Francis Joseph H. Campeña · Severino V. Gervacio

On the fold thickness of graphs

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Abstract The graph G' obtained from a graph G by identifying two nonadjacent vertices in G having at least one common neighbor is called a 1-fold of G . A sequence $G_0, G_1, G_2, \dots, G_k$ of graphs such that $G_0 = G$ and G_i is a 1-fold of G_{i-1} for each $i = 1, 2, \dots, k$ is called a uniform k -folding of G if the graphs in the sequence are all singular or all nonsingular. The fold thickness of G is the largest k for which there is a uniform k -folding of G . We show here that the fold thickness of a singular bipartite graph of order n is $n - 3$. Furthermore, the fold thickness of a nonsingular bipartite graph is 0, i.e., every 1-fold of a nonsingular bipartite graph is singular. We also determine the fold thickness of some well-known families of graphs such as cycles, fans and some wheels. Moreover, we investigate the fold thickness of graphs obtained by performing operations on these families of graphs. Specifically, we determine the fold thickness of graphs obtained from the cartesian product of two graphs and the fold thickness of a disconnected graph whose components are all isomorphic.

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1 Introduction

The notion of folding a graph as defined by Gervacio et al. [2] was motivated by the following situation. Consider a finite number of unit bars joined together at the ends where they are free to turn. Some meter sticks are constructed with this kind of structure as shown in bottom drawing in Fig. 1. Note that this particular meter stick can be folded to look like the top drawing in Fig. 1. The meter stick with this structure can be viewed as a physical model of the path graph P_n . After a sequence of folding of the meter stick, it becomes a physical model of a complete graph K_2 . Based on this observation, they defined the notion of folding a graph as follows.

Let x and y be nonadjacent vertices of a graph that have at least one common neighbor. Obtain a new graph G' from G by identifying x and y and reducing any resulting multiple edges to simple edges. We say that G' is a 1-fold of G . Note that folds of a graph are idempotent homomorphisms and also known as retracts. For more details on the connection between a 1-fold and retracts, please see [6].

Example 1.1 The vertices x and y in the graph G shown in Fig. 2 have two common neighbors, b and c . By identifying x and y , we obtain the graph G' , a 1-fold of G .

Consider a graph G that is not isomorphic to a complete graph. Suppose that G_1 is a 1-fold of G . If G_1 is not a complete graph, then there should be a pair of non-adjacent vertices in G_1 and a graph G_2 which is a 1-fold of G_1 can be obtained. Thus, we can consider a sequence of graphs $G = G_0, G_1, G_2, \dots, G_k$. The

F. J. H. Campeña (✉) · S. V. Gervacio
De La Salle University, Manila, Philippines
E-mail: francis.campeña@dlsu.edu.ph

S. V. Gervacio
E-mail: severino.gervacio@dlsu.edu.ph

Fig. 1 A folded and unfolded meter stick

Fig. 2 Forming a 1-fold of G

Fig. 3 A graph with fold thickness equal to 3

largest integer k for which there exists a k -folding is in the case where G_k is a complete graph. We call this sequence a k -folding of $G = G_0$. An adjacency matrix can be associated with each graph in the sequence denoted by $A(G_i)$. We now say that G_i is singular if the adjacency matrix of the graph is singular; otherwise, we say that the graph is non-singular. A k -folding of G is said to be *uniform* if all the graphs in the sequence are singular or all of them are nonsingular. The largest integer k for which there exists a uniform k -folding of G is called the *fold thickness* of G , denoted by $\mathbf{fold}(G)$.

Remark If $G_0, G_1, G_2, \dots, G_k$ is a k -folding of G , we shall refer to G_k as a k -fold of G .

Example 1.2 The graph G shown in Fig. 3 is singular and $\mathbf{fold}(G) = 3$.

Observe that the two pendant vertices a and b are both adjacent to x and none of them is adjacent to any other vertex. In the adjacency matrix of this graph, the rows corresponding to these vertices are identical. Thus, the determinant of the adjacency matrix is 0 and the graph is singular. We can fold this graph starting at the third pendant vertex and we can repeat the process until we obtain the graph consisting only of a, b and x . This is still singular and by identifying a and b , we get already a nonsingular graph. All the other graphs in the sequence are singular because of the existence of a and b having only x as their neighbor. Thus, $\mathbf{fold}(G) = 3$.

We shall give formulas for the fold thickness of some graphs. Specifically, we shall give the fold thickness of paths, cycles, fans and wheels. Our main result gives the fold thickness of bipartite graphs. Moreover, we investigate the fold thickness of graphs obtained by performing some graphs operations. We note that, in 1979, Cooks et al. have already shown that for a given simple connected graph G , the largest k for a k -folding of G is $|V(G)| - \chi(G)$. This is due to the following theorem.

Theorem 1.3 [1] *Let G be a simple connected graph. The smallest complete graph that G folds into is the complete graph with order $\chi(G)$, where $\chi(G)$ denotes the chromatic number of G .*

Thus, we defined a maximum folding of a graph G on n vertices or simply a max fold of G to be a k -folding of G , where $k = n - \chi(G)$ and $\chi(G)$ denoted the chromatic number of the graph.

The readers are referred to [7, 10, 12] for the terminologies and concepts in graph theory not explicitly defined in this paper.

2 Preliminary results

We shall first state some known useful results and prove other results that we will need in determining the fold thickness of some graphs.

Lemma 2.1 [11] *If the independence number $\alpha(G)$ of a graph G is greater than half the order of G , then G is singular.*

Proof Let G be a graph of order n and let $\alpha(G) = k > n/2$. Let $x_1, x_2, \dots, x_n$ be the vertices of G and without loss of generality, assume that $x_1, x_2, \dots, x_k$ form an independent set. Consider the adjacency matrix $\mathcal{A}(G)$ of G . Let $X_1, X_2, \dots, X_k$ be the first k rows of $\mathcal{A}(G)$ and let V be the subspace of R^n that they span. Let $X'_1, X'_2, \dots, X'_k$ be the vectors in R_{n-k} , where X'_i is the vector whose components are the last $n-k$ components of X_i . If V' is the subspace of R^{n-k} spanned by $X'_1, X'_2, \dots, X'_k$ then obviously V and V' are isomorphic. Since $k > n/2$, it follows that $k > n-k$ and hence the vectors $X'_1, X'_2, \dots, X'_k$ are linearly dependent since their number exceeds the dimension of the vector space R^{n-k} . Consequently, the first k rows of $\mathcal{A}(G)$ are linearly dependent and so G is singular. $\square$

If G and H are graphs, we define $G + H$ to be the graph consisting of G and H (considered to be vertex-disjoint) and all edges of the form $[x, y]$, where x is a vertex of G and y is a vertex of H .

Lemma 2.2 *Let G be an r -regular graph of order n . Then $\det \mathcal{A}(G + K_1) = -\frac{n}{r} \det \mathcal{A}(G)$.*

Proof The adjacency matrix of $G + K_1$ has the form

$$\mathcal{A}(G + K_1) = \begin{bmatrix} & & & 1 \\ & \mathcal{A}(G) & & \vdots \\ 1 & \dots & 1 & 0 \\ \cdot & & & \end{bmatrix}$$

Since G is r -regular, each column in $\mathcal{A}(G)$ has exactly r entries equal to 1. Let us add $-\frac{1}{r}$ times row i to row $n+1$ of $\mathcal{A}(G + K_1)$, for $i = 1, 2, \dots, n$. Then row $n+1$ becomes $(0, 0, \dots, 0, -\frac{n}{r})$. It follows that $\det \mathcal{A}(G + K_1) = -\frac{n}{r} \det \mathcal{A}(G)$ by expanding it along the last row. $\square$

Corollary 2.3 *Let G be an r -regular graph. Then $G + K_1$ is singular (nonsingular) if and only if G is singular (nonsingular).*

Known results on the computation of determinants of adjacency matrices, called reduction formulas, will be needed in our arguments. We shall state some of them without proof.

Notation For any vertex x in a graph G , we shall denote by $N(x)$ the set of all vertices y in G that are adjacent to x . We shall refer to $N(x)$ as the *neighbor set* of x .

Theorem 2.4 [5] *If x and y are vertices in a graph G such that $N(x) = N(y)$, then G is singular.*

Theorem 2.5 [5] *Let x and y be vertices in a graph G such that $N(x) \subseteq N(y)$. If G' is the graph obtained from G by deleting all the edges of the form $[y, z]$, where z is a neighbor of x , then $\det \mathcal{A}(G) = \det \mathcal{A}(G')$.*

Theorem 2.6 [5] *Let x and y be adjacent vertices in a graph G such that $N(x) \setminus \{y\} = N(y) \setminus \{x\}$. Then*

$$\det \mathcal{A}(G) = -2 \det \mathcal{A}(G \setminus \{x\}) - \det \mathcal{A}(G \setminus \{x, y\})$$

The theorem that follows gives an upper bound for the fold thickness of graphs.

Theorem 2.7 *For any connected graph G of order n ,*

$$\text{fold}(G) \leq \begin{cases} n - \chi(G) & \text{if } G \text{ is nonsingular,} \\ n - \chi(G) - 1 & \text{if } G \text{ is singular.} \end{cases}$$

Proof Let $G_0, G_1, G_2, \dots, G_k$ be a uniform k -folding of G for some positive integer k . Let us first consider the case G is nonsingular. If G_k is not the complete graph, we can find 2 non-adjacent vertices that can be identified to form a 1-fold of G_k . This process can be continued until there we obtain a graph G_t which is complete where $t \geq k$. Clearly, the order of G_t is $n - t$. Therefore, $n - t \geq \chi(G)$ and consequently, $k \leq n - \chi(G)$. Now consider the case G is singular. Then G_k is singular and so $k < t$. Therefore, $k \leq t - 1 \leq n - \chi(G) - 1$. $\square$

In view of the above theorem, if there exists a uniform k -folding of a connected graph G where k is equal to the upper bound in the theorem, then k must already be the fold thickness of the graph. This observation will be applied in obtaining the fold thickness of some special graphs.

Consider the path of order n denoted by P_n . Trivially, $\det \mathcal{A}(P_1) = 0$ and $\det \mathcal{A}(P_2) = -1$. Let $n \geq 3$ and assume that x, y, z are vertices of P_n , where x is a pendant vertex, y is the unique neighbor of x and z is adjacent to y . By Theorem 2.5, we can remove the edge $[y, z]$ from P_n to obtain a graph with two components P_2 and P_{n-2} . Thus, $\det \mathcal{A}(P_n) = \det \mathcal{A}(P_2) \det \mathcal{A}(P_{n-2}) = -\det \mathcal{A}(P_{n-2})$. By mathematical induction, the following theorem is established.

Theorem 2.8 For each integer $n \geq 1$,

$$\det \mathcal{A}(P_n) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ (-1)^{n/2} & \text{if } n \text{ is even.} \end{cases}$$

Theorem 2.9 [11] If $[1, 2, 3, 4, 5, 6]$ is an induced path of order 6 in a graph G and G' is the graph obtained from G by deleting the vertices 2, 3, 4, 5 and joining 1 and 6 by an edge, then $\det \mathcal{A}(G) = \det \mathcal{A}(G')$.

By Theorem 2.9 and mathematical induction, we get the next theorem.

Theorem 2.10 For each integer $n \geq 3$,

$$\det \mathcal{A}(C_n) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod{4}, \\ 2 & \text{if } n \equiv 1 \text{ or } 3 \pmod{4}, \\ -4 & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

For convenience in most of our arguments, we shall denote the vertices of the cycle C_n by $1, 2, \dots, n$ and its edges by $[1, 2], [2, 3], \dots, [n-1, n], [n, 1]$. The wheel of order $n+1$ is denoted by W_n . It consists of the cycle C_n and an additional vertex 0 that is adjacent to each of the vertices $1, 2, \dots, n$ of C_n . Thus, $W_n = C_n + K_1$.

Corollary 2.11 For each integer $n \geq 3$,

$$\det \mathcal{A}(W_n) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod{4}, \\ -n & \text{if } n \equiv 1 \text{ or } 3 \pmod{4}, \\ 2n & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

Proof Since C_n is 2-regular and $W_n = C_n + K_1$, the conclusion follows from Lemma 2.2. $\square$

3 Fold thickness of some graphs

In this section, we shall determine the fold thickness of the path P_n , the cycle C_n and the wheel W_n . The cycle C_n is singular if and only if n is divisible by 4. In view of Corollary 2.11, the wheel W_n is singular if and only if n is divisible by 4.

Theorem 3.1 The path of order n , denoted by P_n , has fold thickness given by

$$\text{fold}(P_n) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \max\{0, n-3\} & \text{if } n \text{ is odd.} \end{cases}$$

Proof If n is even, P_n is nonsingular. Any 1-fold of P_n yields a bipartite graph of odd order. Consequently, the independence number of this bipartite graph is more than half its order. Therefore, any 1-fold of P_n is singular and $\text{fold}(P_n) = 0$.

Let us recall that the chromatic number of a path of order greater than 1 is 2. If n is odd, P_n is singular. Therefore, $\text{fold}(P_n) \leq n - 3$. If $n = 1$ or 3, it is easy to see that $\text{fold}(P_n) = 0 = \max\{0, n - 3\}$. Let $n > 3$ and assume that the formula holds for all paths P_k with odd $k < n$. For convenience, let the path have vertices $1, 2, 3, \dots, n$ and edges $[i, i + 1]$, $i = 1, 2, \dots, n - 1$. Let us identify the vertices 2 and 4 to get the graph G_1 . This graph is singular since the vertices 1 and 3 have the same neighbors. Identify 1 and 3 to get P_{n-2} . This is a singular graph since $n - 2$ is odd. By hypothesis of induction, $\text{fold}(P_{n-2}) = \max\{0, n - 5\}$. Therefore, $\text{fold}(P_n) = 2 + \max\{0, n - 5\} = \max\{0, n - 3\}$. $\square$

Theorem 3.2 *The cycle C_n , has fold thickness given by*

$$\text{fold}(C_n) = \begin{cases} 0 & \text{if } n \equiv 2 \pmod{4}, \\ n - 3 & \text{otherwise.} \end{cases}$$

Proof Note that $\chi(C_n) = 2$ if n is even and $\chi(C_n) = 3$ if n is odd. If n is odd, C_n is nonsingular. Therefore, by Theorem 2.7, $\text{fold}(C_n) \leq n - 3$. Obviously, $\text{fold}(C_3) = 0 = 3 - 3$. Let us assume that $n \geq 5$. Any 1-fold of C_n is isomorphic to the graph consisting of C_{n-2} plus one other vertex that is adjacent to exactly one vertex of C_{n-2} . This graph is nonsingular and can be folded onto C_{n-2} which is also nonsingular. The process can be continued and so we see that we have an $(n - 3)$ -uniform folding $G_0, G_1, G_2, \dots, G_{n-3}$ of C_n , where the last graph in the sequence is C_3 . Each graph in the sequence is either an odd cycle or an odd cycle plus another vertex adjacent to exactly one vertex of the odd cycle.

If $n \equiv 0 \pmod{4}$, the cycle C_n is singular. Consider the 4-uniform folding of C_n illustrated in Fig. 4.

Note that the last graph in the 4-folding is a cycle of order $n - 4$. By mathematical induction, we conclude that $\text{fold}(C_n) = n - 3$.

If $n \equiv 2 \pmod{4}$, the cycle C_n is nonsingular. Any 1-fold of C_n is singular. Thus, $\text{fold}(C_n) = 0$. $\square$

The fan of order $n + 1$, denoted by F_n , is the graph obtained from the path P_n by adding a new vertex and making this vertex adjacent to every vertex of P_n .

Lemma 3.3 *If $n > 4$, $\det \mathcal{A}(F_n) = -2 \det \mathcal{A}(P_{n-4}) + \det \mathcal{A}(F_{n-4})$.*

Proof Let F_n be made out of the path P_n with vertices $1, 2, \dots, n$ and edges $[i, i + 1]$, $i = 1, 2, \dots, n - 1$ plus another vertex 0 that is adjacent to each of the n vertices of P_n . Apply Theorem 2.5 to the vertices n and $n - 2$ and call the resulting graph G' . Apply the same Theorem to the vertices $n - 4$ and $n - 2$ and then to the vertices $n - 2$ and 0 to get the graph G'' shown in Fig. 5.

Apply Theorem 2.6 to G'' using the vertices $n - 1$ and n . We see that $\det \mathcal{A}(F_n) = -2 \det \mathcal{A}(L) - \det \mathcal{A}(M)$ where L and M are the graphs shown in Fig. 6.

In the graph L , note that $N(n - 1) \subseteq N(i)$ for $i = 1, 2, \dots, n - 4$. Applying Theorem 2.6 to L successively gives us a graph with three components, namely the paths P_{n-4} , P_2 and P_2 . Thus, $\det \mathcal{A}(F_n) = -2 \det \mathcal{A}(P_{n-4}) + \det \mathcal{A}(F_{n-4})$. $\square$

Fig. 4 A uniform 4-folding of C_n

Fig. 5 Reduced fan after three applications of Theorem 2.5

Fig. 6 The graphs L and M after applying Theorem 2.5 to G''

Fig. 7 A 1-fold of W_n

Lemma 3.4 For each $n \geq 1$,

$$\det \mathcal{A}(F_n) = \begin{cases} -\frac{n}{2} & \text{if } n \equiv 0 \pmod{4}, \\ -1 & \text{if } n \equiv 1 \pmod{4}, \\ 1 + \frac{n}{2} & \text{if } n \equiv 2 \pmod{4}, \\ 0 & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

Proof Let $n \equiv 0 \pmod{4}$. If $n = 4$, direct computation gives $\det \mathcal{A}(F_4) = -2$. Let $n > 4$ and assume that $\det \mathcal{A}(F_k) = -\frac{k}{2}$ for all positive $k < n$ with $k \equiv 0 \pmod{4}$. By Lemma 3.3, $\det \mathcal{A}(F_n) = -2 \det \mathcal{A}(F_{n-4}) + \det \mathcal{A}(F_{n-4})$ and by Theorem 2.8 and hypothesis of induction, $\det \mathcal{A}(F_n) = -2 - \frac{n-4}{2} = -\frac{n}{2}$. The remaining three cases are treated in a similar manner. $\square$

In view of Corollary 2.3, since $W_n = C_n + K_1$, then W_n is singular if and only if n is divisible by 4.

Consider the wheel W_n where n is divisible by 4. If $n = 4$, it is easy to see that $\text{fold}(W_4) = 1$. Let $n > 4$ be divisible by 4. Identify vertices 1 and 5 to form the graph G_1 shown in Fig. 7.

In the graph G_1 , 2 and 4 have the same neighbors and so the graph is singular. Obtain graph G_2 by identifying 6 and 8. The graph G_2 is singular because 2 and 4 still have the same neighbors. Obtain G_3 by identifying 7 and 9. By the same reason, G_3 is singular. We continue the process until we obtain the graph G_{n-5} shown in Fig. 8.

Now, all the graphs from G_0 to G_{n-5} are singular. We can fold G_{n-5} onto $G_{n-4} = W_4$ which is singular and finally, W_4 can be folded onto a singular graph G_{n-3} . This graph has only one pair of nonadjacent vertices but if this two vertices are identified we obtain a nonsingular graph. Therefore, $\text{fold}(W_n) = n - 3$. We now state this result as a theorem.

Fig. 8 The graph G_{n-5} obtained after folding W_n $n-5$ times

Theorem 3.5 *If $n > 0$ is divisible by 4, then $\text{fold}(W_n) = n - 3$.*

Three more cases remain for the fold thickness of the wheel W_n , namely $n \equiv 1, 2$ or $3 \pmod{4}$.

Theorem 3.6 *If $n \equiv 1$ or $3 \pmod{4}$, $\text{fold}(W_n) = n - 3$.*

Proof Consider the wheel W_n , where $n \equiv 1$ or $3 \pmod{4}$. This wheel is nonsingular and $\chi(W_n) = 4$. Therefore, $\text{fold}(W_n) \leq n + 1 - 4 = n - 3$. Identify the vertices 1 and $n - 1$. We obtain the graph G_1 consisting of W_{n-2} plus a vertex n that is adjacent to 0 and 1. To find $\det \mathcal{A}(G_1)$, apply Theorem 2.5. Since $N(n) \subseteq N(n - 2)$, we can delete the edges $[n - 2, 0]$ and $[n - 2, 1]$. Call the resulting graph G'_1 . In G'_1 , $N(n - 2) \subseteq N(n - 4)$ and $N(n - 2) \subseteq N(0)$. By deleting $[n - 4, n - 3]$ and $[0, n - 3]$, we get the graph G''_1 consisting of two components, namely P_2 and F_{n-3} . Both components are nonsingular and hence G_1 is nonsingular. Now, we can fold G_1 onto $G_2 = W_{n-2}$ by identifying n and $n - 2$. Since W_{n-2} is nonsingular, the 2-folding G_0, G_1, G_2 is uniform. Since $n \equiv 1$ or $3 \pmod{4}$, then $n - 2 \equiv 3$ or $1 \pmod{4}$. The process may be continued until we get the uniform k -folding $G_0, G_1, G_2, \dots, G_k = W_3$. Here, $k = n - 3$ and so $\text{fold}(W_n) = n - 3$. $\square$

The case $n \equiv 2 \pmod{4}$ seems to be difficult. It is not difficult to check that $\text{fold}(W_6) = 2$. It can be shown in general that there is a uniform $(n - 4)$ -folding of W_n in this case. In connection with this, we state the following conjecture.

Conjecture 3.7 $\text{fold}(W_n) = n - 4$ if $n \equiv 2 \pmod{4}$.

Using basically the same techniques and theorems, the next result can be obtained easily.

Theorem 3.8 *The fan F_n of order $n + 1$ has fold thickness given by*

$$\text{fold}(F_n) = \max\{0, n - 3\}.$$

We need another lemma before we can state and prove our main result.

Lemma 3.9 *Let G be a connected bipartite graph of order greater than 3. If G is singular, then there exists a 1-fold of G that is singular.*

Proof Let A and B be a partition of the vertices of G into two independent sets. Without loss of generality, assume that $|A| \leq |B|$ and consider the following cases.

Case 1 $|A| = |B|$. If G_1 is any one fold of G , then we may assume without loss of generality that G_1 was obtained by identifying two vertices belonging to A . Clearly, the independence number $\alpha(G_1) \geq |B|$ and this is more than half the order of G_1 . By Lemma 2.1, G_1 is singular.

Case 2 $|A| < |B|$. In case $|A| = 1$, then $|B| \geq 3$. Any 1-fold of G is obtained by identifying two vertices in B . Since $|B| \geq 3$, any 1-fold of G is singular. Consider the case $|A| \geq 2$. Let x and y be any two vertices in A . Since G is connected, there is a path in G joining x to y . The second vertex of such a path must be in B and the third vertex, say z , is in A . Therefore, x and z in A have a common neighbor in B . The 1-fold of G obtained by identifying x and z has independence number more than half its order and hence singular. $\square$

We are now ready to state and prove our main result.

Theorem 3.10 *If G is a connected bipartite graph of order n , then*

$$\mathbf{fold}(G) = \begin{cases} n - 3 & \text{if } G \text{ is singular,} \\ 0 & \text{if } G \text{ is nonsingular.} \end{cases}$$

Proof Let G be a connected bipartite graph. If G is nonsingular consider any partition of the vertices of G into two independent sets A and B . necessarily, $|A| = |B|$ for otherwise, the independence number of G would be greater than half its order and this would mean that G is singular. Now, if G_1 is any 1-fold of G , we may assume that G_1 is obtainable from G by identifying two vertices in A . We see that G_1 has independence number that is greater than or equal to $|B|$ which is more than half the order of G_1 . Therefore, G_1 is singular. Thus, $\mathbf{fold}(G) = 0$.

Consider the case G is singular. Then the order n of G must be at least 3. If $n = 3$, we have $\mathbf{fold}(G) = 0$. If $n > 3$, by Lemma 3.9, G has a singular 1-fold G_1 . By induction, it follows that $\mathbf{fold}(G) = n - 3$. $\square$

Corollary 3.11 *For each $m \geq 2$ and $n \geq 2$,*

$$\mathbf{fold}(K_{m,n}) = m + n - 3.$$

Proof We note that the complete bipartite graph K_{v_1, v_2} is of order $m + n$, where $o(v_1) = m$ and $o(v_2) = n$. By Definition of a complete bipartite graph if $x, y \in v_1$ or $x, y \in v_2$ then $N(x) = N(y)$ and thus $K_{m,n}$ is always singular. From Theorem 3.10, the conclusion easily follows. $\square$

From the above theorem, the fold thickness of a star $K_{1,n}$ is easily computed as $\mathbf{fold}(K_{1,n}) = n - 2$ if $n \geq 2$ and $\mathbf{fold}(K_{1,n}) = 0$ if $n = 1$.

Theorem 3.12 *For m, n and $p \geq 1$,*

$$\mathbf{fold}(K_p + K_{m,n}) = m + n - 3.$$

Proof We note that each vertex from K_p is adjacent to every vertex in $K_{m,n}$, thus we can only identify vertices from $K_{m,n}$. Now, let V_1 and V_2 be a partition of the vertex set of $K_{m,n}$ such that $|V_1| = m$ and $|V_2| = n$. For any pair of vertices $x, y \in V_1$ or V_2 , $N(x) = N(y)$. Thus, $K_p + K_{m,n}$ is singular. By definition of the sum of two graphs, every vertex in $K_{m,n}$ is adjacent to every vertex in K_p . Hence, we are just actually folding $K_{m,n}$ and, by Corollary 3.11, the conclusion follows. $\square$

We now define what we mean by the cartesian product of two disjoint graphs G and H denoted by $G \square H$ or $G \times H$. Let $G = (V_1, E_1)$ and $H = (V_2, E_2)$. Define $G \times H = (V, E)$, where $V = V_1 \times V_2$ and two vertices $u = (u_1, u_2), v = (v_1, v_2)$ are adjacent whenever the following holds: (1) $u_1 = v_1$ and u_2 is adjacent to v_2 or (2) $u_2 = v_2$ and u_1 is adjacent to v_1 . It is well known that the cartesian product of two connected graphs is connected. Moreover, if the two graphs are bipartite then the cartesian product is also a bipartite graph. For more detailed discussion on cartesian product of graphs, see [8]. Hence the following theorem follows easily from Theorem 3.10.

Theorem 3.13 *Let G be a bipartite graph of order m . Then*

$$\mathbf{fold}(P_n \times G) = \begin{cases} nm - 3 & \text{if } P_n \times G \text{ is singular} \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 3.14 *Let G be a bipartite graph of order n . Then*

$$\mathbf{fold}(P_2 \times G) = \begin{cases} 2n - 3 & \text{if } 1 \text{ is an eigenvalue of } A(G) \\ 0 & \text{otherwise.} \end{cases}$$

Proof In [4] Theorem 1.3, it was shown that $G \times K_n$ is singular if and only if 1 or $1 - n$ is an eigenvalue of $A(G)$. Since, $P_2 = K_2$ then, $P_2 \times G$ is singular if and only if 1 is an eigenvalue of $A(G)$. Hence, by Theorem 3.10 the conclusion follows easily. $\square$

Corollary 3.15 *For each $n \geq 2$ and for every $k \geq 1$*

$$\mathbf{fold}(P_k \times P_n) = \begin{cases} kn - 3 & \text{if } \gcd(k + 1, n + 1) > 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof In [3], it was shown that $P_k \times P_n$ is singular if and only if $\gcd(k+1, n+1) > 1$. Since $P_k \times P_n$ is bipartite then by Theorem 3.10 the conclusion easily follows. $\square$

Corollary 3.16 For each $n \geq 2$ and for every $k \geq 1$,

$$\text{fold}(C_{2k} \times P_n) = \begin{cases} 2kn - 3 & \text{if } \gcd(2k, n+1) > 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof In, [3], it was shown that $C_m \times P_n$ is singular if and only if $\gcd(2m, 2n+2) > 2$. Thus, if m is even, that is, $m = 2k$, then $C_m \times P_n$ is singular if and only if

$$\begin{aligned} \gcd(2m, 2n+2) > 2 &\Rightarrow \gcd(4k, 2n+2) > 2 \\ &\Rightarrow 2 \gcd(2k, n+1) > 2 \\ &\Rightarrow \gcd(2k, n+1) > 1. \end{aligned}$$

Hence, the conclusion easily follows from Theorem 3.10. $\square$

Corollary 3.17 For each $n \geq 2$ and for every $k \geq 1$

$$\text{fold}(C_{2n} \times C_{2m}) = 4mn - 3.$$

Proof We first note that a cycle of even length is bipartite. And in [3], it was shown that $C_m \times C_n$ is singular if and only if either m or n is even, thus by Theorem 3.10 the conclusion easily follows. $\square$

4 Fold thickness of a disconnected graph

In this section, we determine the fold thickness of some disconnected graphs; specifically, the fold thickness of a graph with p isomorphic components denoted pG . Explicitly, the fold thickness of pC_n , pF_n , pP_n and pG are determined, where G is a bipartite graph.

Theorem 4.1 Let G be a bipartite graph of order n . Then for $p \geq 1$

$$\text{fold}(pG) = \begin{cases} np - 2p - 1 & \text{if } G \text{ is singular} \\ 0 & \text{if } G \text{ is nonsingular.} \end{cases}$$

Proof Let G be a bipartite graph of order n . For each $p \geq 1$, pG would denote p copies of G , that is pG is a bipartite graph with p components where each component is isomorphic to G . Denote by G_i the i th copy of the graph G where $i = 1, 2, \dots, p$. If G is singular then pG must also be singular. Note that the order of pG is np and since G is a bipartite graph $\chi(G_i) = 2$ for each $i = 1, 2, 3, \dots, p-1$. Now to get the fold thickness of pG we isolate one copy of G and maximize the number of foldings of the other $p-1$ copies of G . By isolating one copy of G , we are assured that no matter how we fold the other $p-1$ copies of G the resulting disconnected graph will still be singular. Now since the maximum fold of G is $n - \chi(G)$, then $\text{fold}(pG) = (p-1)(n - \chi(G)) + \text{fold}(G) = (p-1)(n-2) + n - 3 = np - 2p - 1$. Now suppose the G is non-singular. Since G is non-singular each 1-fold of G is singular and hence $\text{fold}(G) = 0$. This also means that each 1-fold of pG is singular; hence, $\text{fold}(pG) = 0$. $\square$

We recall that the path P_n is singular if n is odd and non-singular if n is even. By Theorem 4.1, the proof for the next corollary follows easily.

Corollary 4.2 For $p \geq 1$ and $n \geq 1$,

$$\text{fold}(pP_n) = \begin{cases} np - 2p - 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Theorem 4.3 For $n \geq 3$,

$$\text{fold}(pC_n) = \begin{cases} 0 & \text{if } n \equiv 2 \pmod{4} \\ np - 2p - 1 & \text{if } n \equiv 0 \pmod{4} \\ p(n-3) & \text{if } n \equiv 1 \text{ or } 3 \pmod{4}. \end{cases}$$

Proof If n is even, then pC_n is a bipartite graph. Suppose $n \equiv 0 \pmod{4}$ then by Theorem 2.10, C_n is singular. Therefore pC_n is also singular. Hence, by Theorem 4.1, $\mathbf{fold}(pC_n) = np - 2p - 1$. Now suppose that $n \equiv 2 \pmod{4}$, by Theorem 2.10 C_n is non-singular. This implies that pC_n is non-singular and, by Corollary 4.1, $\mathbf{fold}(pC_n) = 0$. Consider the case where n is odd; then by Theorem 2.10 C_n nonsingular and hence, pC_n is nonsingular. By Theorem 3.2, $\mathbf{fold}(C_n) = n - 3$, where n is odd, thus $\mathbf{fold}(pC_n) = p(n - 3)$. $\square$

Theorem 4.4 For $n \geq 3$

$$\mathbf{fold}(pF_n) = \begin{cases} np - 2p - 1 & \text{if } n \equiv 3 \pmod{4} \\ p(n - 3) & \text{otherwise} \end{cases}$$

Proof For $n \geq 3$. Let F_n be singular, that is, by Lemma 3.4 $n \equiv 3 \pmod{4}$. We isolate one of the p copies of F_n . To get the $\mathbf{fold}(pF_n)$, we maximize the number of folds of the $(p - 1)F_n$ and add the $\mathbf{fold}(F_n)$. Thus, we have

$$\begin{aligned} \mathbf{fold}(pF_n) &= \text{maximum number of fold on } (p - 1)F_n + \mathbf{fold}(F_n) \\ &= (p - 1)(n + 1 - \chi(F_n)) + n - 3 \\ &= (p - 1)(n + 1 - 3) + n - 3 \\ &= (p - 1)(n - 2) + n - 3 \\ &= np - n - 2p + 2 + n - 3 \\ &= np - 2p - 1. \end{aligned}$$

Suppose F_n is nonsingular. By Theorem 3.8, the $\mathbf{fold}(F_n) = n - 3$ for each copy of F_n . Thus, $\mathbf{fold}(pF_n) = p(n - 3)$. $\square$

5 Conclusion

We end this paper by providing some problems regarding the fold thickness of a graph.

One of the main results of the paper is to determine the fold thickness of a bipartite graph. Another result is the determination of the fold thickness of a disconnected graph with isomorphic components. One aspect of this paper relies on knowing the singularity of a graph which has been studied extensively already, which leads us to examine the following problems:

- Problem 1: Determine the value of $\mathbf{fold}(G)$, where G is a k -partite graph on n vertices.
- Problem 2: Determine the value of $\mathbf{fold}(G)$, where G is a disconnected graph with at least 2 components that are non-isomorphic.

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