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# Regularity results for solutions of linear elliptic degenerate boundary-value problems

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**Abstract** We give an a-priori estimate near the boundary for solutions of a class of higher order degenerate elliptic problems in the general Besov-type spaces $B^{s,\tau}_{p,q}$. This paper extends the results found in Hölder spaces $C^s$, Sobolev spaces $H^s$ and Besov spaces $B^s_{p,q}$, to the more general framework of Besov-type spaces.

**Mathematics Subject Classification** 35J30 · 35J40 · 35B45 · 35B65

## 1 Introduction, definitions, and results

### 1.1 Introduction

The aim of this article is to give an a-priori estimate for solutions of a class of linear degenerate elliptic boundary-value problems in Besov-type spaces involving the differential operator:

$$\tilde{L} = \sum_{h=0}^{\min(k,m)} \varphi^{k-h} P^{m-h}(x, D_x), \tag{1}$$

where $k \in \mathbb{N}$, $m \in \mathbb{N}\backslash\{0\}$, the function $\varphi$ is of class $C^\infty$ from $\mathbb{R}^{n+1}$ to $\mathbb{R}$ and associates with each element of $\Omega$ its distance from the boundary, with $\Omega = \{x \in \mathbb{R}^{n+1}; \varphi(x) > 0\}$, $\partial\Omega = \{x \in \mathbb{R}^{n+1}; \varphi(x) = 0\}$ and $d\varphi \neq 0$ on $\partial\Omega$; $P^{m-h}(x, D_x)$ is a differential operator with smooth coefficients on $\overline{\Omega}$ and of order $\leq m - h$.

These operators were first introduced by Shimakura [16], who obtained a regularity result in Sobolev spaces $H^s$ for solutions of the boundary-value problems associated with $\tilde{L}$. Similar results were found by Bolley and Camus [3]. In addition, the same class was considered by C. Goulaouic and N. Shimakura [14] and also by Bolley et al. [1] in Hölder spaces $C^s$. Later on, Rolland [15] gave an a-priori estimate of (1) in classical Besov spaces $B^s_{p,q}$ with $p = q$.

In this paper, we generalize the previous works to the more general frame of Besov-type spaces $B^{s,\tau}_{p,q}$. They contain all the spaces cited previously: Hölder spaces $C^s$, Sobolev spaces $H^s$, and Besov spaces $B^s_{p,q}$, and include Goldberg spaces bmo and local Morrey–Campanato spaces $l^{2,\lambda}$, as a special case (see Remark 2). In the same spaces, the first author has established a regularity result for solutions of a class of regular elliptic boundary-value problems [12]. In [13], the authors investigated a particular case of operators (1) in $B^{s,\tau}_{p,q}$ spaces, while many researchers were interested in other degenerate operators; for example [4,7,10,20].

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The results of this paper can be useful in several applications, namely, the study of the Lake equation. Indeed, a particular case of operators (1) models this phenomenon. For more details, we refer to [6] and [5, Section 7.2]. In addition, just as in [10, Section 2, Theorem 2.5], these estimates can be employed to prove the regularity of solutions of completely nonlinear boundary-value problems.

The sections of this paper will be tackled in this order, first, we will introduce the definition of $B^{s,\tau}_{p,q}$ spaces, as well as the summary of some important properties of these spaces given in [8,9,18,24], and we state our main result. The second section of this paper will be a presentation of some helpful lemmas. The third section is devoted to the trace characterization for elements of the considered weighted spaces. In the fourth section, we present the proof of the main theorem, which is based on one Peetre's method described in [1,2,12]. In other words, it consists of doing a partial Fourier transformation with respect to the tangential direction on the equation, which leads to an ordinary differential equation. Finally, by applying the isomorphism theorem, we estimate the "almost tangential" derivatives of solutions, and using the interpolation inequalities, we estimate the normal derivatives.

*Notation*

Throughout this paper, for $J \in \mathbb{Z}$, we denote by $B_J$ (resp., $B'_J$), the ball centered at $x_0 \in \mathbb{R}^{n+1}$ (resp., $x'_0 \in \mathbb{R}^n$) and with radius $2^{-J}$.

We set $B_J = \{x \in \mathbb{R}^{n+1} : |x - x_0| < 2^{-J}\}$ (resp., $B'_J = \{x' \in \mathbb{R}^n : |x' - x'_0| < 2^{-J}\}$). For $\alpha > 0$, $\alpha B_J$ (resp., $\alpha B'_J$) denotes the ball centered at $x_0$ (resp., $x'_0$) and of radius $\alpha 2^{-J}$. Likewise, for $\nu \in \mathbb{Z}$, $F_\nu$ (resp., $F'_\nu$) denotes the annulus centered at $x_0 \in \mathbb{R}^{n+1}$, (resp., $x'_0 \in \mathbb{R}^n$), such that $F_\nu = \{x \in \mathbb{R}^{n+1} : \ 2^\nu \leq |x - x_0| \leq 2^{\nu+1}\}$, (resp., $F'_\nu = \{x' \in \mathbb{R}^n : \ 2^\nu \leq |x' - x'_0| \leq 2^{\nu+1}\}$).

We set $J^+ = \max(J, 0)$ and we denote by $|\Omega|$ the measure of $\Omega \subset \mathbb{R}^{n+1}$. $C_0, C_1, C_M, C_N, C_K, C'_K, \epsilon, \varepsilon, \varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots,$ denote various real positive constants, not necessarily the same at each of their occurrences, and the notation $U \lesssim V$ means that there is a positive constant $C$, such that $U \leq CV$.

By $\mathcal{S}$, we denote the Schwartz space of all rapidly decreasing and infinitely differentiable functions on $\mathbb{R}^{n+1}$, by $\mathcal{S}'$ its topological dual, i.e., the collection of all complex-valued tempered distributions on $\mathbb{R}^{n+1}$, and by $C_0^\infty$, the set of all test functions, i.e, the set of all compactly supported and infinitely differentiable functions.

As in [1,12,13], to reach our main goal, we can reduce the problem through a partition of unity, to an a-priori estimate for solutions of degenerate elliptic problems in the upper half space $\mathbb{R}^{n+1}_+ = \{x = (t, x'), t > 0\}$. The operator $\tilde{L}$ will be formulated as:

$$L = \sum_{h=0}^{\min(k,2m)} t^{k-h} P^{2m-h}(x, D_x), \tag{2}$$

where $x = (t, x') = (t, x_1, x_2, \ldots, x_n) \in \mathbb{R}_+ \times \mathbb{R}^n$, with its dual variable: $\xi = (z, \xi') = (z, \xi_1, \xi_2, \ldots, \xi_n)$; $P^{2m-h}(t, x'; D_t, D_{x'}) = \sum_{|\alpha'|+j \leq 2m-h} a_{\alpha',j}(t, x') D^{\alpha'}_{x'} D^j_t$, $\alpha' = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$, $j \in \mathbb{N}$ and $\alpha = (\alpha', j)$. The coefficients $a_{\alpha',j}(t, x')$ belong to $C^\infty(\overline{\mathbb{R}^{n+1}_+})$.

A model of operators (2) is given by $M = t(D_t^2 + D_x^2) + \lambda D_t + \mu D_x$, where $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ and $\lambda, \mu \in \mathbb{C}$.

## 1.2 Definitions

To define the spaces, we will use a Littlewood–Paley partition of unity. Let $x = (t, x') \in \mathbb{R} \times \mathbb{R}^n$, and let $\varphi \in C_0^\infty(\mathbb{R})$ equals to 1 in $[-1, 1]$ and with support in $[-2, 2]$. For $j \in \mathbb{N}$, we set:

$$\begin{aligned} S_j &= \varphi(2^{-j}|D_x|);\ S'_j = \varphi(2^{-j}|D_{x'}|);\ S''_j = \varphi(2^{-j}|D_t|), \\ S_{-1} &= S'_{-1} = S''_{-1} = 0, \end{aligned}$$

and

$$\Delta_j = S_j - S_{j-1};\ \Delta'_j = S'_j - S'_{j-1};\ \Delta''_j = S''_j - S''_{j-1}.$$

Note that: $S_j u = \varphi(2^{-j}|D_x|)u = \mathcal{F}^{-1}(\varphi_j \widehat{u})$, where $\varphi_j(\xi) = \varphi(2^{-j}|\xi|)$.

**Definition 1** (*Inhomogeneous version of Besov-type spaces*) Let $s \in \mathbb{R}$, $\tau \geq 0$ and let $0 < p, q \leq +\infty$. The space $B^{s,\tau}_{p,q}(\mathbb{R}^{n+1})$ denotes the set of all tempered distributions $u \in \mathcal{S}'(\mathbb{R}^{n+1})$, such that:

$$\|u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1})} \equiv \begin{cases} \sup\limits_{B_J} \frac{1}{|B_J|^\tau} \left\{ \sum\limits_{j \geq J^+} 2^{jsq} ||\Delta_j u||^q_{L^p(B_J)} \right\}^{\frac{1}{q}} < +\infty & \text{for } q < \infty, \\ \sup\limits_{B_J} \frac{1}{|B_J|^\tau} \sup_{j \geq J^+} 2^{js} ||\Delta_j u||_{L^p(B_J)} < +\infty & \text{for } q = \infty, \end{cases}$$

where the supremum is taken over all balls $B_J$ of $\mathbb{R}^{n+1}$ for all $J \in \mathbb{Z}$.

*Remark 2* In the sense of equivalence of norms, the following coincidence relations hold, see [11,18,21,24]:

1. $B^{s,0}_{p,q}(\mathbb{R}^{n+1}) = B^s_{p,q}(\mathbb{R}^{n+1})$ the classical Besov spaces;
2. $B^{s,\tau}_{p,p}(\mathbb{R}^{n+1}) = F^{s,\tau}_{p,p}(\mathbb{R}^{n+1})$ Triebel–Lizorkin-type spaces;
3. $B^{s,1}_{p,p}(\mathbb{R}^{n+1}) = \mathrm{bmo}^s_p(\mathbb{R}^{n+1})$ Goldberg spaces;
4. $B^{0,\frac{\lambda}{2(n+1)}}_{2,2}(\mathbb{R}^{n+1}) = l^{2,\lambda}(\mathbb{R}^{n+1})$ local Campanato spaces;
5. $B^{s,0}_{\infty,\infty}(\mathbb{R}^{n+1}) = F^{s,0}_{\infty,\infty}(\mathbb{R}^{n+1}) = C^s(\mathbb{R}^{n+1})$ Hölder–Zygmund spaces, $s > 0$, $s \notin \mathbb{N}$;
6. $B^{s,\frac{1}{p}+\frac{r}{n+1}}_{p,q}(\mathbb{R}^{n+1}) = L^r B^s_{p,q}(\mathbb{R}^{n+1})$ Triebel's hybrid spaces, for $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, and $-\frac{n+1}{p} \leq r < \infty$;
7. $B^{s,\frac{1}{p}-\frac{1}{u}}_{p,\infty}(\mathbb{R}^{n+1}) = \mathcal{N}^s_{u,p,\infty}(\mathbb{R}^{n+1})$ Besov–Morrey spaces, $0 < p \leq u < +\infty$, $s \in \mathbb{R}$;
8. Let $s \in \mathbb{R}, 0 < p \leq +\infty$ then, $B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}) = B^{s+(n+1)(\tau-\frac{1}{p})}_{\infty,\infty}(\mathbb{R}^{n+1})$, for $0 < q < +\infty$ and $\frac{1}{p} < \tau < +\infty$, or $q = +\infty$ and $\frac{1}{p} \leq \tau < +\infty$.

Also, we collect some elementary embeddings; see: [8,9,22,23]:

1. $B^{s_1,\tau}_{p,q}(\mathbb{R}^{n+1}) \hookrightarrow B^{s_2,\tau}_{p,q}(\mathbb{R}^{n+1})$ if $s_2 \leq s_1$;
2. $B^{s,\tau}_{p,q_0}(\mathbb{R}^{n+1}) \hookrightarrow B^{s,\tau}_{p,q_1}(\mathbb{R}^{n+1})$ if $0 < q_0 \leq q_1 < +\infty$ and $0 < p < +\infty$;
3. $B^{s+\frac{n+1}{p}-\frac{\tau(n+1)}{q},\tau}_{p,q}(\mathbb{R}^{n+1}) \hookrightarrow C^s(\mathbb{R}^{n+1})$;
4. Let $0 < t \leq p < +\infty$ then, $B^{s+\varepsilon}_{t,q}(\mathbb{R}^{n+1}) \hookrightarrow B^{s,\tau}_{p,q}(\mathbb{R}^{n+1})$ if and only if $\varepsilon \geq \frac{\tau(n+1)}{q} + \frac{n+1}{t} - \frac{n+1}{p}$.

**Definition 3** (*Anisotropic Besov-type spaces*) Let $s \in \mathbb{R}$, $\tau \geq 0$, and let $0 < p, q \leq +\infty$. The space $L^p(\mathbb{R}; B^{s,\tau}_{p,q}(\mathbb{R}^n))$ denotes the set of all tempered distributions $u \in \mathcal{S}'(\mathbb{R}^{n+1})$, such that:

$$\|u\|_{L^p(\mathbb{R}; B^{s,\tau}_{p,q}(\mathbb{R}^n))} \equiv \begin{cases} \sup\limits_{B'_J} \frac{1}{|B'_J|^\tau} \left\{ \sum\limits_{j \geq J^+} 2^{jsq} ||\Delta'_j u||^q_{L^p(B'_J)} \right\}^{\frac{1}{q}} < +\infty & \text{for } q < \infty, \\ \sup\limits_{B'_J} \frac{1}{|B'_J|^\tau} \sup_{j \geq J^+} 2^{js} ||\Delta'_j u||_{L^p(B'_J)} < +\infty & \text{for } q = \infty, \end{cases}$$

where the supremum is taken over all balls $B'_J$ of $\mathbb{R}^n$ and for all $J \in \mathbb{Z}$.

Below, we introduce the weighted spaces that we will need in this work.

**Definition 4** For $s \in \mathbb{R}$, $\tau \geq 0$, $k, m$ integers, and $1 \leq p, q < +\infty$, we define:

– *Weighted Sobolev spaces* We denote by $W^{2m,p}_k(\mathbb{R})$ the space of all $u \in L^p(\mathbb{R})$, such that $t^{k-h} D^j_t u \in L^p(\mathbb{R})$, where $0 \leq h \leq \min(k, 2m)$ and $0 \leq j \leq 2m - h$. To this space, we associate the following norm:

$$\| u \|_{W^{2m,p}_k(\mathbb{R})} \equiv \left\{ \sum_{h=0}^{\min(k,2m)} \sum_{j=0}^{2m-h} \| t^{k-h} D^j_t u \|^p_{L^p(\mathbb{R})} \right\}^{\frac{1}{p}}.$$

– *Anisotropic weighted Besov-type spaces* We define anisotropic weighted Besov-type spaces, where the properties of differentiability in the directions $x_1, x_2, \ldots, x_n$, are different from those in the direction $t$. We set: $W_k^{2m,p}(\mathbb{R}; B_{p,q}^{s,\tau}(\mathbb{R}^n)) = \Big\{u \in L^p(\mathbb{R}; B_{p,q}^{s+2m-k,\tau}(\mathbb{R}^n)) : t^{k-h}D_t^j u \in L^p(\mathbb{R}; B_{p,q}^{s+2m-h-j,\tau}(\mathbb{R}^n))$ for all $0 \le h \le \min(k, 2m)$ *and* $0 \le j \le 2m - h\Big\}$.
These spaces will allow us to estimate the "almost tangential derivatives" of solutions, which will be useful to the proof of the main theorem. The convenient norm in these spaces is:

$$\| u \|_{W_k^{2m,p}(\mathbb{R}; B_{p,q}^{s,\tau}(\mathbb{R}^n))} \equiv \left\{ \sum_{h=0}^{\min(k,2m)} \sum_{j=0}^{2m-h} \| t^{k-h} D_t^j u \|^p_{L^p(\mathbb{R}; B_{p,q}^{s+2m-h-j,\tau}(\mathbb{R}^n))} \right\}^{\frac{1}{p}}.$$

– *Weighted Besov-type spaces* We introduce the spaces: $B_{p,q,k}^{s+2m,\tau}(\mathbb{R}^{n+1}) = \Big\{u \in B_{p,q}^{s+2m-k,\tau}(\mathbb{R}^{n+1}) : t^{k-h}D_{x'}^{\alpha'}D_t^j u \in B_{p,q}^{s+2m-(|\alpha'|+j)-h,\tau}(\mathbb{R}^{n+1})$ for all $0 \le h \le \min(k, 2m)$ and $0 \le |\alpha'| + j \le 2m - h\Big\}$, with the norm:

$$\| u \|_{B_{p,q,k}^{s+2m,\tau}(\mathbb{R}^{n+1})} \equiv \sum_{h=0}^{\min(k,2m)} \sum_{|\alpha'|+j \le 2m-h} \| t^{k-h} D_{x'}^{\alpha'} D_t^j u \|_{B_{p,q}^{s+2m-(|\alpha'|+j)-h,\tau}(\mathbb{R}^{n+1})}$$

and
$V_{p,q}^{s+2m,\tau}(\mathbb{R}) = \{u \in B_{p,q}^{s+2m-k,\tau}(\mathbb{R}) : t^{k-h}D_t^j u \in B_{p,q}^{s+2m-h-j,\tau}(\mathbb{R})$ where $0 \le h \le \min(k, 2m)$ and $0 \le j \le 2m - h\}$.

The space $W_k^{2m,p}(\mathbb{R}_+)$, $\big[$resp., $V_{p,q}^{s+2m,\tau}(\mathbb{R}_+)\big]$ denotes the set of restrictions of the elements of $W_k^{2m,p}(\mathbb{R})$ $\big[$resp., $V_{p,q}^{s+2m,\tau}(\mathbb{R})\big]$ to $\mathbb{R}_+$. Similarly, the space $B_{p,q,k}^{s+2m,\tau}(\mathbb{R}_+^{n+1})$ $\big[$resp., $W_k^{2m,p}(\mathbb{R}_+; B_{p,q}^{s,\tau}(\mathbb{R}^n))\big]$ is the set of restrictions of the elements of $B_{p,q,k}^{s+2m,\tau}(\mathbb{R}^{n+1})$, $\big[$resp., $W_k^{2m,p}(\mathbb{R}; B_{p,q}^{s,\tau}(\mathbb{R}^n))\big]$ to $\mathbb{R}_+^{n+1}$.

### 1.3 Assumptions and main theorem

Let $k \in \mathbb{N}$, $m \in \mathbb{N}\backslash\{0\}$, $s \in \mathbb{R}$ and $1 \le p < +\infty$.
Let $L^0$ be the "principal part" of the operator $L$ defined by:

$$L^0(t, x'; D_t, D_{x'}) = \sum_{h=0}^{\min(k,2m)} t^{k-h} P_{2m-h}^{2m-h}(t, x'; D_t, D_{x'}), \tag{3}$$

where $P_{2m-h}^{2m-h}(t, x'; D_t, D_{x'}) = \sum_{|\alpha'|+j=2m-h} a_{\alpha',j}(t, x') D_{x'}^{\alpha'} D_t^j$, is the principal part of the operator $P^{2m-h}(t, x'; D_t, D_{x'})$. We suppose that:

(A0) $P^{2m}(t, x'; D_t, D_{x'})$ is a properly elliptic operator in $\overline{\mathbb{R}_+^{n+1}}$.
(A1) For any $x' \in \mathbb{R}^n$ and $\xi' \in \mathbb{R}^n\backslash\{0\}$, the polynomial in the complex variable $z$:

$$P(z) = \sum_{|\alpha'|+j=2m} a_{\alpha',j}(0, x') \xi'^{\alpha'} z^j$$

has exactly $m$ roots with positive imaginary parts and $m$ roots with negative imaginary parts.

(A2) For any $x' \in \mathbb{R}^n$, the $\lambda$-polynomial:

$$p(x', \lambda) = \sum_{h=0}^{\min(k,2m)} (-i)^{2m-h} a_{0,m-h}(0, x')\lambda(\lambda - 1) \cdots (\lambda - k + h + 1)$$

has no roots on the lines $\Re e\lambda = s + \frac{1}{p}$ and $\Re e\lambda = 1 + \frac{1}{p}$.
Let $r(x')$ be the number of roots satisfying $\Re e\lambda > \max(s + \frac{1}{p}, 1 + \frac{1}{p})$, we suppose that $r(x') = r$ is independent of $x'$, and we set $\chi = m - k + r$.

(A3) For any $x' \in \mathbb{R}^n$ and $\omega' \in \mathbb{R}^n \backslash \{0\}$, $|\omega'| = 1$, the problem:

$$\begin{cases} L^0(t, x'; D_t, \omega')u(t) = 0 \\ D_t^l u(0) \qquad\qquad = 0\,, \qquad 0 \le l \le \chi - 1, \end{cases}$$

has the only solution $u \equiv 0$ in the space $\mathcal{S}(\overline{\mathbb{R}}_+)$.

The main result of this paper is the following.

**Theorem 5** *Let $s$ and $\tau$ be two non-negative real numbers and let $1 \le p, q < +\infty$. Assuming* (A0)–(A3)*, for any compact set $K$ in $\overline{\mathbb{R}^{n+1}_+}$, there exists a constant $C_K$, such that for any $u \in B^{s+2m,\tau}_{p,q,k}(\mathbb{R}^{n+1}_+)$ with support in $K$, we have:*

$$\|u\|_{B^{s+2m,\tau}_{p,q,k}(\mathbb{R}^{n+1}_+)} \le C_K \Bigg\{ \|Lu\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} + \sum_{l=0}^{\chi-1} \|\gamma_l u\|_{B^{s+2m-k-l-\frac{1}{p},\tau}_{p,q}(\mathbb{R}^n)} + \|u\|_{B^{s+2m-k-1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} \Bigg\},$$

*where $\gamma_l$ denotes the trace operator on $t = 0$.*

*Remark 6* 1. If $\chi = 0$, the term $\sum_{l=0}^{\chi-1} \|\gamma_l u\|_{B^{s+2m-k-l-\frac{1}{p},\tau}_{p,q}(\mathbb{R}^n)}$ does not appear in the above estimate.

2. Taking $\tau = 0$ and $p = q$, we obtain the regularity in classical Besov spaces [15]. If $k = 0$, we find the results of [12]. Setting $k = 1$ and $m = 1$, we get the estimates proved in [13]. If $s = 0$, $\tau = \frac{\lambda}{2(n+1)}$, and $p = q = 2$, the regularity in local Morrey–Campanato spaces $l^{2,\lambda}$ is proved.

*Example 7* (i) As mentioned above, if $k = 0$, we get the theorem for regular elliptic boundary-value problems described in [12]. If we set in addition $2m = 2$, the following example holds.
If $u$ is a solution of the problem:

$$\begin{cases} u \qquad\quad \in B^{s,\tau}_{p,q}(\mathbb{R}^2_+) \\ -\Delta u = f \ \in B^{s,\tau}_{p,q}(\mathbb{R}^2_+) \\ \gamma_0 u \ \ = g \ \in B^{s+2-\frac{1}{p},\tau}_{p,q}(\mathbb{R}), \end{cases}$$

then $u \in B^{s+2,\tau}_{p,q,\mathrm{loc}}(\mathbb{R}^2_+)$.

(ii) Let $M$ be the following operator given in $\mathbb{R}_+ \times \mathbb{R}$ by:

$$M(t, x; D_t, D_x)u = t(D_t^2 + D_x^2)u + \lambda D_t u + \mu D_x u,$$

where $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, and $\lambda, \mu \in \mathbb{C}$.
If $\mu \ne \pm(2p + i\lambda)$, $p \in \mathbb{N}\backslash\{0\}$, and $\Im m(\lambda) > \max(1 + \frac{1}{p}, s + \frac{1}{p})$, and if $u$ is a solution of the problem:

$$\begin{cases} u \qquad\ \ \in B^{s,\tau}_{p,q}(\mathbb{R}^2_+) \\ Mu = f \in B^{s,\tau}_{p,q}(\mathbb{R}^2_+) \\ \gamma_0 u = g \ \in B^{s+1-\frac{1}{p},\tau}_{p,q}(\mathbb{R}), \end{cases}$$

then $u \in B^{s+2,\tau}_{p,q,1,\mathrm{loc}}(\mathbb{R}^2_+)$, i.e., $u \in B^{s+1,\tau}_{p,q,\mathrm{loc}}(\mathbb{R}^2_+)$, such that $tD_t^2 u$ and $tD_x^2 u \in B^{s,\tau}_{p,q,\mathrm{loc}}(\mathbb{R}^2_+)$.

Let $\Omega$ be a $C^\infty$-bounded open set of $\mathbb{R}^{n+1}$, such that $\Omega = \{x \in \mathbb{R}^{n+1}; \varphi(x) > 0\}$, $\partial\Omega = \{x \in \mathbb{R}^{n+1}; \varphi(x) = 0\}$ and $d\varphi \neq 0$ on $\partial\Omega$ where the function $\varphi$ is of class $C^\infty$ from $\mathbb{R}^{n+1}$ to $\mathbb{R}$ and associates with each element of $\Omega$ its distance from the boundary. We assume that:

(H0) $P^{2m}(x, D_x)$ is a properly elliptic operator in $\overline{\Omega}$.

(H1) For any $x \in \partial\Omega$ and $\xi \in \mathbb{R}^{n+1}\backslash\{0\}$ tangent to $\partial\Omega$ at $x$, the polynomial in the complex variable $z$:

$$P(z) = \sum_{|\alpha|=2m} a_\alpha(x)(\xi + z\nu_x)^\alpha$$

has exactly $m$ roots with positive imaginary parts (and then exactly $m$ roots lying in the lower half plane). Here, $\nu_x$ is the inward unit normal vector to the boundary $\partial\Omega$ at $x$.

(H2) For $x \in \partial\Omega$, we introduce the $\lambda-$polynomial:

$$p(x,\lambda) = \sum_{h=0}^{\min(k,2m)} (-i)^{2m-h} P^{2m-h}_{2m-h}(x, d\varphi(x))\lambda(\lambda-1)\cdots(\lambda-k+h+1),$$

where $P^{2m-h}_{2m-h}(x; D_x)$ is the principal part of the operator $P^{2m-h}(x; D_x)$. We suppose that the polynomial $p(x,\lambda)$ has no roots on the lines $\Re e\lambda = s+\frac{1}{p}$ and $\Re e\lambda = 1+\frac{1}{p}$. Let $r(x)$ be the number of roots verifying $\Re e\lambda > \max(s+\frac{1}{p}, 1+\frac{1}{p})$, we suppose that $r(x) = r$ is independent of $x \in \Omega$, and we set $\chi = m-k+r$.

(H3) For any $x \in \partial\Omega$ and $\xi \in \mathbb{R}^{n+1}$ not colinear to $\mathrm{d}\varphi(x)$, the problem:

$$\begin{cases} L^0(x,\xi;t,D_t)u(t) = \displaystyle\sum_{h=0}^{\min(k,2m)} t^{k-h} P^{m-h}_{m-h}(x, \xi + \mathrm{d}\varphi(x)D_t)u(t) = 0 \\ \gamma(x,\xi;D_t)u(t) \quad = \gamma_l(x, \xi + \mathrm{d}\varphi(x)D_t)u(t) = 0, \quad 0 \le l \le \chi - 1, \end{cases}$$

has the only solution $u \equiv 0$ in the space $u \in \mathcal{S}(\overline{\mathbb{R}}_+)$.

The following result is a consequence of Theorem 5.

**Theorem 8** *Let $s$ and $\tau$ be two non-negative real numbers and let $1 \le p, q < +\infty$. Assuming* (H0)–(H3)*:*

$$\|u\|_{B^{s+2m,\tau}_{p,q,k}(\Omega)} \lesssim \|Lu\|_{B^{s,\tau}_{p,q}(\Omega)} + \sum_{l=0}^{\chi-1} \|\gamma_l u\|_{B^{s+2m-k-l-\frac{1}{p},\tau}_{p,q}(\partial\Omega)} + \|u\|_{B^{s+2m-k-1,\tau}_{p,q}(\Omega)}$$

*holds for any $u \in B^{s+2m,\tau}_{p,q,k}(\Omega)$.*

## 2 Preliminary lemmas

In this section, we recall the most essential lemmas needed for the proof of Theorem 5.

**Lemma 9** [9] *Let $1 \le p < +\infty$ and let $A < 0$. If $(a_{j\nu})_{j,\nu}$ is a sequence of positive real numbers satisfying $(a_{j\nu})_j \in \ell^p$ for any $\nu \ge 1$, then:*

$$\sum_{j\ge1}\left(\sum_{\nu\ge1} 2^{\nu A} a_{j\nu}\right)^p \lesssim \sup_{\nu\ge1}\sum_{j\ge1} a^p_{j\nu}$$

*holds.*

**Lemma 10** [9] *Let $1 \le p \le +\infty$. For any integer $M > 0$, there exists a constant $C_M > 0$, such that for any ball $B_J$, for any $l \in \mathbb{Z}$, and for any $u \in L^p(\mathbb{R}^{n+1})$:*

$$\|A_l u\|_{L^p(B_J)} \le C_M \left\{ \|u\|_{L^p(2B_J)} + \sum_{\nu\ge -J+1} 2^{-(\nu+l)M}\|u\|_{L^p(F_\nu)} \right\}$$

*holds for $A_l = \Delta_l, \Delta'_l, \Delta''_l, S_l, S'_l, S''_l$.*

**Lemma 11** *Let $s$ and $\tau$ be two real numbers, such that $\tau \geq 0$; let $1 \leq p, q \leq +\infty$. For any $\varepsilon > 0$, any integers $k, m \in \mathbb{N}$ and for any $u \in W_k^{2m,p}(\mathbb{R}; B_{p,q}^{s,\tau}(\mathbb{R}^n))$:*

$$\| t^{k-h} D_t^r u \|_{L^p(\mathbb{R}; B_{p,q}^{s+2m-h-r,\tau}(\mathbb{R}^n))} \lesssim \varepsilon \| t^k D_t^{2m} u \|_{L^p(\mathbb{R}; B_{p,q}^{s,\tau}(\mathbb{R}^n))} + \varepsilon^{-\frac{r+h}{2m-r}} \| t^k u \|_{L^p(\mathbb{R}; B_{p,q}^{s+2m,\tau}(\mathbb{R}^n))}$$

*holds for $0 \leq h \leq \min(k, 2m)$ and $0 \leq r \leq 2m - h$, with $r \neq 2m$.*

*Proof* According to [17], it is well known that if $t^k D_t^{2m} u$ and $t^k u$ belong to $L^p(\mathbb{R})$, then for $0 \leq h \leq \min(k, 2m)$ and $0 \leq r \leq 2m - h$, with $r \neq 2m$, $t^{k-h} D_t^r u \in L^p(\mathbb{R})$, and:

$$\| t^{k-h} D_t^r u \|_{L^p(\mathbb{R})} \lesssim \| t^k D_t^{2m} u \|_{L^p(\mathbb{R})} + \| t^k u \|_{L^p(\mathbb{R})} \tag{4}$$

applying inequality (4) to $u(\lambda t)$ with $\lambda > 0$, we deduce:

$$\| t^{k-h} D_t^r u \|_{L^p(\mathbb{R})} \lesssim \lambda^{2m-h-r} \| t^k D_t^{2m} u \|_{L^p(\mathbb{R})} + \lambda^{-h-r} \| t^k u \|_{L^p(\mathbb{R})} . \tag{5}$$

Let $u \in W_k^{2m,p}(\mathbb{R}; B_{p,q}^{s,\tau}(\mathbb{R}^n))$; we apply inequality (5) to $\Delta'_j u(t, x')$, $j \in \mathbb{N}$, and $x' \in B'_J$:

$$\begin{aligned} &\| t^{k-h} D_t^r \Delta'_j u(\cdot, x') \|_{L^p(\mathbb{R})} \\ &\lesssim \lambda^{2m-h-r} \| t^k D_t^{2m} \Delta'_j u(\cdot, x') \|_{L^p(\mathbb{R})} + \lambda^{-h-r} \| t^k \Delta'_j u(\cdot, x') \|_{L^p(\mathbb{R})} \end{aligned} \tag{6}$$

replacing in (6) $\lambda$ with $\varepsilon^{\frac{1}{2m-h-r}} 2^{-j}$, integrating with respect to $x'$ over a ball $B'_J$, and multiplying each side of the preceding inequality by $\frac{2^{j(s+2m-h-r)}}{|B'_J|^\tau}$, we obtain:

$$\begin{aligned} &\frac{2^{j(s+2m-h-r)}}{|B'_J|^\tau} \| t^{k-h} D_t^r \Delta'_j u \|_{L^p(\mathbb{R}\times B'_J)} \\ &\lesssim \varepsilon \frac{2^{js}}{|B'_J|^\tau} \| t^k D_t^{2m} \Delta'_j u \|_{L^p(\mathbb{R}\times B'_J)} + \varepsilon^{-\frac{h+r}{2m-h-r}} \frac{2^{j(s+2m)}}{|B'_J|^\tau} \| t^k \Delta'_j u \|_{L^p(\mathbb{R}\times B'_J)}; \end{aligned} \tag{7}$$

summing over $j \geq J^+$ in (7) and applying the $l^q-$norm, we deduce Lemma 11. □

The next three lemmas are shown in [12] for $p = q$. In the same way, we deduce them for $p \neq q$.

**Lemma 12** [12] *Let $\tau$ be a positive real number, $1 \leq p, q < +\infty$, $m$ be an integer $\geq 1$, and let $s$ be a real number $< m$. If $u \in L^p(\mathbb{R}; B_{p,q}^{s,\tau}(\mathbb{R}^n))$, such that $D_t^m u \in L^p(\mathbb{R}; B_{p,q}^{s-m,\tau}(\mathbb{R}^n))$, then $u \in B_{p,q}^{s,\tau}(\mathbb{R}^{n+1})$ and:*

$$\|u\|_{B_{p,q}^{s,\tau}(\mathbb{R}^{n+1})} \lesssim \|D_t^m u\|_{L^p(\mathbb{R}; B_{p,q}^{s-m,\tau}(\mathbb{R}^n))} + \|u\|_{L^p(\mathbb{R}; B_{p,q}^{s,\tau}(\mathbb{R}^n))}$$

*holds.*

**Lemma 13** [12] *Let $s \in \mathbb{R}$, $\tau \geq 0$ and let $1 \leq p, q < +\infty$. There exists $C_0 > 0$, such that for any $\varphi \in \mathcal{S}(\mathbb{R}^{n+1})$, there exists $C_1 > 0$ satisfying for any $u \in B_{p,q}^{s,\tau}(\mathbb{R}^{n+1})$ $\left[resp., \ L^p(\mathbb{R}; B_{p,q}^{s,\tau}(\mathbb{R}^n))\right]$:*

$$\|\varphi u\|_{B_{p,q}^{s,\tau}(\mathbb{R}^{n+1})} \leq C_0 \|\varphi\|_{L^\infty(\mathbb{R}^{n+1})} \|u\|_{B_{p,q}^{s,\tau}(\mathbb{R}^{n+1})} + C_1 \|u\|_{B_{p,q}^{s-1,\tau}(\mathbb{R}^{n+1})},$$

$$\Big[\text{resp., } \|\varphi u\|_{L^p(\mathbb{R}, B_{p,q}^{s,\tau}(\mathbb{R}^n))} \leq C_0 \|\varphi\|_{L^\infty(\mathbb{R}^{n+1})} \|u\|_{L^p(\mathbb{R}; B_{p,q}^{s,\tau}(\mathbb{R}^n))} + C_1 \|u\|_{L^p(\mathbb{R}; B_{p,q}^{s-1,\tau}(\mathbb{R}^n))}\Big].$$

**Lemma 14** [12] *Let $s_1 \leq s_2 < s_3$ be three real numbers, $\tau \geq 0$, and let $1 \leq p, q < +\infty$. For any $\varepsilon > 0$ and $u \in B_{p,q}^{s_3,\tau}(\mathbb{R}^{n+1})$ $\left[resp., \ L^p(\mathbb{R}; B_{p,q}^{s_3,\tau}(\mathbb{R}^n))\right]$, we get:*

$$\|u\|_{B_{p,q}^{s_2,\tau}(\mathbb{R}^{n+1})} \lesssim \varepsilon \|u\|_{B_{p,q}^{s_3,\tau}(\mathbb{R}^{n+1})} + \varepsilon^{-\frac{s_2-s_1}{s_3-s_2}} \|u\|_{B_{p,q}^{s_1,\tau}(\mathbb{R}^{n+1})},$$

$$\Big[\text{resp., } \|u\|_{L^p(\mathbb{R}; B_{p,q}^{s_2,\tau}(\mathbb{R}^n))} \lesssim \varepsilon \|u\|_{L^p(\mathbb{R}; B_{p,q}^{s_3,\tau}(\mathbb{R}^n))} + \varepsilon^{-\frac{s_2-s_1}{s_3-s_2}} \|u\|_{L^p(\mathbb{R}; B_{p,q}^{s_1,\tau}(\mathbb{R}^n))}\Big].$$

## 3 Trace of elements of $B^{s+2m,\tau}_{p,q,k}(\mathbb{R}^{n+1}_+)$ and $W^{2m,p}_k(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))$

**Theorem 15** *Let $s$ and $\tau$ be two real numbers, such that $\tau \geq 0$ and let $1 \leq p, q < +\infty$. For $u \in W^{2m,p}_{k,\mathrm{loc}}(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))$ and $l \in \{0, \ldots, 2m-1\}$, the series $\sum_{j\geq 0} D^l_t \Delta'_j u(0,\cdot)$ converges in $\mathcal{S}'(\mathbb{R}^n)$ and defines an element $\gamma_l u$ belonging to $B^{s+2m-k-l-\frac{1}{p},\tau}_{p,q}(\mathbb{R}^n)$.*

*In addition, the mapping $u \mapsto \gamma_l u$ is continuous and surjective from $W^{2m,p}_{k,\mathrm{loc}}(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))$ to $B^{s+2m-k-l-\frac{1}{p},\tau}_{p,q}(\mathbb{R}^n)$.*

*Also, there exists an extension operator $R_l$ from $B^{s+2m-k-l-\frac{1}{p},\tau}_{p,q}(\mathbb{R}^n)$ to $W^{2m,p}_k(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))$, such that $\gamma_l o R_l = Id_{B^{s+2m-k-l-\frac{1}{p},\tau}_{p,q}(\mathbb{R}^n)}$.*

*In particular, if $s \geq 0$, the operator $\gamma_l$ is bounded and surjective from $B^{s+2m,\tau}_{p,q,k}(\mathbb{R}^{n+1}_+)$ to $B^{s+2m-k-l-\frac{1}{p},\tau}_{p,q}(\mathbb{R}^n)$.*

*Proof* To prove the theorem, it suffices to show that for $0 \leq l \leq 2m-1$

(i) The operator $\gamma_l$ is bounded from $W^{2m,p}_{k,\mathrm{loc}}(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))$ to $B^{s+2m-k-l-\frac{1}{p},\tau}_{p,q}(\mathbb{R}^n)$.

(ii) There exists an extension operator $R_l$ which is bounded from $B^{s+2m-k-l-\frac{1}{p},\tau}_{p,q}(\mathbb{R}^n)$ to $W^{2m,p}_k(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))$.

Let us show the first assertion (i).
From the Sobolev embeddings, we have:

$$W^{2m,p}_{k,\mathrm{loc}}(\mathbb{R}_+) \subset W^{2m,p}_{\mathrm{loc}}(\mathbb{R}_+) \hookrightarrow C^{2m-1}_{\mathrm{loc}}(\mathbb{R}_+).$$

Then:

$$| D^l_t v(0) | \lesssim \sum_{h=0}^{\min(k,2m)} \sum_{r=0}^{2m-h} \| t^{k-h} D^r_t \varphi v \|_{L^p(\mathbb{R}_+)} \tag{8}$$

for any integer $l$, such that $0 \leq l \leq 2m-1$, and any $\varphi \in C^\infty_0(\mathbb{R}_+)$ and $v \in W^{2m,p}_k(\mathbb{R}_+)$.
Changing in (8) $v(t)$ by $v(\lambda t)$ for any $\lambda > 0$, we get:

$$\lambda^l | D^l_t v(0) | \lesssim \sum_{h=0}^{\min(k,2m)} \sum_{r=0}^{2m-h} \lambda^{r-k+h-\frac{1}{p}} \| t^{k-h} D^r_t \varphi v \|_{L^p(\mathbb{R}_+)}. \tag{9}$$

Let $u \in W^{2m,p}_k(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))$. For $j \in \mathbb{N}$, we set:

$$u_j(t, x') = \Delta'_j u(t, x') \in W^{2m,p}_{k,\mathrm{loc}}(\mathbb{R}_+; C^\infty(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)).$$

Applying inequality (9) to $u_j$, choosing $\lambda = 2^{-j}$, integrating over a ball $B'_J$, and multiplying the both sides by $\frac{2^{j(s+2m-k-\frac{1}{p})}}{|B'_J|^\tau}$, we get:

$$\begin{aligned} &\frac{2^{j(s+2m-k-l-\frac{1}{p})}}{| B'_J |^\tau} \| D^l_t \Delta'_j u(0,\cdot) \|_{L^p(B'_J)} \\ &\lesssim \sum_{h=0}^{\min(k,2m)} \sum_{r=0}^{2m-h} \frac{2^{j(s+2m-r-h)}}{| B'_J |^\tau} \| t^{k-h} D^r_t \varphi \Delta'_j u \|_{L^p(\mathbb{R}_+ \times B'_J)}. \end{aligned} \tag{10}$$

Since $\varphi$ and $\Delta'_j$ commute and taking the $l^q$-norm on each side of (10), we deduce the first assertion (i).

Now, let $u \in B_{p,q}^{s+2m-k-l-\frac{1}{p},\tau}(\mathbb{R}^n)$, $0 \le l \le 2m-1$. Let $\varphi_0 \in C_0^\infty(\mathbb{R})$ equals to 1 in a neighborhood of 0, with $\varphi_l(t) = \frac{t^l}{l!}\varphi_0(t)$, and then, $\partial_t^l \varphi_l(0) = 1$ and $\partial_t^k \varphi_l(0) = 0$ for $k \ne l$, $0 \le k \le 2m-1$. For $0 \le l \le 2m-1$, we set:

$$R_l u(t,x') = \sum_{j=0}^{+\infty} 2^{-jl}\varphi_l(2^j t)\Delta'_j u(0,x').$$

The second assertion (ii) follows from the inequality:

$$\| R_l u \|_{W_k^{2m,p}(\mathbb{R}_+;B_{p,q}^{s,\tau}(\mathbb{R}^n))} \lesssim \| u \|_{B_{p,q}^{s+2m-k-l-\frac{1}{p},\tau}(\mathbb{R}^n)}. \tag{11}$$

We have

$$R_l u(t,x') = \sum_{j=0}^{+\infty} 2^{-jl}\varphi_l(2^j t)\Delta'_j u(0,x');$$

then

$$t^{k-h} D_t^r R_l u(t,x') = \sum_{j=0}^{+\infty} 2^{-jl} \times 2^{jr} \times t^{k-h}(D_t^r \varphi_l)(2^j t)\Delta'_j u(0,x') \tag{12}$$

applying the operator $\Delta'_i$ to (12) and since $\Delta'_i\Delta'_j \ne 0$ for $i \sim j$, we obtain:

$$\Delta'_i t^{k-h} D_t^r R_l u(t,x') = \sum_{i\sim j} 2^{-jl} \times 2^{jr} \times t^{k-h}(D_t^r \varphi_l)(2^j t)\Delta'_i\Delta'_j u(0,x');$$

integrating with respect to $t \in \mathbb{R}_+$, next with respect to $x'$ over a ball $B'_J$, we get:

$$\| \Delta'_i t^{k-h} D_t^r R_l u \|_{L^p(\mathbb{R}_+\times B'_J)} \lesssim 2^{i(r-l+h-k-\frac{1}{p})} \sum_{i\sim j} \| \Delta'_i\Delta'_j u \|_{L^p(B'_J)}.$$

Lemma 10 implies that:

$$\begin{aligned}\| \Delta'_i t^{k-h} D_t^r R_l u \|_{L^p(\mathbb{R}_+\times B'_J)} &\lesssim 2^{i(r-l+h-k-\frac{1}{p})} \| \Delta'_i u \|_{L^p(2B'_J)} \\ &\quad + 2^{i(r-l+h-k-\frac{1}{p})} \sum_{\nu\ge -J+1} 2^{-(\nu+i)M} \| \Delta'_i u \|_{L^p(F'_\nu)};\end{aligned} \tag{13}$$

multiplying both sides of (13) by $\frac{2^{i(s+2m-h-r)}}{|B'_J|^\tau}$, summing over $i \ge J^+$, and taking the $l^q$-norm, we get:

$$\begin{aligned}&\| R_l u \|_{W_k^{2m,p}(\mathbb{R}_+;B_{p,q}^{s,\tau}(\mathbb{R}^n))} \\ &\lesssim \frac{1}{|B'_J|^\tau}\left\{\sum_{i\ge J^+} 2^{iq(s+2m-k-l-\frac{1}{p})} \| \Delta'_i u \|_{L^p(2B'_J)}^q\right\}^{\frac{1}{q}} \\ &\quad + \frac{1}{|B'_J|^\tau}\left\{\sum_{i\ge J^+} 2^{iq(s+2m-k-l-\frac{1}{p})}\left(\sum_{\nu\ge -J+1} 2^{-(\nu+i)M} \| \Delta'_i u \|_{L^p(F'_\nu)}\right)^q\right\}^{\frac{1}{q}} \\ &\lesssim I_1 + I_2.\end{aligned}$$

We bound the term $I_1$ with $\| u \|_{B_{p,q}^{s+2m-k-l-\frac{1}{p},\tau}(\mathbb{R}^n)}$.

For $I_2$, we set $\mu = \nu + J$. Since $\mid F'_{\mu-J} \mid\sim 2^{n(\mu-J)}$:

$$I_2^q \lesssim \frac{1}{\mid B'_J \mid^{\tau q}} \sum_{i \geq J^+} \left( \sum_{\mu \geq 1} 2^{n\tau(\mu-J)} \times 2^{-(\mu-J+i)M} \right.$$
$$\left. \times \frac{2^{i(s+2m-k-l-\frac{1}{p})}}{\mid F'_{\mu-J} \mid^{\tau}} \parallel \Delta'_i u \parallel_{L^p(F'_{\mu-J})} \right)^q$$
$$\lesssim \sum_{i \geq J^+} 2^{(J-i)Mq} \sum_{\mu \geq 1} \left( 2^{\mu(n\tau-M)} \times \frac{2^{i(s+2m-k-l-\frac{1}{p})}}{\mid F'_{\mu-J} \mid^{\tau}} \parallel \Delta'_i u \parallel_{L^p(F'_{\mu-J})} \right)^q ;$$

for $M$ sufficiently large, we apply Lemma 9, we deduce:

$$I_2^q \lesssim \sup_{\mu \geq 1} \frac{1}{\mid F'_{\mu-J} \mid^{\tau q}} \sum_{i \geq J^+} \left( 2^{i(s+2m-k-l-\frac{1}{p})} \parallel \Delta'_i u \parallel_{L^p(F'_{\mu-J})} \right)^q$$
$$\lesssim \sup_{\mu \geq 1} \frac{1}{\mid F'_{\mu-J} \mid^{\tau q}} \sum_{i \geq (J-\mu+1)^+} \left( 2^{i(s+2m-k-l-\frac{1}{p})} \parallel \Delta'_i u \parallel_{L^p(F'_{\mu-J})} \right)^q$$
$$\lesssim \parallel u \parallel_{B_{p,q}^{s+2m-k-l-\frac{1}{p},\tau}(\mathbb{R}^n)}.$$

Then, the estimate (11) is proved, and it is not hard to show that

$$\gamma_l o R_l = Id_{B_{p,q}^{s+2m-k-l-\frac{1}{p},\tau}(\mathbb{R}^n)}.$$

Accordingly, assertion (ii) is proved. □

## 4 Proof of Theorem 5

*An ordinary differential equation* Let us consider the following class of ordinary differential operators, defined on $\mathbb{R}_+$ by:

$$L = L(t, D_t) = \sum_{h=0}^{\min(k,2m)} t^{k-h} P^{2m-h}(D_t)$$

where $P^{2m-h}(D_t) = \sum_{j=0}^{2m-h} a_j^{2m-h} D_t^j$ with $a_j^{2m-h} \in \mathbb{C}$. We assume that:

(C1) For any $z \in \mathbb{R}$, the polynomial $P^{2m}(z)$ has exactly $m$ roots with positive imaginary parts and $m$ roots with negative imaginary parts.

(C2) Let $s \in \mathbb{R}$, $1 \leq p < +\infty$ and let $\phi(\lambda) = 0$ be the characteristic equation associated to the operator $L$ in $t = 0$:

$$\phi(\lambda) = \sum_{h=0}^{\min(k,2m)} (-i)^{2m-h} a_{2m-h}^{2m-h} \lambda(\lambda-1)\cdots(\lambda-k+h+1),$$

which is also the characteristic equation of the principal operator:

$$L^0(t, D_t) = \sum_{h=0}^{\min(k,2m)} a_{2m-h}^{2m-h} t^{k-h} D_t^{2m-h}.$$

We assume that the equation $\phi(\lambda) = 0$ has no roots on the lines: $\Re e(\lambda) = 1 + \frac{1}{p}$ and $\Re e(\lambda) = s + \frac{1}{p}$ and let $r$ denotes the number of the roots satisfying $\Re e(\lambda) > \max(1 + \frac{1}{p}, s + \frac{1}{p})$.

The following theorem holds.

**Theorem 16** [1,15] *Let $s$ and $\tau$ be two non-negative real numbers and let $1 \leq p, q < +\infty$. Under hypotheses $(C1)$ and $(C2)$, the operator $L$ from $W_k^{2m,p}(\mathbb{R}_+)$ to $L^p(\mathbb{R}_+)$ and from $V_{p,q}^{s+2m,\tau}(\mathbb{R}_+)$ to $B_{p,q}^{s,\tau}(\mathbb{R}_+)$ is a Fredholm operator and its index is equal to $m-k+r$.*

To establish Theorem 5, we will have to go through two steps: first, we will prove the following proposition which will allow us to give an estimate of the "almost tangential" derivatives of solutions, and the second step will be evaluating the normal derivatives using Lemma 18.

**Proposition 17** *Let $s$ and $\tau$ be two non-negative real numbers, and let $1 \leq p, q < +\infty$. Under hypotheses* (A0)–(A3), *for any compact set $K$ of $\overline{\mathbb{R}_+^{n+1}}$, there exists a constant $C_K > 0$, such that for any $u \in W_k^{2m,p}(\mathbb{R}_+; B_{p,q}^{s,\tau}(\mathbb{R}^n))$ with* $\operatorname{supp} u \subset K$*:*

$$\|u\|_{W_k^{2m,p}(\mathbb{R}_+;B_{p,q}^{s,\tau}(\mathbb{R}^n))} \leq C_K \left\{ \|Lu\|_{L^p(\mathbb{R}_+;B_{p,q}^{s,\tau}(\mathbb{R}^n))} + \sum_{l=0}^{\chi-1} \|\gamma_l u\|_{B_{p,q}^{s+2m-k-l-\frac{1}{p},\tau}(\mathbb{R}^n)} \right.$$
$$\left. + \|u\|_{L^p(\mathbb{R}_+;B_{p,q}^{s+2m-k-1,\tau}(\mathbb{R}^n))} \right\}$$

*holds.*

*Proof* The idea of the proof is the same as that in [1,10,12,13]. We decompose the operator $L$ as follows: $L = L^0 + L^1 + L^2$, where:

$$L^0(t,x';D_t,D_{x'}) = t^k D_t^{2m} + \sum_{h=0}^{\min(k,2m)} t^{k-h} \sum_{\substack{|\alpha'|+j=2m-h \\ j\neq 2m}} a_{\alpha',j}(0,0) D_{x'}^{\alpha'} D_t^j,$$

$$L^1(t,x';D_t,D_{x'}) = \sum_{h=0}^{\min(k,2m)} t^{k-h} \sum_{\substack{|\alpha'|+j=2m-h \\ j\neq 2m}} \left(a_{\alpha',j}(t,x') - a_{\alpha',j}(0,0)\right) D_{x'}^{\alpha'} D_t^j,$$

and

$$L^2(t,x';D_t,D_{x'}) = \sum_{h=0}^{\min(k,2m)} t^{k-h} \sum_{|\alpha'|+j=0}^{2m-h-1} a_{\alpha',j}(t,x') D_{x'}^{\alpha'} D_t^j.$$

Set $\gamma = (\gamma_l)_{0\leq l\leq \chi-1}$. As in [1] and by means of Theorem 16, under assumptions (A0)–(A3), for every $\xi' \in \mathbb{R}^n\backslash\{0\}$, the operator $(L^0(t,0;D_t,\xi'),\gamma)$ is invertible from $W_k^{2m,p}(\mathbb{R}_+)$ onto $L^p(\mathbb{R}_+)\times\mathbb{C}^\chi$, and if $K_{\xi'}$ denotes its inverse, then the mapping $\xi' \mapsto K_{\xi'}$ is of class $C^\infty$ from $\mathbb{R}^n\backslash\{0\}$ to $\mathcal{L}\big(L^p(\mathbb{R}_+)\times\mathbb{C}^\chi; W_k^{2m,p}(\mathbb{R}_+)\big)$, and for any multi-index $\alpha'$, there exists $C_{\alpha'} > 0$, such that for any $\xi'$ with $\frac{1}{2} \leq |\xi'| \leq 2$ and any $(f,g) \in L^p(\mathbb{R}_+)\times\mathbb{C}^\chi$:

$$\|D_{\xi'}^{\alpha'} K_{\xi'}(f,g)\|_{W_k^{2m,p}(\mathbb{R}_+)} \leq C_{\alpha'} \|(f,g)\|_{L^p(\mathbb{R}_+)\times\mathbb{C}^\chi} \tag{14}$$

holds.

First of all, we prove that for any $N \geq 1$ large enough and any ball $B'_J$ of $\mathbb{R}^n$:

$$\|u\|_{L^p(B'_J;W_k^{2m,p}(\mathbb{R}_+))}$$
$$\lesssim \|L^0u\|_{L^p(2B'_J;L^p(\mathbb{R}_+))} + \sum_{l=0}^{\chi-1} \|\gamma_l u\|_{L^p(2B'_J)} + |B'_J|^{\frac{1}{p}} \sum_{\nu\geq -J+1} 2^{-2\nu N} |F'_\nu|^{1-\frac{1}{p}}$$
$$\times \left( \|L^0u\|_{L^p(F'_\nu;L^p(\mathbb{R}_+))} + \sum_{l=0}^{\chi-1} \|\gamma_l u\|_{L^p(F'_\nu)} \right) \tag{15}$$

holds for any $u \in \mathcal{S}(\mathbb{R}^n; W_k^{2m,p}(\mathbb{R}_+))$ with tangential spectrum belonging to the annulus $\frac{1}{2} \leq |\xi'| \leq 2$.

We apply the operator $(L^0(t, 0; D_t, \xi'), \gamma(\xi'))$ to the relation:

$$\widehat{u}(\cdot, \xi') = \int_{y' \in \mathbb{R}^n} e^{-iy'\cdot\xi'} u(\cdot, y') dy'$$

to get the system:

$$\begin{cases} L^0(t, 0; D_t, \xi')\widehat{u}(\cdot, \xi') = \widehat{L^0 u}(\cdot, \xi') = \int \mathrm{e}^{-iy'\cdot\xi'} L^0 u(\cdot, y')\mathrm{d}y' \\ \gamma_l \widehat{u}(\xi') \quad = \quad \widehat{\gamma_l u}(\xi') = \int \mathrm{e}^{-iy'\cdot\xi'} \gamma_l u(y')\mathrm{d}y', \end{cases} \tag{16}$$

for $l \in \{0, 1, \ldots, \chi - 1\}$. Applying $K_{\xi'}$ to this system, we obtain:

$$\widehat{u}(\cdot, \xi') = \int e^{-iy'\cdot\xi'} K_{\xi'} \left(L^0 u(\cdot, y'), \gamma u(y')\right) \mathrm{d}y'.$$

Let $\phi(\xi') \in C_0^\infty(\mathbb{R}^n)$ equals to 1 on $\frac{1}{2} \leq |\xi'| \leq 2$ and its support belongs to an annulus. Then:

$$\begin{aligned} &u(\cdot, x') \\ &= \int \mathrm{e}^{ix'\cdot\xi'} \phi(\xi')\widehat{u}(\cdot, \xi') \frac{\mathrm{d}\xi'}{(2\pi)^n} \\ &= \int\int \mathrm{e}^{i(x'-y')\cdot\xi'} \Big\{\phi(\xi')K_{\xi'}\big(L^0 u(\cdot, y'), \gamma u(y')\big)\Big\} dy' \frac{d\xi'}{(2\pi)^n} \\ &= \int\int \frac{\mathrm{e}^{i(x'-y')\cdot\xi'}}{\big(1 + |x' - y'|^2\big)^N} \big(I - \Delta_{\xi'}\big)^N \Big\{\phi(\xi')K_{\xi'}\big(L^0 u(\cdot, y'), \gamma u(y')\big)\Big\} dy' \frac{\mathrm{d}\xi'}{(2\pi)^n}; \end{aligned}$$

applying the inequality (14), we deduce:

$$\begin{aligned} &\|u(\cdot, x')\|_{W_k^{2m,p}(\mathbb{R}_+)} \\ &\leq C_N \int_{y' \in \mathbb{R}^n} \frac{1}{1 + |x' - y'|^{2N}} \left\|\left(L^0 u(\cdot, y'), \gamma u(y')\right)\right\|_{L^p(\mathbb{R}_+)\times\mathbb{C}^\chi} \mathrm{d}y'; \end{aligned} \tag{17}$$

integrating in (17) with respect to $x'$ over $B'_J$, we obtain:

$$\begin{aligned} &\|u\|_{L^p(B'_J; W_k^{2m,p}(\mathbb{R}_+))} \\ &\leq C_N \left\{ \int_{B'_J} \left( \int_{y' \in \mathbb{R}^n} \frac{1}{1 + |x' - y'|^{2N}} \left\|\left(L^0 u(\cdot, y'), \gamma u(y')\right)\right\|_{L^p(\mathbb{R}_+)\times\mathbb{C}^\chi} \mathrm{d}y' \right)^p \mathrm{d}x' \right\}^{\frac{1}{p}}. \end{aligned}$$

Now, we decompose $\mathbb{R}^n$: $\mathbb{R}^n = 2B'_J \cup \bigcup_{\nu \geq -J+1} F'_\nu$. Then:

$$\begin{aligned} &\|u\|_{L^p(B'_J; W_k^{2m,p}(\mathbb{R}_+))} \\ &\lesssim \left\{ \int_{B'_J} \left( \int_{y' \in \mathbb{R}^n} \frac{1}{1 + |x' - y'|^{2N}} \chi_{2B'_J}(y') \right.\right. \\ &\quad \left.\left. \times \left\|\left(L^0 u(\cdot, y'), \gamma u(y')\right)\right\|_{L^p(\mathbb{R}_+)\times\mathbb{C}^\chi} \mathrm{d}y' \right)^p \mathrm{d}x' \right\}^{\frac{1}{p}} \\ &\quad + \left\{ \int_{B'_J} \left( \sum_{\nu \geq -J+1} \int_{y' \in F'_\nu} \frac{1}{1 + |x' - y'|^{2N}} \right.\right. \\ &\quad \left.\left. \times \left\|\left(L^0 u(\cdot, y'), \gamma u(y')\right)\right\|_{L^p(\mathbb{R}_+)\times\mathbb{C}^\chi} \mathrm{d}y' \right)^p \mathrm{d}x' \right\}^{\frac{1}{p}} \\ &\lesssim I'_1 + I'_2. \end{aligned} \tag{18}$$

The first term $I_1'$ is an $L^p$-norm of a convolution product between a function of $L^1(\mathbb{R}^n)$ (for $N$ large) and a function of $L^p(\mathbb{R}^n)$. Then, Young's inequality yields:

$$I_1' \leq C_N\big\{\|L^0u\|_{L^p(2B_J';L^p(\mathbb{R}_+))} + \|\gamma u\|_{L^p(2B_J')}\big\}. \tag{19}$$

For $I_2'$, since $x' \in B_J'$ and $y' \in F_\nu'$, we have $|x'-y'| \sim 2^\nu$. Then:

$$I_2' \lesssim |B_J'|^{\frac{1}{p}} \sum_{\nu \geq -J+1} 2^{-2\nu N} \int_{y'\in F_\nu'} \left\|\left(L^0u(\cdot,y'), \gamma u(y')\right)\right\|_{L^p(\mathbb{R}_+)\times\mathbb{C}^\chi} dy'$$

Hölder's inequality yields:

$$I_2' \lesssim |B_J'|^{\frac{1}{p}} \sum_{\nu \geq -J+1} 2^{-2\nu N}|F_\nu'|^{1-\frac{1}{p}} \left(\int_{y'\in F_\nu'} \left\|\left(L^0u(\cdot,y'), \gamma u(y')\right)\right\|^p_{L^p(\mathbb{R}_+)\times\mathbb{C}^\chi} dy'\right)^{\frac{1}{p}}. \tag{20}$$

Inequalities (18), (19), and (20) yield:

$$\begin{aligned}
&\|u\|_{L^p(B_J';W_k^{2m,p}(\mathbb{R}_+))}\\
&\quad\lesssim \|L^0u\|_{L^p(2B_J';L^p(\mathbb{R}_+))} + \sum_{l=0}^{\chi-1} \|\gamma_l u\|_{L^p(2B_J')} + |B_J'|^{\frac{1}{p}} \sum_{\nu\geq -J+1} 2^{-2\nu N}|F_\nu'|^{1-\frac{1}{p}}\\
&\quad\quad\times\left(\int_{y'\in F_\nu'} \left\|\left(L^0u(\cdot,y'), \gamma u(y')\right)\right\|^p_{L^p(\mathbb{R}_+)\times\mathbb{C}^\chi} dy'\right)^{\frac{1}{p}}.
\end{aligned}\tag{21}$$

Let $u \in W_k^{2m,p}(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))$ with $\operatorname{supp} u \subset K$, where $K$ is a compact set of $\overline{\mathbb{R}^{n+1}_+}$.

For $j \in \mathbb{N}$, we set $u_j(t,x') = \Delta_j' u(2^{-j}t, 2^{-j}x')$, and then, $u_j \in \mathcal{S}(\mathbb{R}^n; W_k^{2m,p}(\mathbb{R}_+))$ for $j \geq 1$, with tangential spectrum belonging to the annulus $\{\frac{1}{2} \leq |\xi'| \leq 2\}$. Since:

$$\begin{cases} (L^0u)_j = 2^{j(2m-k)}L^0u_j\\ (\gamma_l u)_j \;= 2^{jl}\gamma_l u_j, \end{cases}$$

and by applying inequality (21) for each $u_j$ with $j \geq 1$, we obtain:

$$\begin{aligned}
&\|u_j\|^q_{L^p(B_J';W_k^{2m,p}(\mathbb{R}_+))}\\
&\quad\lesssim \|L^0u_j\|^q_{L^p(2B_J';L^p(\mathbb{R}_+))} + \sum_{l=0}^{\chi-1}\|\gamma_l u_j\|^q_{L^p(2B_J')} + |B_J'|^{\frac{q}{p}}\left\{\sum_{\nu\geq -J+1} 2^{-2\nu N}|F_\nu'|^{1-\frac{1}{p}}\right.\\
&\quad\quad\left.\times\left(\|L^0u_j\|_{L^p(F_\nu';L^p(\mathbb{R}_+))} + \sum_{l=0}^{\chi-1}\|\gamma_l u_j\|_{L^p(F_\nu')}\right)\right\}^q\\
&\quad\lesssim 2^{-jq(2m-k)}\|(L^0u)_j\|^q_{L^p(2B_J';L^p(\mathbb{R}_+))} + \sum_{l=0}^{\chi-1} 2^{-jlq}\|(\gamma_l u)_j\|^q_{L^p(2B_J')}\\
&\quad\quad + |B_J'|^{\frac{q}{p}}\left\{\sum_{\nu\geq -J+1} 2^{-2\nu N}|F_\nu'|^{1-\frac{1}{p}}\left(2^{-j(2m-k)}\|(L^0u)_j\|_{L^p(F_\nu';L^p(\mathbb{R}_+))}\right.\right.\\
&\quad\quad\left.\left.+\sum_{l=0}^{\chi-1} 2^{-jl}\|(\gamma_l u)_j\|_{L^p(F_\nu')}\right)\right\}^q.
\end{aligned}$$

Hence:

$$
\begin{aligned}
&\sum_{h=0}^{\min(k,2m)} \sum_{r=0}^{2m-h} 2^{jq(k-h-r)} \parallel t^{k-h} D_t^r \Delta'_j u \parallel^q_{L^p(\mathbb{R}_+ \times 2^{-j} B'_J)} \\
&\lesssim 2^{-jq(2m-k)} \| \Delta'_j L^0 u \|^q_{L^p(\mathbb{R}_+ \times 2^{-j+1} B'_J)} + \sum_{l=0}^{\chi-1} 2^{-jq(l+\frac{1}{p})} \| \Delta'_j \gamma_l u \|^q_{L^p(2^{-j+1} B'_J)} \\
&\quad + |B'_J|^{\frac{q}{p}} \left\{ \sum_{\nu \geq -J+1} 2^{-2\nu N} |F'_\nu|^{1-\frac{1}{p}} \left( 2^{-j(2m-k)} \| \Delta'_j L^0 u \|_{L^p(\mathbb{R}_+ \times 2^{-j} F'_\nu)} \right.\right. \\
&\quad \left.\left. + \sum_{l=0}^{\chi-1} 2^{-j(l+\frac{1}{p})} \parallel \Delta'_j \gamma_l u \parallel_{L^p(2^{-j} F'_\nu)} \right) \right\}^q ;
\end{aligned} \tag{22}
$$

multiplying both sides of (22) by $2^{jq(s+2m-k)}$, we obtain:

$$
\begin{aligned}
&\sum_{h=0}^{\min(k,2m)} \sum_{r=0}^{2m-h} 2^{jq(s+2m-h-r)} \parallel t^{k-h} D_t^r \Delta'_j u \parallel^q_{L^p(\mathbb{R}_+ \times 2^{-j} B'_J)} \\
&\lesssim 2^{jqs} \| \Delta'_j L^0 u \|^q_{L^p(\mathbb{R}_+ \times 2^{-j+1} B'_J)} + \sum_{l=0}^{\chi-1} 2^{jq(s+2m-k-l-\frac{1}{p})} \| \Delta'_j \gamma_l u \|^q_{L^p(2^{-j+1} B'_J)} \\
&\quad + |B'_J|^{\frac{q}{p}} \left\{ \sum_{\nu \geq -J+1} 2^{-2\nu N} |F'_\nu|^{1-\frac{1}{p}} \left( 2^{js} \| \Delta'_j L^0 u \|_{L^p(\mathbb{R}_+ \times 2^{-j} F'_\nu)} \right.\right. \\
&\quad \left.\left. + \sum_{l=0}^{\chi-1} 2^{j(s+2m-k-l-\frac{1}{p})} \parallel \Delta'_j \gamma_l u \parallel_{L^p(2^{-j} F'_\nu)} \right) \right\}^q \lesssim I''_1 + I''_2,
\end{aligned} \tag{23}
$$

where

$$
I''_1 = 2^{jqs} \| \Delta'_j L^0 u \|^q_{L^p(\mathbb{R}_+ \times 2^{-j+1} B'_J)} + \sum_{l=0}^{\chi-1} 2^{jq(s+2m-k-l-\frac{1}{p})} \| \Delta'_j \gamma_l u \|^q_{L^p(2^{-j+1} B'_J)}
$$

and

$$
\begin{aligned}
I''_2 = |B'_J|^{\frac{q}{p}} &\left\{ \sum_{\nu \geq -J+1} 2^{-2\nu N} |F'_\nu|^{1-\frac{1}{p}} \left( 2^{js} \| \Delta'_j L^0 u \|_{L^p(\mathbb{R}_+ \times 2^{-j} F'_\nu)} \right.\right. \\
&\left.\left. + \sum_{l=0}^{\chi-1} 2^{j(s+2m-k-l-\frac{1}{p})} \parallel \Delta'_j \gamma_l u \parallel_{L^p(2^{-j} F'_\nu)} \right) \right\}^q .
\end{aligned}
$$

We set $K = J + j$ and $\mu = \nu - j$. Then:

$$
I''_1 = 2^{jqs} \| \Delta'_j L^0 u \|^q_{L^p(\mathbb{R}_+ \times 2B'_K)} + \sum_{l=0}^{\chi-1} 2^{jq(s+2m-k-l-\frac{1}{p})} \| \Delta'_j \gamma_l u \|^q_{L^p(2B'_K)}. \tag{24}
$$

On the other hand, since $|F'_\nu|^{1-\frac{1}{p}} \sim 2^{n\nu(1-\frac{1}{p})}$ and setting $\mu' = \mu + K$:

$$\begin{aligned}
I''_2 &\le |B'_J|^{\frac{q}{p}} 2^{jq(n(1-\frac{1}{p})-2N)} \left\{ \sum_{\mu \ge -K+1} 2^{\mu(n(1-\frac{1}{p})-2N)} \left( 2^{js} \| \Delta'_j L^0 u \|_{L^p(\mathbb{R}_+ \times F'_\mu)} \right.\right. \\
&\quad \left.\left. + \sum_{l=0}^{\chi-1} 2^{j(s+2m-k-l-\frac{1}{p})} \| \Delta'_j \gamma_l u \|_{L^p(F'_\mu)} \right) \right\}^q \\
&\le |B'_J|^{\frac{q}{p}} 2^{(j-K)q(n(1-\frac{1}{p})-2N)} 2^{-Kn\tau q} \left\{ \sum_{\mu' \ge 1} 2^{\mu'(n(1-\frac{1}{p})+n\tau-2N)} \left( \frac{2^{js}}{|F'_{\mu'-K}|^\tau} \right.\right. \\
&\quad \left.\left. \times \| \Delta'_j L^0 u \|_{L^p(\mathbb{R}_+ \times F'_{\mu'-K})} + \sum_{l=0}^{\chi-1} \frac{2^{j(s+2m-k-l-\frac{1}{p})}}{|F'_{\mu'-K}|^\tau} \| \Delta'_j \gamma_l u \|_{L^p(F'_{\mu'-K})} \right) \right\}^q \\
&\le 2^{(j-K)q(n-2N)} 2^{-Kn\tau q} \left\{ \sum_{\mu' \ge 1} 2^{\mu'(n(1-\frac{1}{p})+n\tau-2N)} \left( \frac{2^{js}}{|F'_{\mu'-K}|^\tau} \right.\right. \\
&\quad \left.\left. \times \| \Delta'_j L^0 u \|_{L^p(\mathbb{R}_+ \times F'_{\mu'-K})} + \sum_{l=0}^{\chi-1} \frac{2^{j(s+2m-k-l-\frac{1}{p})}}{|F'_{\mu'-K}|^\tau} \| \Delta'_j \gamma_l u \|_{L^p(F'_{\mu'-K})} \right) \right\}^q ; \qquad (25)
\end{aligned}$$

considering inequalities (23)–(25), multiplying by $\frac{1}{|B'_K|^{\tau q}}$, and summing over $j \ge \max(K, 1)$, we obtain:

$$\begin{aligned}
&\sum_{h=0}^{\min(k,2m)} \sum_{r=0}^{2m-h} \sum_{j \ge \max(K,1)} \frac{2^{jq(s+2m-h-r)}}{|B'_K|^{\tau q}} \| t^{k-h} D_t^r \Delta'_j u \|^q_{L^p(\mathbb{R}_+ \times B'_K)} \\
&\lesssim \sum_{j \ge \max(K,1)} \frac{2^{jqs}}{|B'_K|^{\tau q}} \| \Delta'_j L^0 u \|^q_{L^p(\mathbb{R}_+ \times 2B'_K)} + \sum_{l=0}^{\chi-1} \sum_{j \ge \max(K,1)} \frac{2^{jq(s+2m-k-l-\frac{1}{p})}}{|B'_K|^{\tau q}} \\
&\quad \times \| \Delta'_j \gamma_l u \|^q_{L^p(2B'_K)} + \sum_{j \ge \max(K,1)} \frac{2^{-Kn\tau q}}{|B'_K|^{\tau q}} 2^{(j-K)(n-2N)q} \\
&\quad \times \left\{ \sum_{\mu' \ge 1} 2^{\mu'(n(1-\frac{1}{p})+n\tau-2N)} \left( \frac{2^{js}}{|F'_{\mu'-K}|^\tau} \| \Delta'_j L^0 u \|_{L^p(\mathbb{R}_+ \times F'_{\mu'-K})} \right.\right. \\
&\quad \left.\left. + \sum_{l=0}^{\chi-1} \frac{2^{j(s+2m-k-l-\frac{1}{p})}}{|F'_{\mu'-K}|^\tau} \| \Delta'_j \gamma_l u \|_{L^p(F'_{\mu'-K})} \right) \right\}^q ;
\end{aligned}$$

since $K^+ \le \max(K, 1)$, Lemma 9 gives:

$$\begin{aligned}
&\sum_{h=0}^{\min(k,2m)} \sum_{r=0}^{2m-h} \sum_{j \ge \max(K,1)} \frac{2^{jq(s+2m-h-r)}}{|B'_K|^{\tau q}} \| t^{k-h} D_t^r \Delta'_j u \|^q_{L^p(\mathbb{R}_+ \times B'_K)} \\
&\lesssim \| L^0 u \|^q_{L^p(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))} + \sum_{l=0}^{\chi-1} \| \gamma_l u \|^q_{B^{s+2m-k-l-\frac{1}{p},\tau}_{p,q}(\mathbb{R}^n)} + \sup_{\mu' \ge 1} \frac{1}{|F'_{\mu'-K}|^{\tau q}} \qquad (26) \\
&\quad \times \sum_{j \ge K^+} \left( 2^{jqs} \| \Delta'_j L^0 u \|^q_{L^p(\mathbb{R}_+ \times F'_{\mu'-K})} + \sum_{l=0}^{\chi-1} 2^{jq(s+2m-k-l-\frac{1}{p})} \| \Delta'_j \gamma_l u \|^q_{L^p(F'_{\mu'-K})} \right);
\end{aligned}$$

we add the terms associated with $j = 0$ and we replace the condition on the right-hand side of (26) $j \geq K^+$ with $j \geq (K - \mu' + 1)^+$; we obtain:

$$\begin{aligned} &\|u\|^q_{W^{2m,p}_k(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))} \\ &\quad \lesssim C_K \left\{ \|L^0 u\|^q_{L^p(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))} + \sum_{l=0}^{\chi-1} \|\gamma_l u\|^q_{B^{s+2m-k-l-\frac{1}{p},\tau}_{p,q}(\mathbb{R}^n)} \right\} + R_0, \end{aligned} \tag{27}$$

where

$$R_0 = \sum_{h=0}^{\min(k,2m)} \sum_{r=0}^{2m-h} \frac{1}{|B'_K|^{\tau q}} \parallel t^{k-h} D_t^r \Delta'_0 u \parallel^q_{L^p(\mathbb{R}_+ \times B'_K)}.$$

Now, we will estimate the "remainder term" $R_0$. First, we have:

$$\begin{aligned} R_0 = &\sum_{h=0}^{\min(k,2m)} \sum_{\substack{r \leq 2m-h \\ r \neq 2m}} \frac{1}{|B'_K|^{\tau q}} \parallel t^{k-h} D_t^r \Delta'_0 u \parallel^q_{L^p(\mathbb{R}_+ \times B'_K)} \\ &+ \frac{1}{|B'_K|^{\tau q}} \parallel t^k D_t^{2m} \Delta'_0 u \parallel^q_{L^p(\mathbb{R}_+ \times B'_K)}. \end{aligned} \tag{28}$$

Applying Lemma 11 to the first term on the right-hand side of (28), we get:

$$\begin{aligned} &\sum_{h=0}^{\min(k,2m)} \sum_{\substack{r \leq 2m-h \\ r \neq 2m}} \frac{1}{|B'_K|^{\tau q}} \parallel t^{k-h} D_t^r \Delta'_0 u \parallel^q_{L^p(\mathbb{R}_+ \times B'_K)} \\ &\lesssim \varepsilon \frac{1}{|B'_K|^{\tau q}} \parallel t^k D_t^{2m} \Delta'_0 u \parallel^q_{L^p(\mathbb{R}_+ \times B'_K)} + \varepsilon^{-\frac{r+h}{2m-h}} \frac{1}{|B'_K|^{\tau q}} \parallel t^k \Delta'_0 u \parallel^q_{L^p(\mathbb{R}_+ \times B'_K)}; \end{aligned}$$

since $\operatorname{supp} u \subset K$:

$$\begin{aligned} &\sum_{h=0}^{\min(k,2m)} \sum_{\substack{r \leq 2m-h \\ r \neq 2m}} \frac{1}{|B'_K|^{\tau q}} \parallel t^{k-h} D_t^r \Delta'_0 u \parallel^q_{L^p(\mathbb{R}_+ \times B'_K)} \\ &\lesssim \varepsilon \frac{1}{|B'_K|^{\tau q}} \parallel t^k D_t^{2m} \Delta'_0 u \parallel^q_{L^p(\mathbb{R}_+ \times B'_K)} + C_{K,\varepsilon} \frac{1}{|B'_K|^{\tau q}} \parallel \Delta'_0 u \parallel^q_{L^p(\mathbb{R}_+ \times B'_K)} \\ &\lesssim \varepsilon \frac{1}{|B'_K|^{\tau q}} \parallel t^k D_t^{2m} \Delta'_0 u \parallel^q_{L^p(\mathbb{R}_+ \times B'_K)} + C_{K,\varepsilon} \parallel u \parallel^q_{L^p(\mathbb{R}_+; B^{s+2m-k-1,\tau}_{p,q}(\mathbb{R}^n))}. \end{aligned} \tag{29}$$

To estimate the last term of the right-hand side of (28), we write:

$$t^k D_t^{2m} u = L^0 u - \sum_{h=0}^{\min(k,2m)} t^{k-h} \sum_{\substack{|\alpha'|+j=2m-h \\ j \neq 2m}} a_{\alpha',j}(0,0) D_{x'}^{\alpha'} D_t^j u;$$

hence:

$$\begin{aligned} &\frac{1}{|B'_K|^{\tau}} \parallel t^k D_t^{2m} \Delta'_0 u \parallel_{L^p(\mathbb{R}_+ \times B'_K)} \\ &\quad \lesssim \frac{1}{|B'_K|^{\tau}} \parallel \Delta'_0 L^0 u \parallel_{L^p(\mathbb{R}_+ \times B'_K)} \\ &\qquad + \frac{1}{|B'_K|^{\tau}} \sum_{h=0}^{\min(k,2m)} \sum_{\substack{j \leq 2m-h \\ j \neq 2m}} \parallel t^{k-h} D_t^j \Delta'_0 u \parallel_{L^p(\mathbb{R}_+ \times B'_K)}; \end{aligned}$$

due to Lemma 11 and since $\operatorname{supp} u \subset K$, we get:

$$\frac{1}{|B'_K|^{\tau q}} \parallel t^k D_t^{2m} \Delta'_0 u \parallel^q_{L^p(\mathbb{R}_+ \times B'_K)} \lesssim \parallel L^0 u \parallel^q_{L^p(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))} + C_{K,\varepsilon} \parallel u \parallel^q_{L^p(\mathbb{R}_+; B^{s+2m-k-1,\tau}_{p,q}(\mathbb{R}^n))}. \tag{30}$$

Inequalities (27)–(30) imply the proposition 17 for the operator $L^0$.

Now, we estimate the terms of the operator $L^1$:

$$\parallel L^1 u \parallel_{L^p(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))} \lesssim \sum_{h=0}^{\min(k,2m)} \sum_{\substack{|\alpha'|+j=2m-h \\ j\neq 2m}} \parallel \left(a_{\alpha',j}(t,x') - a_{\alpha',j}(0,0)\right) t^{k-h} D^{\alpha'}_{x'} D^j_t u \parallel_{L^p(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))};$$

Lemma 13 yields:

$$\begin{aligned}\parallel L^1 u \parallel_{L^p(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))} &\lesssim \parallel \left(a_{\alpha',j}(t,x') - a_{\alpha',j}(0,0)\right) \parallel_{L^\infty(\mathbb{R}^{n+1}_+)} \parallel t^{k-h} D^{\alpha'}_{x'} D^j_t u \parallel_{L^p(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))} \\ &\quad + C \parallel t^{k-h} D^{\alpha'}_{x'} D^j_t u \parallel_{L^p(\mathbb{R}_+; B^{s-1,\tau}_{p,q}(\mathbb{R}^n))} \\ &\lesssim \epsilon \parallel t^{k-h} D^{\alpha'}_{x'} D^j_t u \parallel_{L^p(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))} + C \parallel t^{k-h} D^{\alpha'}_{x'} D^j_t u \parallel_{L^p(\mathbb{R}_+; B^{s-1,\tau}_{p,q}(\mathbb{R}^n))}\end{aligned}$$

for $0 \leq h \leq \min(k, 2m)$, $0 \leq j \leq 2m - h$ with $j \neq 2m$ and for $u$ with support included in a half-ball of center $(0, 0)$ and with a small enough radius $\epsilon$. Afterwards, since $D^{\alpha'}_{x'}$ maps continuously from $L^p(\mathbb{R}_+; B^{s+|\alpha'|-1,\tau}_{p,q}(\mathbb{R}^n))$ to $L^p(\mathbb{R}_+; B^{s-1,\tau}_{p,q}(\mathbb{R}^n))$ and with the aid of Lemma 11, we obtain:

$$\begin{aligned}\parallel L^1 u \parallel_{L^p(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))} &\lesssim \epsilon \parallel t^{k-h} D^j_t u \parallel_{L^p(\mathbb{R}_+; B^{s+2m-h-j,\tau}_{p,q}(\mathbb{R}^n))} \\ &\quad + C\Big(\varepsilon_0 \parallel t^k D^{2m}_t u \parallel_{L^p(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))} + \varepsilon_0^{-\frac{j+h}{2m-j}} \parallel t^k u \parallel_{L^p(\mathbb{R}_+; B^{s+2m-1,\tau}_{p,q}(\mathbb{R}^n))}\Big).\end{aligned}$$

Thus, Lemma 14 implies that:

$$\begin{aligned}\parallel L^1 u \parallel_{L^p(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))} &\lesssim \epsilon \parallel t^{k-h} D^j_t u \parallel_{L^p(\mathbb{R}_+; B^{s+2m-h-j,\tau}_{p,q}(\mathbb{R}^n))} \\ &\quad + C\Big(\varepsilon_0 \parallel t^k D^{2m}_t u \parallel_{L^p(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))} \\ &\quad + C_{\varepsilon_0}\varepsilon_1 \parallel t^k u \parallel_{L^p(\mathbb{R}_+; B^{s+2m,\tau}_{p,q}(\mathbb{R}^n))} + C_{\varepsilon_0,\varepsilon_1,K} \parallel u \parallel_{L^p(\mathbb{R}_+; B^{s+2m-k-1,\tau}_{p,q}(\mathbb{R}^n))}\Big).\end{aligned} \tag{31}$$

Finally, we use Lemmas 11 and 14 to control $L^2$; we deduce:

$$\begin{aligned}\parallel L^2 u \parallel_{L^p(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))} &\lesssim C_K \varepsilon \parallel t^k D^{2m}_t u \parallel_{L^p(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))} \\ &\quad + C_{K,\varepsilon}\Big(\varepsilon_1 \parallel t^k u \parallel_{L^p(\mathbb{R}_+; B^{s+2m,\tau}_{p,q}(\mathbb{R}^n))} + C_{\varepsilon_1} \parallel u \parallel_{L^p(\mathbb{R}_+; B^{s+2m-k-1,\tau}_{p,q}(\mathbb{R}^n))}\Big).\end{aligned} \tag{32}$$

Inequalities (31) and (32) complete the proof for the operator $L = L^0 + L^1 + L^2$, and so, the proof of Proposition 17 for $u$ with a small enough support around $(0, 0)$ denoted $\epsilon$ is done. In the same way, the estimate is proved around the point $(0, x_0)$ of $K$. Otherwise, the assumption $(A0)$ yields the same estimation in the neighborhood of the point $(t_0, x_0)$ with $t_0 \neq 0$ of $K$. Finally, the general a-priori estimate is obtained by the use of a partition of unity. □

To complete the proof of Theorem 5, we need the following lemma.

**Lemma 18** *Let $s$ and $\tau$ be two non-negative real numbers and let $1 \le p, q < +\infty$. For any compact $K$ of $\overline{\mathbb{R}^{n+1}_+}$, there exists a constant $C_K > 0$, such that for any $u \in B^{s+2m,\tau}_{p,q,k}(\mathbb{R}^{n+1}_+)$ with $\operatorname{supp} u \subset K$, we have:*

$$\|u\|_{B^{s+2m,\tau}_{p,q,k}(\mathbb{R}^{n+1}_+)} \le C_K \left\{ \|Lu\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} + \|u\|_{W^{2m,p}_k(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))} + \|u\|_{B^{s+2m-k-1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} \right\}.$$

*Proof* We restrict ourselves to the case where $0 \le s < 1$. The case $s \ge 0$ can be shown by induction on $r$ where $s = r + \sigma$, such that $r$ is a non-negative integer and $\sigma \in [0, 1[$.

The proof is essentially based on Lemma 12, which allows us to control the normal derivatives of solutions by the “almost tangential" derivatives.

As previously, we write $L = L^0 + L^1 + L^2$, and we first prove the lemma for $L^0$. For this, we estimate the different terms of the norm of $u$ in $B^{s+2m,\tau}_{p,q,k}(\mathbb{R}^{n+1}_+)$. We recall:

$$\| u \|_{B^{s+2m,\tau}_{p,q,k}(\mathbb{R}^{n+1}_+)} \equiv \sum_{h=0}^{\min(k,2m)} \sum_{|\alpha'|+j \le 2m-h} \| t^{k-h} D^{\alpha'}_{x'} D^j_t u \|_{B^{s+2m-(|\alpha'|+j)-h,\tau}_{p,q}(\mathbb{R}^{n+1}_+)};$$

since $D^{\alpha'}_{x'}$ maps continuously from $B^{s+|\alpha'|,\tau}_{p,q}(\mathbb{R}^{n+1}_+)$ to $B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)$, we obtain:

$$\| u \|_{B^{s+2m,\tau}_{p,q,k}(\mathbb{R}^{n+1}_+)} \lesssim \sum_{h=0}^{\min(k,2m)} \sum_{j \le 2m-h} \| t^{k-h} D^j_t u \|_{B^{s+2m-j-h,\tau}_{p,q}(\mathbb{R}^{n+1}_+)};$$

then, we decompose:

$$\| u \|_{B^{s+2m,\tau}_{p,q,k}(\mathbb{R}^{n+1}_+)} \lesssim \sum_{h=0}^{\min(k,2m)} \sum_{j \le 2m-h-1} \| t^{k-h} D^j_t u \|_{B^{s+2m-j-h,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} + \sum_{h=0}^{\min(k,2m)} \| t^{k-h} D^{2m-h}_t u \|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} . \tag{33}$$

For the first term of the right-hand side of inequality (33), we fix $j = 2m - h - 1$ and we estimate the term: $\sum_{h=0}^{\min(k,2m)} \| t^{k-h} D^{2m-h-1}_t u \|_{B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)}$. There are two cases to investigate.

We consider at first the case $\min(k, 2m) = k$, then:

$$\sum_{h=0}^{k} \| t^{k-h} D^{2m-h-1}_t u \|_{B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} = \| D^{2m-k-1}_t u \|_{B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} + \sum_{h=0}^{k-1} \| t^{k-h} D^{2m-h-1}_t u \|_{B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} . \tag{34}$$

To bound the first term of (34), we write:

$$\| D^{2m-k-1}_t u \|_{B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} \equiv \| D_{x'} D^{2m-k-1}_t u \|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} + \| D_t D^{2m-k-1}_t u \|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)};$$

Lemma 12 provides:

$$\begin{aligned} &\| D^{2m-k-1}_t u \|_{B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} \\ &\lesssim \| D_{x'} D^{2m-k}_t u \|_{L^p(\mathbb{R}_+; B^{s-1,\tau}_{p,q}(\mathbb{R}^n))} + \| D_{x'} D^{2m-k-1}_t u \|_{L^p(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))} \\ &\quad + \| D^{2m-k}_t u \|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} \\ &\lesssim \| u \|_{W^{2m,p}_k(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))} + \| D^{2m-k}_t u \|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} . \end{aligned} \tag{35}$$

For the second term of (34), we use again Lemma 12, and hence:

$$
\begin{aligned}
&\sum_{h=0}^{k-1} \| t^{k-h} D_t^{2m-h-1} u \|_{B_{p,q}^{s+1,\tau}(\mathbb{R}_+^{n+1})} \\
&\quad \equiv \sum_{h=0}^{k-1} \left\{ \| D_{x'} t^{k-h} D_t^{2m-h-1} u \|_{B_{p,q}^{s,\tau}(\mathbb{R}_+^{n+1})} + \| D_t (t^{k-h} D_t^{2m-h-1}) u \|_{B_{p,q}^{s,\tau}(\mathbb{R}_+^{n+1})} \right\} \\
&\quad \lesssim \sum_{h=0}^{k-1} \left\{ \| D_{x'} t^{k-h} D_t^{2m-h-1} u \|_{B_{p,q}^{s,\tau}(\mathbb{R}_+^{n+1})} + \| t^{k-h-1} D_t^{2m-h-1} u \|_{B_{p,q}^{s,\tau}(\mathbb{R}_+^{n+1})} \right. \\
&\qquad \left. + \| t^{k-h} D_t^{2m-h} u \|_{B_{p,q}^{s,\tau}(\mathbb{R}_+^{n+1})} \right\} \\
&\quad \lesssim \| u \|_{W_k^{2m,p}(\mathbb{R}_+; B_{p,q}^{s,\tau}(\mathbb{R}^n))} + \sum_{h=0}^{k} \| t^{k-h} D_t^{2m-h} u \|_{B_{p,q}^{s,\tau}(\mathbb{R}_+^{n+1})} . \qquad (36)
\end{aligned}
$$

Inequalities (34)–(36) yield:

$$
\begin{aligned}
&\sum_{h=0}^{k} \| t^{k-h} D_t^{2m-h-1} u \|_{B_{p,q}^{s+1,\tau}(\mathbb{R}_+^{n+1})} \\
&\quad \lesssim \| u \|_{W_k^{2m,p}(\mathbb{R}_+; B_{p,q}^{s,\tau}(\mathbb{R}^n))} + \sum_{h=0}^{k} \| t^{k-h} D_t^{2m-h} u \|_{B_{p,q}^{s,\tau}(\mathbb{R}_+^{n+1})} . \qquad (37)
\end{aligned}
$$

Now, we suppose that $\min(k, 2m) = 2m$, using Lemma 12, we obtain:

$$
\begin{aligned}
&\sum_{h=0}^{2m} \| t^{k-h} D_t^{2m-h-1} u \|_{B_{p,q}^{s+1,\tau}(\mathbb{R}_+^{n+1})} \\
&\quad \equiv \sum_{h=0}^{2m-1} \left\{ \| D_{x'} t^{k-h} D_t^{2m-h-1} u \|_{B_{p,q}^{s,\tau}(\mathbb{R}_+^{n+1})} \right. \\
&\qquad \left. + \| D_t (t^{k-h} D_t^{2m-h-1}) u \|_{B_{p,q}^{s,\tau}(\mathbb{R}_+^{n+1})} \right\} \\
&\quad \lesssim \sum_{h=0}^{2m-1} \left\{ \| D_{x'} t^{k-h} D_t^{2m-h-1} u \|_{B_{p,q}^{s,\tau}(\mathbb{R}_+^{n+1})} + \| t^{k-h-1} D_t^{2m-h-1} u \|_{B_{p,q}^{s,\tau}(\mathbb{R}_+^{n+1})} \right. \\
&\qquad \left. + \| t^{k-h} D_t^{2m-h} u \|_{B_{p,q}^{s,\tau}(\mathbb{R}_+^{n+1})} \right\} \\
&\quad \lesssim \| u \|_{W_k^{2m,p}(\mathbb{R}_+; B_{p,q}^{s,\tau}(\mathbb{R}^n))} + \sum_{h=0}^{2m} \| t^{k-h} D_t^{2m-h} u \|_{B_{p,q}^{s,\tau}(\mathbb{R}_+^{n+1})} . \qquad (38)
\end{aligned}
$$

Gathering estimates (33), (37), and (38), we deduce:

$$
\begin{aligned}
\| u \|_{B_{p,q,k}^{s+2m,\tau}(\mathbb{R}_+^{n+1})} &\lesssim \| u \|_{W_k^{2m,p}(\mathbb{R}_+; B_{p,q}^{s,\tau}(\mathbb{R}^n))} \\
&\quad + \sum_{h=0}^{\min(k,2m)} \| t^{k-h} D_t^{2m-h} u \|_{B_{p,q}^{s,\tau}(\mathbb{R}_+^{n+1})} \qquad (39)
\end{aligned}
$$

holds for $j = 2m - h - 1$. We can proceed similarly for all other terms of (33).

It remains now to estimate the term: $\sum_{h=0}^{\min(k,2m)} \| t^{k-h} D_t^{2m-h} u \|_{B_{p,q}^{s,\tau}(\mathbb{R}_+^{n+1})}$ of (33).

We use the remark that if the support of $v$ is included in a compact set $K$, we have: $v \in B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)$, if and only if $v \in L^p(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))$ and $v \in L^p(\mathbb{R}^n; B^{s,\tau}_{p,q}(\mathbb{R}_+))$. Therefore:

$$\| v \|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} \equiv \| v \|_{L^p(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))} + \| v \|_{L^p(\mathbb{R}^n; B^{s,\tau}_{p,q}(\mathbb{R}_+))} .$$

We have:

$$\sum_{h=0}^{\min(k,2m)} \| t^{k-h} D_t^{2m-h} u \|_{L^p(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))} \lesssim \|u\|_{W_k^{2m,p}(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))}. \tag{40}$$

By returning to the ordinary differential equation as well as using Theorem 16, we deduce that the operator $\left(L^0(0,t;e_1,D_t),\gamma\right)$ is invertible from $V^{s+2m,\tau}_{p,q}(\mathbb{R}_+)$ to $B^{s,\tau}_{p,q}(\mathbb{R}_+)\times\mathbb{C}^{\chi}$, with $e_1=(1,0,\ldots,0)\in\mathbb{R}^n$. Then, for any $v \in V^{s+2m,\tau}_{p,q}(\mathbb{R}_+)$, we have:

$$\|v\|_{V^{s+2m,\tau}_{p,q}(\mathbb{R}_+)} \lesssim \|L^0(t,0;D_t,e_1)v\|_{B^{s,\tau}_{p,q}(\mathbb{R}_+)} + \sum_{l=0}^{\chi-1} |D_t^l v(0)|. \tag{41}$$

We write:

$$\begin{aligned} L^0(t,0;D_t,e_1)u = L^0u &+ \sum_{h=0}^{\min(k,2m)} t^{k-h} \sum_{\substack{\alpha_1+j=2m-h \\ j\le 2m-h-1}} a_{\alpha_1,j}(0,0) D_t^j u \\ &- \sum_{h=0}^{\min(k,2m)} t^{k-h} \sum_{\substack{|\alpha'|+j=2m-h \\ j\le 2m-h-1}} a_{\alpha',j}(0,0) D_{x'}^{\alpha'} D_t^j u; \end{aligned}$$

according to (41), we deduce:

$$\begin{aligned} &\sum_{h=0}^{\min(k,2m)} \| t^{k-h} D_t^{2m-h} u \|_{L^p(\mathbb{R}^n; B^{s,\tau}_{p,q}(\mathbb{R}_+))} \\ &\lesssim \| L^0 u \|_{L^p(\mathbb{R}^n; B^{s,\tau}_{p,q}(\mathbb{R}_+))} + \sum_{h=0}^{\min(k,2m)} \Bigg\{ \sum_{j\le 2m-h-1} \| t^{k-h} D_t^j u \|_{L^p(\mathbb{R}^n; B^{s,\tau}_{p,q}(\mathbb{R}_+))} \\ &\quad + \sum_{\substack{|\alpha'|+j=2m-h \\ j\le 2m-h-1}} \| t^{k-h} D_{x'}^{\alpha'} D_t^j u \|_{L^p(\mathbb{R}^n; B^{s,\tau}_{p,q}(\mathbb{R}_+))} \Bigg\} \\ &\lesssim \|L^0 u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} + \sum_{h=0}^{\min(k,2m)} \Bigg\{ \sum_{j\le 2m-h-1} \| t^{k-h} D_t^j u \|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} \\ &\quad + \sum_{\substack{|\alpha'|+j=2m-h \\ j\le 2m-h-1}} \| t^{k-h} D_{x'}^{\alpha'} D_t^j u \|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} \Bigg\} \\ &\lesssim \|L^0 u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} + \|u\|_{W_k^{2m,p}(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))}. \end{aligned} \tag{42}$$

Both (40) and (42) give:

$$\sum_{h=0}^{\min(k,2m)} \| t^{k-h} D_t^{2m-h} u \|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} \lesssim \|L^0 u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} + \|u\|_{W_k^{2m,p}(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))}. \tag{43}$$

Considering estimates (39) and (43), we deduce the Lemma 18 for the operator $L^0$ and for $s \in [0, 1[$.

Now, for $L = L^0 + L^1$, we estimate the terms of the operator $L^1$ in $B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)$ by assuming again that $\operatorname{supp} u$ is included in a half-ball with center (0, 0) and radius $\epsilon$ small enough. Lemma 13 yields:

$$\begin{aligned}
&\| L^1 u \|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} \\
&\lesssim \epsilon \sum_{h=0}^{\min(k,2m)} \sum_{\substack{|\alpha'|+j=2m-h \\ j\neq 2m}} \| t^{k-h} D^{\alpha'}_{x'} D^j_t u \|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} \\
&\quad + C \sum_{h=0}^{\min(k,2m)} \sum_{\substack{|\alpha'|+j=2m-h \\ j\neq 2m}} \| t^{k-h} D^{\alpha'}_{x'} D^j_t u \|_{B^{s-1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} \\
&\lesssim \epsilon \sum_{h=0}^{\min(k,2m)} \sum_{\substack{|\alpha'|+j=2m-h \\ j\neq 2m}} \| t^{k-h} D^{\alpha'}_{x'} D^j_t u \|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} + C \| u \|_{B^{s+2m-k-1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} .
\end{aligned}$$

Therefore, the Lemma 18 is shown for the operator $L = L^0 + L^1$. Finally, in the same way as before, we evaluate the terms of $L^2 u$ in $B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)$, we get:

$$\begin{aligned}
\| L^2 u \|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} &\lesssim \sum_{h=0}^{\min(k,2m)} \sum_{|\alpha'|+j\leq 2m-h-1} \| a_{\alpha',j}(t,x') t^{k-h} D^{\alpha'}_{x'} D^j_t u \|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} \\
&\lesssim C_K \sum_{h=0}^{\min(k,2m)} \sum_{|\alpha'|+j\leq 2m-h-1} \| t^{k-h} D^{\alpha'}_{x'} D^j_t u \|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} \\
&\lesssim C_K \| u \|_{B^{s+2m-k-1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} .
\end{aligned}$$

The Lemma 18 is proved for $u$ with a small enough support around the origin (0,0). As previously, the general a-priori estimate holds true by the use of a partition of unity.

Finally, Proposition 17 with Lemma 18 leads to Theorem 5. □

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