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Statistical Tauberian theorems for Cesàro integrability mean based on post-quantum calculus

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Abstract The notion of statistical convergence is more general than the classical convergence. Tauberian theorems via different ordinary summability means have been established by many researchers. In the present work, we have established some new Tauberian theorems based on post-quantum calculus via statistical Cesàro summability mean of real-valued continuous function of one variable under oscillating behavior and De la Vallée Poussin mean of a single integral. Moreover, some remarks and corollaries are provided here to support our theorems.

Mathematics Subject Classification 40C10 · 40G05 · 40E05

1 Introduction and motivation

It makes no sense to speak of the sum of a divergent series. Nevertheless, a series that is not “too badly divergent” can be assigned a generalized sum in a variety of natural ways. We are familiar with the notion of summability in connection with the theorems of Abel and Cesàro, which asserts that every convergent series is Abel or Cesàro summable to its ordinary sum. More generally, an Abelian theorem or Cesàro theorem work to the effect that a method of summability assigns to each convergent series its ordinary sum. A Tauberian theorem goes in the opposite direction and asserts that every summable series, which is not too badly divergent is actually convergent.

Tauber [33] introduced the first Tauberian theorem for single sequence, that an Abel summable sequence is convergent with some suitable conditions. A number of authors such as Landau [18], Hardy and Littlewood [7], and Schmidt [25] obtained some classical Tauberian theorems for Cesàro and Abel summability methods of single sequence. Recently, Çanak and Totur [2], and Jena et al. [9] investigated and studied several Tauberian theorems for single sequence. Knopp [16] obtained some classical type Tauberian theorems for Abel and $(C, 1, 1)$ summability methods of double sequences and proved that these methods hold for the set of bounded sequences. Móricz [19] proved some Tauberian theorems for Cesàro summable double sequences and deduced Tauberian theorems of Landau [17] and Hardy [6] type. Recently, Totur [34] extended some classical type Tauberian theorems for single and double sequences in connection with one-sided Tauberian theorems.

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Furthermore, Çanak and Totur [1] proved a Tauberian theorem for Cesàro summability of single integral and also, they established the alternative proofs of some classical type Tauberian theorems for the Cesàro summability of single integral. Moreover, in the year 2017, Jena et al. [11] proved the Tauberian theorems for Harmonic summability of double-integrable real-valued function over $\mathbb{R}^2$ and also established the inclusion relation between the statistical convergence and classical convergence. Very recently, Çanak et al. [3] introduced and studied the concept of Tauberian theorem for Cesàro integrability mean based upon quantum calculus via usual convergence.

Motivated essentially by the above-mentioned investigations and results, here we prove the statistical versions of Tauberian theorems via Cesàro integrability mean based upon post-quantum calculus of a real-valued continuous function of one variable under slow oscillation and De la Vallée Poussin mean of the integral. In fact, we extend here the result of Çanak [3] using the idea of Tauberian theorems for Cesàro integrability mean based on quantum calculus via usual summability.

2 Preliminaries and definitions

The notion of statistical convergence was first introduced and studied by Fast [5] and Steinhaus [32] independently. For more recent works in this direction, one may refer [4, 8–15, 21–24, 26–31].

Let $f(x)$ be a function in $\mathbb{R}$ with the partial sum

$$s(x) = \int_0^x f(\zeta) d\zeta, \quad (0 < x < \infty).$$

The $(C, 1)$ mean of $f(x)$ is (see [20]),

$$\sigma(s(x)) = \frac{1}{x} \int_0^x s(\zeta) d\zeta.$$

The integral $\int_0^x s(\zeta) d\zeta$ is $(C, 1)$ -summable to a finite number ℓ , if

$$\lim_{x \rightarrow \infty} \sigma(s(x)) \rightarrow \ell. \quad (2.1)$$

The integral $\int_0^x s(\zeta) d\zeta$ is statistically $(C, 1)$ -summable to a finite number ℓ if, for each $\epsilon > 0$,

$$\lim_{u \rightarrow \infty} \frac{1}{u} |\{0 < x \leq u \text{ and } |\sigma(s(x)) - \ell| \geq \epsilon\}| = 0.$$

In this case, we write

$$\text{stat} \lim_{x \rightarrow \infty} \sigma(s(x)) = \ell. \quad (2.2)$$

If $\lim_{x \rightarrow \infty} s(x) = \ell$ exists, then the relations (2.1) and (2.2) hold. However, in general, the converse is not true. To prove the sufficient part, we use the oscillatory behavior and De la Vallée Poussin mean of the above integral over $\mathbb{R}$. Such a condition is called a Tauberian condition and the resulting theorem is called a Tauberian theorem.

Quantum calculus (or q -calculus) is the modern name for the investigation of calculus which is focused on the idea of deriving q -analogues of results belonging to standard calculus without using limits. The q -calculus served as a bridge between mathematics and physics. Furthermore, there is a possibility of extension of the q -calculus to post-quantum calculus (or (p, q) -calculus). It has gained noticeable importance and popularity during the past 3 decades due mainly to its applications in different mathematical areas such as number theory, combinatorics, orthogonal polynomials, basic hypergeometric series and other sciences such as statistical mechanics, quantum theory, and the theory of relativity.

We recall some basic concepts of (p, q) -calculus and some associated properties of (p, q) -Cesàro integrability mean method. For any $n \in \mathbb{N}$, the (p, q) -integer $[n]_{p,q}$ is defined by

$$[n]_{p,q} = \begin{cases} \frac{p^n - q^n}{p - q} & (n \geq 1) \\ 0 & (n = 0) \end{cases}$$

where $0 < q, p \leq 1$.

The (p, q) -factorial is defined by

$$[n]!_{p,q} = \begin{cases} [1]_{p,q}[2]_{p,q}\dots[n]_{p,q} & (n \geq 1) \\ 1 & (n = 0). \end{cases}$$

The (p, q) -binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]!_{p,q}}{[k]!_{p,q} [n-k]!_{p,q}} \quad \text{for all } n, k \in \mathbb{N} \text{ and } n \geq k.$$

We also recall that, suppose $0 < q < p \leq 1$ and the (p, q) -derivative of the function f is defined as

$$D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p-q)x}, \quad x \neq 0.$$

As a special case, when $p = 1$, (p, q) -derivative reduces to q -derivative and also, the (p, q) -derivative fulfills the following product derivative properties:

$$D_{p,q}(f(x)g(x)) = f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x)$$

and

$$D_{p,q}(f(x)g(x)) = g(px)D_{p,q}f(x) + f(qx)D_{p,q}g(x).$$

Next, let f be a real-valued continuous function and let a be a real number. The (p, q) -integral of $f(x)$ on $[0, a]$ is defined as

$$\int_0^a f(x) d_{p,q}x = (p-q)a \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}}a\right), \quad \text{if } \left|\frac{q}{p}\right| < 1$$

and

$$\int_0^{\infty} f(x) d_{p,q}x = (p-q) \sum_{k=-\infty}^{\infty} \frac{q^k}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}}\right), \quad \text{if } \left|\frac{q}{p}\right| < 1$$

provided the sums converge absolutely.

A function f is said to be (p, q) -integrable on $[0, \infty)$, if the series

$$\sum_{k=-\infty}^{\infty} \frac{q^k}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}}\right)$$

converges absolutely. Moreover, we assume that, $f(t)$ is a function defined on $[0, \infty)$ and satisfying $\int_0^x |f(t) d_{p,q}t| < \infty$ and also, let the partial sum of $f(t)$ be defined as

$$\bar{s}(x) = \int_0^x f(t) d_{p,q}t. \quad (2.3)$$

Next, we define the (p, q) Cesàro mean of $f(x)$ as follows:

$$t(\bar{s}(x)) = \frac{1}{x} \int_0^x \bar{s}(\zeta) d_{p,q}\zeta. \quad (2.4)$$

The function $\bar{s}(x)$ is (p, q) -Cesàro integrable to ℓ , if

$$\lim_{x \rightarrow \infty} t(\bar{s}(x)) = \ell.$$

Furthermore, the function $\bar{s}(x)$ is (p, q) -statistically Cesàro summable to ℓ if, for every $\epsilon > 0$,

$$\lim_{u \rightarrow \infty} \frac{1}{u} |\{0 < x \leq u \text{ and } |t(\bar{s}(x)) - \ell| \geq \epsilon\}| = 0.$$

In this case, we write

$$\text{stat} \lim_{x \rightarrow \infty} t(\bar{s}(x)) = \ell. \quad (2.5)$$

If $\lim_{x \rightarrow \infty} \bar{s}(x) = \ell$ exists, then the relations (2.4) and (2.5) hold. However, in general, the converse is not true. To prove the oscillatory behavior and De la vallée Poussin mean based on sufficient part, we use the (p, q) -calculus of the above integral over $\mathbb{R}$. Such a condition is called a Tauberian condition and the resulting theorem is called a Tauberian theorem under post-quantum calculus.

For each non-negative integer k we define

$$\sigma^{(k)}(s(x)) = \begin{cases} \frac{1}{x} \int_0^x \sigma^{(k-1)}s(\zeta) d\zeta & (k \geq 1) \\ \int_0^x s(\zeta) d\zeta & (k = 0). \end{cases}$$

An integral $\int_0^\infty f(x)dx$ is said to be statistically (C, k) -summable to ℓ if and only if $\sigma^{(k)}(s(x))$ is statistically summable to ℓ .

Remark 2.1 If $k = 1$, then (C, k) -summability mean is same as the $(C, 1)$ -summability mean.

Next, for (p, q) -calculus and for each non-negative integer k , we define

$$t^{(k)}(\bar{s}(x)) = \begin{cases} \frac{1}{x} \int_0^x t^{(k-1)}\bar{s}(\zeta) d_{p,q}\zeta & (k \geq 1) \\ \int_0^x \bar{s}(\zeta) d_{p,q}\zeta & (k = 0). \end{cases}$$

An integral $\int_0^\infty f(x)dx$ is said to be statistically (C, k) -summable to ℓ under post-quantum calculus if and only if $t^{(k)}(\bar{s}(x))$ is statistically summable to ℓ .

Remark 2.2 If $k = 1$ and $p = 1$, then (C, k) -summability mean reduces $(C, 1)$ -summability mean in q -calculus. Furthermore, if $k = 1$ and $q = p = 1$, then (C, k) -summability mean reduces to simply $(C, 1)$ -summability mean.

Next we have, the partial sum of the function is

$$s(x) = \int_0^x f(\zeta) d\zeta$$

and consequently,

$$s(x) - \sigma(s(x)) = v(f(x)), \quad (2.6)$$

where

$$v(f(x)) = \frac{1}{x} \int_0^x \zeta f(\zeta) d\zeta.$$

Notice that

$$\sigma(s(x)) = \frac{v(f(x))}{x}.$$

Now, let us define for each non-negative integer k ,

$$v^{(k)}(f(x)) = \begin{cases} \frac{1}{x} \int_0^x v^{(k-1)}\zeta f(\zeta) d\zeta, & \text{for } k \geq 1 \\ \int_0^x \zeta f(\zeta) d\zeta, & \text{for } k = 0. \end{cases}$$

Here, the integral $\int_0^\infty xf(x)dx$ is statistically (C, k) -summable to ℓ if and only if $v^{(k)}(f(x))$ is statistically summable to ℓ .

Similarly, for post-quantum calculus, the partial sum of the function is

$$\bar{s}(x) = \int_0^x f(\zeta) d_{p,q}\zeta.$$

Also, we have

$$\bar{s}(x) - t(\bar{s}(x)) = (p, q)\bar{v}(f(x)), \quad (2.7)$$

where

$$\bar{v}(f(x)) = \frac{1}{x} \int_0^x \zeta f(\zeta) d_{p,q}\zeta.$$

Notice that

$$D_{p,q}(\bar{t}(\bar{s}(x))) = \frac{\bar{v}(f(x))}{x}.$$

Now, we define for each non negative integer k ,

$$\bar{v}^{(k)}(f(x)) = \begin{cases} \frac{1}{x} \int_0^x \bar{v}^{(k-1)} \zeta f(\zeta) d_{p,q}\zeta, & \text{for } k \geq 1 \\ \int_0^x \zeta f(\zeta) d_{p,q}\zeta, & \text{for } k = 0. \end{cases}$$

Here, the integral $\int_0^\infty xf(x)dx$ is statistically (C, k) -summable to ℓ under post-quantum calculus if and only if $\bar{v}^{(k)}(f(x))$ is summable to ℓ .

Now recalling the De la Vallée Poussin mean (see [3]) of the integral $\int_0^x f(\zeta) d\zeta$, we have

$$\tau(s(x)) = \frac{1}{(x - \lambda x)} \int_{\lambda x}^x s(\zeta) d\zeta, \quad \lambda \in (0, 1).$$

In the similar way, we define the De la Vallée Poussin mean for the post-quantum calculus of the integral $\int_0^x f(\zeta) d\zeta$ as

$$\tau(\bar{s}(x)) = \frac{1}{(p - q)x} \int_{qx}^{px} s(\zeta) d_{p,q}\zeta, \quad (0 < q < p \leq 1).$$

Note that, an integral $\int_0^x f(x)dx$ belonging to $\mathbb{R}$ is oscillating slowly if [20]

$$\lim_{\lambda \rightarrow 1^-} \limsup_{x \rightarrow \infty} \max_{\lambda x \leq \zeta \leq x} |s(\zeta) - s(x)| = 0. \quad (2.8)$$

In the same way, under post-quantum calculus the integral $\int_0^x f(x)dx$ belonging to $\mathbb{R}$ is oscillating slowly if

$$\limsup_{x \rightarrow \infty} \max_{qx \leq \zeta \leq px} |s(\zeta) - s(x)| = 0. \quad (2.9)$$

3 Main results

Theorem 3.1 *If $\bar{s}(x)$ is (p, q) -statistically $(C, 1)$ -summable to ℓ and $\bar{s}(x)$ is (p, q) -oscillating slowly, then $\bar{s}(x) \rightarrow \ell$, as $x \rightarrow \infty$.*

The proof of this theorem requires the following lemmas

Lemma 3.2 *The sequence of partial sum $\bar{s}(x)$ of an integrable function $f(x)$ over $\mathbb{R}$ is (p, q) -oscillating slowly if and only if $\bar{v}(f(x))$ is bounded and oscillating slowly.*

Proof Let $\bar{s}(x)$ be (p, q) -oscillating slowly. Initially, let us show that $\bar{v}(f(x)) = O(1)$ as $x \rightarrow \infty$.

We have

$$\int_0^x wf(w) d_{p,q}w = \sum_{i=0}^{\infty} \int_{x/2^{i+1}}^{x/2^i} wf(w) d_{p,q}w. \quad (3.1)$$

Now it follows from Eq. (2.7) that

$$\begin{aligned} \int_{\alpha}^{\beta} wf(w) d_{p,q}w &= \int_{\alpha}^{\beta} w\bar{s}(w) d_{p,q}w \\ &= \left[w(\bar{s}(w))_{\alpha}^{\beta} - \int_{\alpha}^{\beta} \bar{s}(w) d_{p,q}w \right] \\ &= - \int_{\alpha}^{\beta} \bar{s}(w) d_{p,q}w + \beta(\bar{s}(\beta)) - \alpha(\bar{s}(\alpha)) - \alpha(\bar{s}(\beta)) + \alpha(\bar{s}(\beta)) \\ &\quad - \int_{\alpha}^{\beta} \bar{s}(w) d_{p,q}w + (\beta - \alpha)\bar{s}(\beta) + \alpha(\bar{s}(\beta) - \bar{s}(\alpha)) \\ &= - \int_{\alpha}^{\beta} (\bar{s}(w) - \bar{s}(\beta)) d_{p,q}w + \alpha(\bar{s}(\beta) - \bar{s}(\alpha)) \\ &= (\beta - \alpha) \max_{q\alpha \leq x \leq p\beta} |\bar{s}(x) - \bar{s}(\beta)| + \alpha |\bar{s}(\beta) - \bar{s}(\alpha)|. \end{aligned}$$

If we choose $\beta = x/2^i$ and $\beta/\alpha \leq 2$, we obtain

$$\int_0^x wf(w) d_{p,q}w \leq A \sum_{i=0}^{\infty} \frac{x}{2^i} = O(x), \quad \text{as } x \rightarrow \infty.$$

Now we have to show that $t(\bar{s}(x))$ is oscillating slowly. Since

$$D_{p,q}(\bar{t}(\bar{s}(x))) = \frac{\bar{v}(f(x))}{x},$$

we get

$$\begin{aligned} |t(\bar{s}(\zeta)) - t(\bar{s}(x))| &= \left| \int_x^{\zeta} \bar{t}(\bar{s}(w)) d_{p,q}w \right| \\ &= \left| \int_x^{\zeta} f(w) d_{p,q}w \right| \\ &\leq C \int_x^{\zeta} \frac{d_{p,q}w}{w} \\ &= C \log(\zeta/x), \quad \text{for any } qx \leq \zeta \leq px. \end{aligned}$$

Clearly, we have

$$\max_{qx \leq \zeta \leq px} |t(\bar{s}(\zeta)) - t(\bar{s}(x))| \leq C$$

and

$$\lim_{x \rightarrow \infty} \sup_{q x \leq \zeta \leq p x} |t(\bar{s}(\zeta)) - t(\bar{s}(x))| = 0.$$

That implies $\bar{v}(f(x))$ is oscillating slowly by Kronecker identity.

To prove the converse part, suppose that $\bar{v}(f(x))$ is bounded and oscillating slowly. The boundedness of $\bar{v}(f(x))$ implies that $t(\bar{s}(x))$ is oscillating slowly. Since $\bar{v}(f(x))$ is oscillating slowly, so $\bar{s}(x)$ is oscillating slowly by Kronecker identity. This establishes Lemma 3.2. $\square$

Lemma 3.3 For $0 < q < p \leq 1$,

$$\bar{s}(x) - t(\bar{s}(x)) = \frac{q}{(p-q)} (t(\bar{s}(x)) - \bar{s}(x)) - \frac{q}{(p-q)x} \int_{qx}^{px} (\bar{s}(x) - \bar{s}(\zeta)) d_{p,q} \zeta.$$

Proof We have by De la Vallée Poussin mean of $\bar{s}(x)$,

$$\begin{aligned} \tau(\bar{s}(x)) &= \frac{q}{(p-q)x} \int_{qx}^{px} \bar{s}(\zeta) d_{p,q} \zeta \\ &= \frac{q}{x(p-q)} \left(\int_0^{qx} \bar{s}(\zeta) d\zeta - \int_0^{px} \bar{s}(\zeta) d\zeta \right). \end{aligned}$$

Again, since

$$t(\bar{s}(x)) = \frac{1}{x} \int_0^{qx} \bar{s}(\zeta) d_{p,q} \zeta,$$

and

$$t(\bar{s}(x)) = \frac{1}{x} \int_0^{px} \bar{s}(\zeta) d_{p,q} \zeta,$$

we have

$$\begin{aligned} \tau(\bar{s}(x)) &= \frac{q^2}{(p-q)^2} t(\bar{s}(x)) - \frac{1}{(p-q)^2} t(\bar{s}(x)) \\ &= \left(1 + \frac{1}{(p-q)} \right)^2 t(\bar{s}(x)) - \frac{1}{(p-q)^2} t(\bar{s}(x)). \end{aligned}$$

Now

$$\begin{aligned} \tau(s(x)) - t(\bar{s}(x)) &= \frac{1}{(p-q)^2} t(\bar{s}(x)) + \frac{2}{(p-q)} t(\bar{s}(x)) \\ &\quad - \frac{1}{(p-q)^2} \sigma(s(x)). \end{aligned} \tag{3.2}$$

Subtracting $t(\bar{s}(x))$ from the identity, also

$$\bar{s}(x) = \tau(\bar{s}(x)) - \frac{1}{(p-q)x} \int_{qx}^{px} (\bar{s}(\zeta) - \bar{s}(x)) d_{p,q} \zeta,$$

we have

$$\begin{aligned} \bar{s}(x) - t(\bar{s}(x)) &= \tau(\bar{s}(x)) - t(\bar{s}(x)) \\ &\quad - \frac{q}{(p-q)x} \int_{qx}^{px} (\bar{s}(\zeta) - \bar{s}(x)) d_{p,q} \zeta. \end{aligned} \tag{3.3}$$

From Eqs. (3.2) and (3.3), we get

$$\begin{aligned} \bar{s}(x) - t(\bar{s}(x)) &= \frac{1}{(p-q)^2} (t(\bar{s}(x)) - \sigma(\bar{s}(x))) \\ &\quad + \frac{2}{(p-q)} t(\bar{s}(x)) - \frac{q}{(p-q)x} \int_{qx}^{px} (\bar{s}(\zeta) - \bar{s}(x)) d_{p,q} \zeta. \end{aligned}$$

This establish Lemma 2. $\square$

Proof of Theorem 3.1.

Proof Let $\bar{s}(x)$ be statistically $(C, 1, 1)$ -summable to ℓ , this implies $t(\bar{s}(x))$ is $(C, 1)$ -summable to ℓ . Now from Eq. (2.7), we have $\bar{v}(f(x))$ is statistically $(C, 1)$ -summable to zero. Thus, by Lemma 3.2, $\bar{v}(f(x))$ is oscillating slowly. Furthermore, by Lemma 3.3, we get

$$\begin{aligned} \bar{v}(f(x)) - t(\bar{v}(f(x))) &= \frac{q}{(p-q)^2} ((t(\bar{v}(f(x)))) - t(\bar{v}(f(x)))) \\ &\quad + \frac{2}{(p-q)} t(\bar{v}(f(x))) - \frac{q}{(p-q)x} \\ &\quad \cdot \int_{qx}^{px} (\bar{v}(f(\zeta)) - \bar{v}(f(x))) d_{p,q}\zeta. \end{aligned} \quad (3.4)$$

Next, by (3.4)

$$\begin{aligned} |\bar{v}(f(x)) - t(\bar{v}(f(x)))| &\leq \frac{1}{(p-q)^2} |t(\bar{v}(f(x))) - t(\bar{v}(f(x)))| \\ &\quad + \frac{2}{(p-q)} |t(\bar{v}(f(x)))| \\ &\quad + \max_{qx \leq \zeta \leq px} |\bar{v}(f(\zeta)) - \bar{v}(f(x))|. \end{aligned} \quad (3.5)$$

Now taking $\limsup$ on both sides of Eq. (3.5) as $x \rightarrow \infty$, we obtain

$$\begin{aligned} \limsup_{x \rightarrow \infty} |\bar{v}(f(x)) - t(\bar{v}(f(x)))| &\leq \limsup_{x \rightarrow \infty} \frac{1}{(p-q)^2} |t(\bar{v}(f(x))) - t(\bar{v}(f(x)))| \\ &\quad + \limsup_{x \rightarrow \infty} \frac{2}{(p-q)} |t(\bar{v}(f(x)))| \\ &\quad + \limsup_{x \rightarrow \infty} \max_{qx \leq \zeta \leq px} |\bar{v}(f(\zeta)) - \bar{v}(f(x))|. \end{aligned} \quad (3.6)$$

Furthermore, as $t(\bar{v}(f(x)))$ converges, the first and second terms in the right-hand side of Eq. (3.6) must vanish.

This implies

$$\limsup_{x \rightarrow \infty} |\bar{v}(f(x)) - t(\bar{v}(f(x)))| \leq \limsup_{x \rightarrow \infty} \max_{qx \leq \zeta \leq px} |\bar{v}(f(\zeta)) - \bar{v}(f(x))|. \quad (3.7)$$

As $0 < q < p \leq 1$ in (3.7), so we get

$$\limsup_{x \rightarrow \infty} |\bar{v}(f(x)) - t(\bar{v}(f(x)))| \leq 0.$$

It implies that, $\bar{v}(f(x)) = o(1)$ as $x \rightarrow \infty$. Since $\bar{s}(x)$ is statistically summable to ℓ by Cesàro mean and $\bar{v}(f(x)) = o(1)$ as $x \rightarrow \infty$, so $\lim_{x \rightarrow \infty} \bar{s}(x) = \ell$. $\square$

Theorem 3.4 Let $p_n = q_n = 1$, if $s(x)$ is statistically $(C, 1)$ -summable to ℓ and $s(x)$ is oscillating slowly, then $s(x) \rightarrow \ell$, as $x \rightarrow \infty$.

Proof Let $s(x)$ is statistically $(C, 1)$ -summable to ℓ , this implies $\sigma(s(x))$ is $(C, 1)$ -summable to ℓ . Now from Eq. (2.6), we have $v(f(x))$ is statistically $(C, 1)$ -summable to zero. Thus, by Lemma 3.2, $v(f(x))$ is oscillating slowly. Again by Lemma 3.3, we get

$$\begin{aligned} v(f(x)) - \sigma(v(f(\lambda x))) &= \frac{1}{(\lambda - 1)^2} ((\sigma(v(f(\lambda x)))) - \sigma(v(f(x)))) \\ &\quad + \frac{2}{(\lambda - 1)} \sigma(v(f(\lambda x))) - \frac{1}{(\lambda x - x)} \\ &\quad \cdot \int_x^{\lambda x} (v(f(\zeta)) - v(f(x))) d\zeta. \end{aligned} \quad (3.8)$$

Next, by (3.8)

$$\begin{aligned} |v(f(x)) - \sigma(v(f(x)))| &\leq \frac{1}{(\lambda - 1)^2} |\sigma(v(f(\lambda x))) - \sigma(v(f(x)))| \\ &\quad + \frac{2}{(\lambda - 1)} |\sigma(v(f(\lambda x)))| + \max_{x \leq \zeta \leq \lambda x} |v(f(\zeta)) - v(f(x))|. \end{aligned} \quad (3.9)$$

Now taking $\limsup$ on both sides of Eq. (3.9) as $x \rightarrow \infty$, we obtain

$$\begin{aligned} \limsup_{x \rightarrow \infty} |v(f(x)) - \sigma(v(f(x)))| &\leq \limsup_{x \rightarrow \infty} \frac{1}{(\lambda - 1)^2} |\sigma(v(f(\lambda x))) - \sigma(v(f(x)))| \\ &\quad + \limsup_{x \rightarrow \infty} \frac{2}{(\lambda - 1)} |\sigma(v(f(\lambda x)))| \\ &\quad + \limsup_{x \rightarrow \infty} \max_{x \leq \zeta \leq \lambda x} |v(f(\zeta)) - v(f(x))|. \end{aligned} \quad (3.10)$$

Furthermore, as $\sigma(v(f(\lambda x)))$ converges, the first and second terms in the right-hand side of Eq. (3.10) must vanish.

This implies

$$\begin{aligned} \limsup_{x \rightarrow \infty} |v(f(x)) - \sigma(v(f(x)))| &\leq \limsup_{x \rightarrow \infty} \max_{x \leq \zeta \leq \lambda x} |v(f(\zeta)) - v(f(x))|. \end{aligned} \quad (3.11)$$

As $\lambda \rightarrow 1^+$ in (3.11), so we get

$$\limsup_{x, y \rightarrow \infty} |v(f(x)) - \sigma(v(f(x)))| \leq 0.$$

It implies that $v(f(x)) = o(1)$ as $x \rightarrow \infty$. Since $s(x)$ is statistically summable to ℓ by Cesàro mean and $v(f(x)) = o(1)$ as $x \rightarrow \infty$, so $\lim_{x \rightarrow \infty} s(x) = \ell$. $\square$

Corollary 3.5 *If $\bar{s}(x)$ is statistically (C, k) -summable to ℓ and $\bar{s}(x)$ is (p, q) -oscillating slowly, then $\bar{s}(x) \rightarrow \ell$ as $x \rightarrow \infty$.*

Proof By Lemma 3.2, $\bar{s}(x)$ is oscillating slowly and also $t^{(k)}(\bar{s}(x))$ is oscillating slowly. Furthermore, from Theorem 3.1, $\bar{s}(x)$ being statistically (C, k) -summable to ℓ , so

$$\text{stat} \lim_{x \rightarrow \infty} t^{(k)}(\bar{s}(x)) = \ell. \quad (3.12)$$

Next, from the definition,

$$t^{(k)}(\bar{s}(x)) = t^{(1)}(\bar{s}(x)) (t^{(k-1)}(\bar{s}(x))). \quad (3.13)$$

Clearly, Eqs. (3.12) and (3.13) implies that $\bar{s}(x)$ is statistically $(C, k-1)$ -summable to ℓ . Again by Lemma 3.2, $t^{(k-1)}(\bar{s}(x))$ is also oscillating slowly.

Thus, Theorem 3.1 implies

$$\lim_{x \rightarrow \infty} t^{(k-1)}(\bar{s}(x)) = \ell.$$

Continuing in this way, we get $\lim_{x \rightarrow \infty} \bar{s}(x) = \ell$. $\square$

Theorem 3.6 *If $\bar{s}(x)$ is (p, q) -statistically $(C, 1)$ -summable to ℓ and $\bar{v}(f(x))$ is (p, q) -oscillating slowly, then $\bar{s}(x) \rightarrow \ell$ ($x \rightarrow \infty$).*

Proof The proof of this theorem is similar to the proof of Theorem 3.1. $\square$

Theorem 3.7 *If $s(x)$ is statistically $(C, 1)$ -summable to ℓ and $v(f(x))$ is oscillating slowly, then $s(x) \rightarrow \ell$ as $x \rightarrow \infty$.*

Proof The proof of this theorem is similar to the proof of Theorem 3.4. $\square$

Corollary 3.8 *If $\bar{s}(x)$ is statistically (C, k) -summable to ℓ and $\bar{v}(f(x))$ is oscillating slowly, then $\bar{s}(x) \rightarrow \ell$ as $x \rightarrow \infty$.*

Proof As $\bar{v}(f(x))$ is oscillating slowly, setting $\bar{v}(f(x))$ in place of $\bar{s}(x)$; $t^{(k)}(\bar{v}(f(x)))$ is oscillating slowly by Lemma 3.2. Again as $\bar{v}(f(x))$ is statistically (C, k) summable to ℓ , so Theorem 3.6 implies

$$\text{stat} \lim_{x \rightarrow \infty} t^{(k)}(\bar{v}(f(x))) = \ell. \quad (3.14)$$

Next by definition,

$$\lim_{x \rightarrow \infty} t^{(k)}(\bar{v}(f(x))) = t^{(1)}(\bar{v}(f(x)))t^{(k-1)}(\bar{v}(f(x))). \quad (3.15)$$

From (3.14) and (3.15), it is clear that $\bar{v}(f(x))$ is statistically $(C, k-1)$ -summable to ℓ . Moreover, by Lemma 3.2, $\sigma^{k-1}(\bar{v}(f(x)))$ being oscillating slowly, by Theorem 3.6, we have $\lim_{x \rightarrow \infty} t^{k-1}(\bar{v}(f(x))) = \ell$. Continuing in this way, we obtain $\lim_{x \rightarrow \infty} \bar{v}(f(x)) = \ell$. $\square$

4 Conclusion

Tauberian theorems for single sequences as well as for functions of single variable have achieved a high degree of development; however, it is still in its infancy for double sequences and functions of two or more variables. The result established here that the statistical versions of Tauberian theorems for a real-valued continuous function of one variable under the post-quantum calculus (or (p, q) -calculus) of integrals via statistical Cesàro summability mean generalizes some earlier existing Tauberian theorems for the function of a single variable in quantum calculus (or (q) -calculus) via classical Cesàro summability mean. Further, it will be encouraging if one can extend the result for functions of two or more variables using the post-quantum calculus (or (p, q) -calculus) of integrals via the statistical Cesàro mean and so also for other different statistical versions of summability means.

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