f(s, u(s), {}^cD_{0+}^\beta u(s))(t) \\
 & + (v_6 t^2 - v_1 t) I_{0+}^{\alpha+\sigma_1} f(s, u(s), {}^cD_{0+}^\beta u(s))(\eta_1) \\
 & + (v_2 t - v_5 t^2) I_{0+}^{\alpha+\sigma_2} f(s, u(s), {}^cD_{0+}^\beta u(s))(\eta_2) \\
 & + (v_8 t^2 - v_4 t) I_{0+}^{\alpha-\beta_1} f(s, u(s), {}^cD_{0+}^\beta u(s))(1) \\
 & + (1 - v_3 t - v_7 t^2) g(u) \}.
 \end{aligned}$$

With hypotheses (H3)–(H6), we can write:

$$r_0 \leq (M_1 + M_2)\|p\|_{L^1}\psi(r_0) + (N_1 + N_2)\omega r_0, \tag{25}$$

which implies that:

$$\frac{r_0}{(M_1 + M_2)\|p\|_{L^1}\psi(r_0)} \leq \frac{1}{1 - (N_1 + N_2)\omega}, \tag{26}$$

which contradicts (16). Consequently, we have proved that the operators T_1 and T_2 satisfy all the conditions in Lemma 3.5. Hence, the operator T has at least one fixed point $u \in \overline{B}_{r_0}$, which is the solution of the boundary value problem (1). □

Our next result relies on the following fixed point theorem due to Burton and Kirk [8]

Theorem 3.7 *Let E be a Banach space, and $A, B : E \rightarrow E$ be two operators, such that A is a contraction and B is completely continuous. Then, either*

- the operator equation $u = A(u) + B(u)$ has a solution, or
- the set $\Omega = \{u \in E : \lambda A(\frac{u}{\lambda}) + \lambda B(u) = u\}$ is unbounded for $\lambda \in (0, 1)$.

Then, there exists $z \in U$, such that $z = Az + Bz$.

Theorem 3.8 *Assume that (H1), (H3), (H4), and (H7) hold. In addition, we suppose that:*

$$(M_1 + M_2)\|p\|_{L^1} + (N_1 + N_2)\omega < 1.$$

Then, the fractional boundary value problem (1) has at least one solution on J .

Proof We define the map $T : E \rightarrow E$ by:

$$Tu(t) = T_1u(t) + T_2u(t), t \in J,$$

where T_1 and T_2 are defined by (11) and (12), respectively. Notice that Problem (1) is equivalent to the fixed point problem $T(u) = u$. We shall prove that T_1 and T_2 satisfy the conditions of Theorem 3.7. For clarity, we will divide the remain of the proof into several steps.

Step 1: T_1 is continuous. The continuity of f implies that the operator T_1 is continuous.

Step 2: The operator T_1 maps bounded sets into bounded sets in E . In a similar manner as previous proof of Theorem 3.6, we can find that:

$$\|T_1u\|_E \leq (M_1 + M_2)\|p\|_{L^1}(1 + \|u\|_E).$$

Step 3: The operator T_1 maps bounded sets into equi-continuous sets. By a discussion similar to that of Theorem 3.6, we can get:

$$|(T_1u)(t_2) - (T_1u)(t_1)| \rightarrow 0.$$

And

$$|{}^cD_{0+}^\beta T_1y(t_2) - {}^cD_{0+}^\beta T_1y(t_1)| \rightarrow 0,$$

as $t_2 \rightarrow t_1$. This implies that:

$$\|(Tu)(t_2) - (Tu)(t_1)\|_E \rightarrow 0, \text{ as } t_2 \rightarrow t_1.$$

Step 4: The operator T_2 is a contraction. This was established in Step 2 of Theorem 3.6.

Step 5: A priori bounds on solutions. Now, it remains to show that the set $\Omega = \{u \in E : \lambda T_2\left(\frac{u}{\lambda}\right) + \lambda T_1(u) = u\}$ is unbounded for $\lambda \in (0, 1)$. Let $\lambda \in (0, 1)$, and $u \in E$ be a solution of the integral equation:

$$\begin{aligned} u(t) = \lambda \left\{ & I_{0+}^\alpha f(s, u(s), {}^cD_{0+}^\beta u(s))(t) \right. \\ & + (v_6 t^2 - v_1 t) I_{0+}^{\alpha+\sigma_1} f(s, u(s), {}^cD_{0+}^\beta u(s))(\eta_1) \\ & + (v_2 t - v_5 t^2) I_{0+}^{\alpha+\sigma_2} f(s, u(s), {}^cD_{0+}^\beta u(s))(\eta_2) \\ & + (v_8 t^2 - v_4 t) I_{0+}^{\alpha-\beta_1} f(s, u(s), {}^cD_{0+}^\beta u(s))(1) \\ & \left. + (1 - v_3 t - v_7 t^2) g\left(\frac{u}{\lambda}\right) \right\}. \end{aligned}$$

This implies by (H3), (H4), and (H7) that, for each $t \in J$, we have:

$$\begin{aligned} |u(t)| \leq & I_{0+}^\alpha |f(s, u(s), {}^cD_{0+}^\beta u(s))|(t) \\ & + (|v_6| + |v_1|) I_{0+}^{\alpha+\sigma_1} |f(s, u(s), {}^cD_{0+}^\beta u(s))|(\eta_1) \\ & + (|v_2| + |v_5|) I_{0+}^{\alpha+\sigma_2} |f(s, u(s), {}^cD_{0+}^\beta u(s))|(\eta_2) \\ & + (|v_8| + |v_4|) I_{0+}^{\alpha-\beta_1} |f(s, u(s), {}^cD_{0+}^\beta u(s))|(1) \\ & + (1 + |v_3| + |v_7|) |g\left(\frac{u(t)}{\lambda}\right)| \\ \leq & (1 + \|u\|_E) \left(I_{0+}^\alpha p(s)(t) + (|v_6| + |v_1|) I_{0+}^{\alpha+\sigma_1} p(s)(\eta_1) \right. \\ & \left. + (|v_2| + |v_5|) I_{0+}^{\alpha+\sigma_2} p(s)(\eta_2) + (|v_8| + |v_4|) I_{0+}^{\alpha-\beta_1} p(s)(1) \right) \\ & + (1 + |v_3| + |v_7|) \omega \|u\|. \end{aligned}$$

According to the Lemmas 2.7 and 2.8, we have:

$$\|u\|_\infty \leq M_1 \|p\|_{L^1} (1 + \|u\|_E) + N_1 \omega \|u\|, \tag{27}$$

for all $t \in J$. In a similar manner, we obtain:

$$\|{}^cD_{0+}^\beta u\|_\infty \leq M_2 \|p\|_{L^1} (1 + \|u\|_E) + N_2 \omega \|u\|. \tag{28}$$

Combining (27) with (28), we get:

$$\|u\|_E \leq \frac{(M_1 + M_2) \|p\|_{L^1}}{1 - ((M_1 + M_2) \|p\|_{L^1} + (N_1 + N_2) \omega)}.$$

This shows that the set Ω is bounded, since $(M_1 + M_2) \|p\|_{L^1} + (N_1 + N_2) \omega < 1$. Hence, T has a fixed point by Theorem 3.7, and consequently, the problem (1) has a solution. This completes the proof. \square

4 An example

Example 4.1 Let us consider Problem (1) with specific data:

$$\begin{aligned}\alpha &= \frac{5}{2}, \beta = \frac{1}{2}, \beta_1 = \frac{1}{4}, \\ a &= 1, b = 2, \sigma_1 = \frac{5}{2}, \\ \sigma_2 &= \frac{7}{2}, \eta_1 = \frac{1}{4}, \eta_2 = \frac{1}{2}.\end{aligned}\quad (29)$$

Using the given values of the parameters in (6) and (14), by the Matlab program, we find that:

$$\begin{aligned}v_1 &= -1.0062, v_2 = 0.0027, v_3 = -0.0535, \\ v_4 &= 0.0014, v_5 = -1.6060, v_6 = -0.8803, \\ v_7 &= 0.0457, v_8 = -0.8030, M_1 = 1.5542, \\ M_2 &= 2.1903, M_1 + M_2 = 3.7445, N_1 = 1.0993, \\ N_2 &= 0.1292, N_1 + N_2 = 1.2285, \Delta = 0.6179, .\end{aligned}\quad (30)$$

To illustrate Theorem 3.4, we take:

$$\begin{aligned}f(t, u(t), {}^cD_{0+}^{\frac{1}{4}}u(t)) &= \frac{(1-t)^2}{10} \left(\frac{1}{2} \left(x(t) + \sqrt{x^2(t) + 1} \right) \right. \\ &\quad \left. + \frac{\left| {}^cD_{0+}^{\frac{1}{4}}u(t) \right|}{1 + \left| {}^cD_{0+}^{\frac{1}{4}}u(t) \right|} \right) + e^{-t}, \\ g(u) &= \frac{1}{10}(1 - \cos u),\end{aligned}\quad (31)$$

in (1) and note that:

$$\begin{aligned}|f(t, u, v) - f(t, u_1, v_1)| &\leq \frac{(1-t)^2}{10} \left(\left| \frac{1}{2} \left(u - u_1 + \sqrt{1+u^2} - \sqrt{1+u_1^2} \right) \right| \right. \\ &\quad \left. + \left| \frac{|v|}{1+|v|} - \frac{|v_1|}{1+|v_1|} \right| \right) \\ &\leq \frac{(1-t)^2}{10} \left(\left| \frac{1}{2}(u - u_1) \left(1 + \frac{u + u_1}{\sqrt{1+u^2} + \sqrt{1+u_1^2}} \right) \right| \right. \\ &\quad \left. + \frac{|v - v_1|}{(1+|v|)(1+|v_1|)} \right) \\ &\leq \frac{(1-t)^2}{10} (|u - u_1| + |v - v_1|),\end{aligned}$$

for each $(t, u, v) \in [0, 1] \times \mathbb{R} \times \mathbb{R}$ and

$$|g(u) - g(v)| \leq \frac{1}{10}|u - v|,$$

for each $(u, v) \in \mathbb{R} \times \mathbb{R}$.

Here, $\omega = \frac{1}{10}$, $g(t) = \frac{(1-t)^2}{10}$ with $\|g\|_{L^1} = \frac{1}{30}$. Using the values of $M_1 + M_2$ and $N_1 + N_2$ given by (30), we find that $\gamma_1 = (M_1 + M_2)\|g\|_{L^1} + (N_1 + N_2)\omega = 0.2477 < 1$. As all the conditions of Theorem 3.4 are satisfied, therefore, there exists a unique solution of Problem (1) with the data (29) and (31) on J .



Conclusion

We have established the existence and uniqueness of solutions for fractional boundary value problem under more general assumptions. Using the techniques of fixed point theorems such as Banach contraction principle, O'Regan fixed point theorem, and Burton and Kirk fixed point theorem, we establish the main results of the paper. Finally, we conclude the paper with an example to make our results clear.

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