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Study of a boundary value problem for fractional order ψ -Hilfer fractional derivative

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Abstract This manuscript is devoted to the existence theory of a class of random fractional differential equations (RFDEs) involving boundary condition (BCs). Here we take the corresponding derivative of arbitrary order in ψ -Hilfer sense. By utilizing classical fixed point theory and nonlinear analysis we establish some basic results of the qualitative theory such as existence, uniqueness and stability of solutions to the considered boundary value problem of RFDEs. Further, for the justification of our analysis we provide two examples.

Mathematics Subject Classification 15B52 · 30E25 · 34A12 · 34D35

1 Introduction

In the last few decades the area of fractional calculus has been given proper attention by many researchers. This important branch of mathematics was founded by Newton and Leibniz in seventeenth century. But due to some complexity of the fractional differential operator, the concerned branch got attention very poorly in eighteenth century. However during that time some notable definitions about fractional derivatives were given by some famous mathematicians like Riemann, Liouville, Grönwal, Letnikove, Hadamard and many others, see for detail [16]. After 1970 the aforesaid area got attention. In the last few decades significant work has been done on various aspects of fractional calculus. This is due to the fact that, the modeling of various phenomenons in the fields of science and engineering is done more precisely using FDEs as compared to ordinary differential equations. Therefore FDEs are considered valuable tools to model many real world problems. Since boundary value problem (BVPs) of differential equations represent an important class of applied analysis, therefore the said area was given more importance, see [3, 5, 8, 21, 37] and references therein. Different aspects have been considered in literature for investigating FDEs with BCs like existence theory, stability analysis, numerical analysis and optimization, etc. For the aforesaid analysis mostly the researchers have given attention to study FDEs by taking Caputo or Riemann–Liouville derivative. They established for the aforementioned equations existence theory of solutions and stability analysis mostly, see [4, 15, 24] and the references therein. An important characteristic is that engineers and scientists have developed some new models that involve FDEs. These models have been applied successfully, for instance in mechanics (theory of viscoelasticity and

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viscoplasticity), (bio-)chemistry (modelling of polymers and proteins), electrical engineering (transmission of ultrasound waves), medicine (modelling of human tissue under mechanical loads), etc. Since it is clear that dealing Riemann–Liouville derivative in various applied problems is very difficult, therefore certain modifications were introduced to avoid the above-mentioned difficulties. In this regard some new type fractional order derivative operators were introduced in literature like Caputo, Hadamard, etc. Recently, Hilfer [12] initiated an extended Riemann–Liouville fractional derivative, named Hilfer fractional derivative, which interpolates Caputo fractional derivative and Riemann–Liouville fractional derivative. This said operator arose in the theoretical simulation of dielectric relaxation in glass-forming materials. Further Hilfer et al. [13] initially presented linear differential equations with the new Hilfer fractional derivative and applied operational calculus to solve such generalized fractional differential equations.

In last few years, many researchers have studied FDEs using other definitions of fractional derivative like Hilfer, Hadamard, etc, see [1, 12]. The mentioned FDEs were studied by many authors in last few years, see [9, 11] and the references therein. The authors have mainly considered initial value problems corresponding to Hilfer derivative. But to the best of our knowledge investigation of BVPs corresponding to RFDEs is very rarely considered. The concerned FDEs involving Hilfer derivative have many applications, we refer [26] and the references cited therein. There are actual world occurrences with uncharacteristic dynamics such as atmospheric diffusion of pollution, signals transmissions through strong magnetic fields, the effect of theory on the profitability of stocks in economic markets, network traffic, and so on. The area devoted to RFDEs is a natural extension of such deterministic situations, occurring in several applications and has been examined by many mathematicians, see [1, 19, 34] and the references therein. The mentioned study has addressed existence and stability of solutions to RFDEs.

On the other hand stability analysis has been establish for various kinds of FDEs. Since the mentioned aspect is very important from numerical and optimization point of view. Different kinds of stability like exponential, Lyapunov and Mittag-Leffler type have been studied very well for functional, integral and differential equations, see [18, 29, 30]. Another kind of stability which has been given much attention in last few years for FDEs is known as Ulam–Hyers (HU), generalized Ulam–Hyers (g-UH), Ulam–Hyers–Rassias (UHR) and generalized Ulam–Hyers–Rassias (g-UHR) stability, see [22, 31, 32]. The mentioned stability has been very well studied for FDEs involving Caputo and Riemann–Liouville fractional derivative, see [6, 20, 35] and the references therein. UH stability concept is relatively important in practical problems in mathematical analysis in different fields such as biology and economics. Thus, the generalized results of the considered stability have been discussed in many books and papers, see [7, 14, 23, 36]. For detailed definitions of the said stability and its generalization, we suggest [10, 28, 33] about HFDEs. In this work we establish different kinds of the aforesaid stability for the given BVP of RFDEs as:

$$\begin{cases} \mathcal{D}^{v,\beta;\psi} \hbar(t, \rho) = \eta(t, \rho, \hbar(t, \rho)), & t \in J := [0, T], \\ a \mathcal{I}^{1-\mu;\psi} \hbar(t, \rho)|_{t=0} + b \mathcal{I}^{1-\mu;\psi} \hbar(t, \rho)|_{t=T} = c. \end{cases} \quad (1)$$

Here $\mathcal{D}^{v,\beta;\psi}$ represents ψ -HFD and $\mathcal{I}^{1-\mu;\psi}$ is ψ -integral of orders $1 - \mu$ ($\mu = v + \beta - v\beta$). Let R be a Banach space, $\eta : J \times \Omega \times R \rightarrow R$ is a given continuous function and where a , b and c are some constants. The detailed study and development of ψ -HFD can be seen in [27]. The equivalent integral equation of Eq. (1) is set as

$$\begin{aligned} \hbar(t, \rho) = & \left(c - b \int_0^T \mathcal{I}_T^{1-\beta+v\beta}(s) \eta(s, \rho, \hbar(s, \rho)) ds \right) \frac{\psi_0^{\mu-1}(t)}{(a+b)\Gamma(\mu)} \\ & + \int_0^t \mathcal{I}_t^v(s) \eta(s, \rho, \hbar(s, \rho)) ds, \end{aligned} \quad (2)$$

where

$$\mathcal{I}_t^v(s) = \frac{\psi'(s)(\psi(t) - \psi(s))^{\nu-1}}{\Gamma(\nu)}, \quad \psi_0^{\mu-1}(t) = (\psi(t) - \psi(0))^{\mu-1}.$$

We list some hypotheses to prove our required results.

(H1) There exist constants $n, m : J \times \Omega \rightarrow R$, such that

$$|\eta(\cdot, \rho, \hbar_1(\cdot, \rho))| \leq n(\cdot, \rho) + m(\cdot, \rho) |\hbar(\cdot, \rho)|$$

for every $t \in J$ and $\rho \in \Omega$. Denote $N(\rho) = \sup n(\cdot)$ and $M(\rho) = \sup m(\cdot)$.

(H2) For every constant $\ell > 0$, we have

$$|\eta(\cdot, \rho, \hbar(\cdot, \rho)) - \eta(\cdot, \rho, v(\cdot, \rho))| \leq \ell(\cdot, \rho) |\hbar(\cdot, \rho) - v(\cdot, \rho)|$$

(H3) Thus $\lambda_\varphi > 0$, we have

$$\mathcal{I}^{\nu; \psi} \varphi(t, \rho) \leq \lambda_\varphi \varphi(t, \rho).$$

2 Preliminaries

Definitions and results to obtain the solution are established in this section, see [2]. Let C be the Banach space of all continuous functions $\hbar : J \times \Omega \rightarrow R$ endowed with

$$\|\hbar\|_C = \sup\{|\hbar(t, \rho)| : t \in J\}.$$

Consider weighted space

$$C_{\mu, \psi}(J, R) = \{\hbar : J \times \Omega \rightarrow R : \psi_0^\mu(t) \hbar(t, \rho) \in C\}, \quad 0 \leq \mu < 1,$$

with the norm

$$\|\eta\|_{C_{\mu, \psi}} = \sup_{t \in J} |\psi_0^\mu(t) \eta(t, \rho)|.$$

Definition 2.1 [27] The ψ -fractional integral of a function $\hbar(t)$ is defined by

$$(\mathcal{I}^{\nu; \psi}) \hbar(t) = \int_a^t \mathcal{I}_t^\nu(s) \hbar(s) ds, \quad t > a. \quad (3)$$

Definition 2.2 [27] The ψ -HFD in Riemann–Liouville sense of a function $\hbar$ with respect to ψ of order ν is defined by

$$\mathcal{D}^{\nu; \psi} \hbar(t) = \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n \int_a^t \mathcal{I}_t^{n-\nu}(s) \hbar(s) ds, \quad (4)$$

where $n = [\nu] + 1$.

Definition 2.3 [27] The left ψ -Caputo derivative of order ν is given by

$$\mathcal{D}^{\nu; \psi} \hbar(t) = \mathcal{I}^{n-\nu; \psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n \hbar(t). \quad (5)$$

Definition 2.4 [27] The ψ -HFD of function $\hbar$ is given by

$$\mathcal{D}^{\nu, \beta; \psi} \hbar(t) = \mathcal{I}^{\beta(1-\nu); \psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right) \mathcal{I}^{(1-\beta)(1-\nu)\psi} \hbar(t). \quad (6)$$

The ψ -HFD can be written in another form as

$$\mathcal{D}^{\nu, \beta; \psi} \hbar(t) = \mathcal{I}^{\mu-\nu; \psi} \mathcal{D}^{\mu; \psi} \hbar(t).$$

Next, we shall give the definitions g-UHR stable for the problem

$$\mathcal{D}^{\nu, \beta; \psi} \hbar(t, \rho) = \eta(t, \rho, \hbar(t, \rho)), \quad t \in J. \quad (7)$$

for the continuous function $\varphi : J \times \Omega \rightarrow \mathbb{R}^+$ satisfies the inequality

$$|\mathcal{D}^{\nu, \beta; \psi} v(t, \rho) - \eta(t, \rho, v(t, \rho))| \leq \varphi(t). \quad (8)$$

Definition 2.5 [33] Equation (7) is g-UHR stable with respect to φ if there exists a real number $C_{f, \varphi} > 0$ such that for each solution $v \in C_{1-\mu, \psi}$ of the inequality (8) there exists a solution $\hbar \in C_{1-\mu, \psi}$ of Eq. (7) with

$$|v(t, \rho) - \hbar(t, \rho)| \leq C_{f, \varphi} \varphi(t, \rho), \quad t \in J.$$

Lemma 2.6 (Grönwall's lemma) [28] Suppose $v > 0$, $a(t, \rho)$ is a nonnegative function locally integrable on $J \times \Omega$ (some $T \leq \infty$), and let $g(t, \rho)$ be a nonnegative, nondecreasing continuous function defined on $J \times \Omega$, such that $g(t, \rho) \leq K$ for some constant K . Further let $\mathfrak{h}(t, \rho)$ be a nonnegative locally integrable on $J \times \Omega$ function with

$$\mathfrak{h}(t, \rho) \leq a(t, \rho) + g(t, \rho) \int_a^t \mathcal{T}_t^v(s) \mathfrak{h}(s, \rho) ds, \quad (t, \rho) \in J \times \Omega,$$

with some $v > 0$. Then

$$\mathfrak{h}(t, \rho) \leq a(t, \rho) + \int_a^t \left[\sum_{n=1}^{\infty} (g(t, \rho) \Gamma(v))^n \mathcal{T}_t^{nv}(s) \right] a(s, \rho) ds, \quad (t, \rho) \in J \times \Omega.$$

3 Existence and stability results

Theorem 3.1 Under the hypothesis [H1] and [H2], the Eq. (1) has at least one solution.

Proof Consider the operator $\mathcal{P} : C_{1-\mu, \psi} \rightarrow C_{1-\mu, \psi}$. The operator form of Eq. (2) is as follows:

$$\mathcal{P}\mathfrak{h}(t, \rho) = \left(c - b \int_0^T \mathcal{T}_T^{1-\beta+v\beta}(s) \mathfrak{h}(s, \rho, \mathfrak{h}(s, \rho)) ds \right) \frac{\psi_0^{\mu-1}(t)}{(a+b)\Gamma(\mu)} + \int_0^t \mathcal{T}_t^v(s) \mathfrak{h}(s, \rho, \mathfrak{h}(s, \rho)) ds. \quad (9)$$

Step 1: $\mathcal{P}$ is continuous.

Let $\mathfrak{h}_n$ be a sequence with $\mathfrak{h}_n \rightarrow \mathfrak{h}$ in $C_{1-\mu, \psi}$. Thus

$$\begin{aligned} & \left| (\mathcal{P}\mathfrak{h}_n(t, \rho) - \mathcal{P}\mathfrak{h}(t, \rho)) \psi_0^{1-\mu}(t) \right| \\ & \leq \left(b \int_0^T \mathcal{T}_T^{1-\beta+v\beta}(s) |\mathfrak{h}(s, \rho, \mathfrak{h}_n(s, \rho)) - \mathfrak{h}(s, \rho, \mathfrak{h}(s, \rho))| ds \right) \frac{1}{(a+b)\Gamma(\mu)} \\ & \quad + \psi_0^{1-\mu}(t) \int_0^t \mathcal{T}_t^v(s) |\mathfrak{h}(s, \rho, \mathfrak{h}_n(s, \rho)) - \mathfrak{h}(s, \rho, \mathfrak{h}(s, \rho))| ds \\ & \leq \left(\left(\frac{bB(\mu, 1-\beta+v\beta)}{(a+b)\Gamma(\mu)\Gamma(1-\beta+v\beta)} \psi_0^v(T) \right) + \frac{\psi_0^{1-\mu}(t)}{\Gamma(v)} B(\mu, v) \psi_0^{v+\mu-1}(t) \right) \\ & \quad \|\mathfrak{h}(\cdot, \rho, \mathfrak{h}_n(\cdot, \rho)) - \mathfrak{h}(\cdot, \rho, \mathfrak{h}(\cdot, \rho))\|_{C_{1-\mu, \psi}} \\ & \leq \left(\left(\frac{bB(\mu, 1-\beta+v\beta)}{(a+b)\Gamma(\mu)\Gamma(1-\beta+v\beta)} \psi_0^v(T) \right) + \frac{B(\mu, v)}{\Gamma(v)} \psi_0^v(T) \right) \|\mathfrak{h}(\cdot, \rho, \mathfrak{h}_n(\cdot, \rho)) - \mathfrak{h}(\cdot, \rho, \mathfrak{h}(\cdot, \rho))\|_{C_{1-\mu, \psi}} \end{aligned}$$

since $\mathfrak{h}$ is continuous, then we have

$$\|\mathcal{P}\mathfrak{h}_n - \mathcal{P}\mathfrak{h}\|_{C_{1-\mu, \psi}} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Step 2: $\mathcal{P}$ maps bounded sets into bounded sets in $C_{1-\mu, \psi}$.

Indeed, it is enough to show that for $r > 0$, there exists a positive constant l such that $B_r = \{\mathfrak{h} \in C_{1-\mu, \psi} : \|\mathfrak{h}\|_{C_{1-\mu, \psi}} \leq r\}$, we have $\|\mathcal{P}\mathfrak{h}\|_{C_{1-\mu, \psi}} \leq l$

$$\begin{aligned} & \left| \mathcal{P}\mathfrak{h}(t, \rho) \psi_0^{1-\mu}(t) \right| \\ & \leq \frac{c}{(a+b)\Gamma(\mu)} + \frac{b}{(a+b)\Gamma(\mu)} \int_0^T \mathcal{T}_T^{1-\beta+v\beta}(s) |\mathfrak{h}(s, \rho, \mathfrak{h}(s, \rho))| ds \\ & \quad + \psi_0^{1-\mu}(t) \int_0^t \mathcal{T}_t^v(s) |\mathfrak{h}(s, \rho, \mathfrak{h}(s, \rho))| ds \\ & \leq \frac{c}{(a+b)\Gamma(\mu)} + \frac{b}{(a+b)\Gamma(\mu)} \int_0^T \mathcal{T}_T^{1-\beta+v\beta}(s) (n(s, \rho) + m(s, \rho) |\mathfrak{h}(s, \rho)|) ds \end{aligned}$$

$$\begin{aligned}
 & + \psi_0^{1-\mu}(t) \int_0^t \mathcal{T}_t^v(s) (n(s, \rho) + m(s, \rho) |\hbar(s, \rho)|) ds \\
 & \leq \frac{c}{(a+b)\Gamma(\mu)} + \frac{bN(\rho)}{(a+b)\Gamma(\mu)\Gamma(2-\beta+v\beta)} \psi_0^{1-\beta+v\beta}(T) \\
 & \quad + \frac{bM(\rho)B(\mu, 1-\beta+v\beta)}{(a+b)\Gamma(\mu)\Gamma(1-\beta+v\beta)} \psi_0^v(T) \|\hbar\|_{C_{1-\mu, \psi}} \\
 & \quad + \frac{N(\rho)}{\Gamma(v+1)} \psi_0^{v+1-\mu}(t) + \frac{M(\rho)\psi_0^{1-\mu}(t)}{\Gamma(v)} B(\mu, v) \psi_0^{v+\mu-1}(t) \|\hbar\|_{C_{1-\mu, \psi}} \\
 & \leq \frac{c}{(a+b)\Gamma(\mu)} + \frac{bN(\rho)}{(a+b)\Gamma(\mu)\Gamma(2-\beta+v\beta)} \psi_0^{1-\beta+v\beta}(T) + \frac{N(\rho)}{\Gamma(v+1)} \psi_0^{v+1-\mu}(T) \\
 & \quad + \left(\frac{bM(\rho)}{(a+b)\Gamma(\mu)\Gamma(1-\beta+v\beta)} B(\mu, 1-\beta+v\beta) \psi_0^v(T) + \frac{M(\rho)}{\Gamma(v)} B(\mu, v) \psi_0^v(T) \right) r \\
 & := l.
 \end{aligned}$$

Step 3: $\mathcal{P}$ maps bounded sets into equi-continuous set of $C_{1-\mu, \psi}$.

Let $t_1, t_2 \in J, t_1 > t_2$. Thus we obtain

$$\begin{aligned}
 & \left| \psi_0^{1-\mu}(t_1) (\mathcal{P}(t_1, \rho) - \psi_0^{1-\mu}(t_2) \mathcal{P}(t_2, \rho)) \right| \\
 & \leq \left| \psi_0^{1-\mu}(t_1) \int_0^{t_1} \mathcal{T}_{t_1}^v(s) \eta(s, \rho, \hbar(s, \rho)) ds + \psi_0^{1-\mu}(t_2) \int_0^{t_2} \mathcal{T}_{t_2}^v(s) \eta(s, \rho, \hbar(s, \rho)) ds \right|.
 \end{aligned}$$

The right hand side is likely to be zero as $t_1 \rightarrow t_2$. As an outcome of Steps 1–3, in concert with Arzelà-Ascoli theorem, it proves that $\mathcal{P}$ is continuous and completely continuous.

Step 4: A priori bounds.

Finally to prove η is bounded, where

$$\eta = \{\hbar \in C_{1-\mu, \psi} : \hbar = \delta \mathcal{P} \hbar, 0 < \delta < 1\}.$$

We obtain

$$\hbar(t, \rho) = \delta \left[\left(c - b \int_0^T \mathcal{T}_T^{1-\beta+v\beta}(s) \eta(s, \rho, \hbar(s, \rho)) ds \right) \frac{\psi_0^{\mu-1}(t)}{(a+b)\Gamma(\mu)} + \int_0^t \mathcal{T}_t^v(s) \eta(s, \rho, \hbar(s, \rho)) ds \right].$$

Hence the theorem is concluded. $\square$

Theorem 3.2 Under the hypotheses [H2] and if

$$\left(\frac{bL(\rho)B(\mu, 1-\beta+v\beta)}{(a+b)\Gamma(\mu)\Gamma(1-\beta+v\beta)} \psi_0^v(T) + \frac{L(\rho)B(\mu, v)}{\Gamma(v)} \psi_0^v(T) \right) < 1$$

then Eq. (1) has a unique solution.

Theorem 3.3 Under the assumptions [H2] and [H3], the solution of Eq. (1) is g-UHR stable.

Proof Let v be solution of inequality 8, there exists $\hbar$, a unique solution by Theorem 3.2 of the problem

$$\begin{aligned}
 & \mathcal{D}^{v, \beta; \psi} \hbar(t, \rho) = \eta(t, \rho, \hbar(t, \rho)), \quad t \in J, \\
 & a \mathcal{J}^{1-\mu; \psi} \hbar(t, \rho)|_{t=0} + b \mathcal{J}^{1-\mu; \psi} \hbar(t, \rho)|_{t=T} = c
 \end{aligned}$$

given by

$$\hbar(t, \rho) = A_{\hbar} + \int_0^t \mathcal{T}_t^v(s) \eta(s, \rho, \hbar(s, \rho)) ds,$$

where

$$A_{\hbar} = \left(c - b \frac{1}{\Gamma(v)} \int_0^T \mathcal{T}_T^{1-\beta+v\beta}(s) \eta(s, \rho, \hbar(s, \rho)) ds \right) \frac{\psi_0^{\mu-1}(t)}{(a+b)\Gamma(\mu)}.$$

Thus $A_{\hbar} = A_v$.

By differentiating inequality (8), we have

$$\begin{aligned} & \left| v(t, \rho) - A_v - \frac{1}{\Gamma(v)} \int_0^t \mathcal{T}_t^v(s) \eta(s, \rho, v(s, \rho)) ds \right| \\ & \leq \frac{1}{\Gamma(v)} \int_0^t \mathcal{T}_t^v(s) \varphi(s, \rho) ds \\ & \leq \lambda_{\varphi} \varphi(t, \rho). \end{aligned}$$

Hence it follows

$$\begin{aligned} |v(t, \rho) - \hbar(t, \rho)| & \leq \left| v(t, \rho) - A_{\hbar} - \frac{1}{\Gamma(v)} \int_0^t \mathcal{T}_t^v(s) \eta(s, \rho, \hbar(s, \rho)) ds \right| \\ & \leq \left| v(t, \rho) - A_{\hbar} - \frac{1}{\Gamma(v)} \int_0^t \mathcal{T}_t^v(s) \eta(s, \rho, v(s)) ds \right| \\ & \quad + \frac{1}{\Gamma(v)} \int_0^t \mathcal{T}_t^v(s) |\eta(s, \rho, v(s)) - \eta(s, \rho, \hbar(s, \rho))| ds \\ & \leq \lambda_{\varphi} \varphi(t, \rho) + \frac{L(\rho)}{\Gamma(v)} \int_0^t \mathcal{T}_t^v(s) |v(s, \rho) - \hbar(s, \rho)| ds. \end{aligned}$$

By Lemma 2.6, it shows that for a constant $\mathcal{M} > 0$ independent of $\lambda_{\varphi} \varphi(t, \rho)$, we get

$$|v(t, \rho) - \hbar(t, \rho)| \leq \mathcal{M} \lambda_{\varphi} \varphi(t, \rho) := C_{f, \varphi} \varphi(t, \rho).$$

Thus, Eq. (1) is g-UHR stable. $\square$

4 Some test problems

Here we provide some test problems to demonstrate the established results.

Example 4.1 For $\psi(t) = t$, we obtain the particular case of Eq. (1) is as follows:

$$\begin{aligned} \mathcal{D}^{v, \beta; t} \hbar(t, \rho) &= \eta(t, \rho, \hbar(t, \rho)), \quad t \in J := [0, \frac{4}{5}], \\ \mathcal{I}^{1-\mu; t} \mathfrak{h}(0, \rho) &= \rho \end{aligned}$$

we choose $v = \frac{1}{2}$, $\beta = \frac{1}{2}$, and $\mu = \frac{3}{4}$. Here

$$\eta(t, \rho, \hbar(t, \rho)) = \frac{1}{9e^t} \frac{\hbar^2(t, \rho)}{1 + \hbar^2(t, \rho)}.$$

For $\hbar$, $v \in R$, we have

$$|\eta(t, \rho, \hbar(t, \rho)) - \eta(t, \rho, v(t, \rho))| \leq \frac{1}{9} |\hbar - v|.$$

Thus assumptions of Theorem 3.2 are satisfied, provides unique solution. Next, set $\varphi(t, \rho) = e^{t+\rho}$

$$\mathcal{I}^{\frac{1}{2}; t} \varphi(t, \rho) \leq \frac{1}{\Gamma(\frac{3}{2})} \varphi(t, \rho) = \lambda_{\varphi} \varphi(t, \rho).$$

It is easy to prove that the solutions fulfill the conditions of various stabilities like UH, g-UH, UHR, and g-UHR stability.

Example 4.2 Consider the Problem

$$\begin{aligned}\mathcal{D}^{\nu,\beta;t}\hbar(t,\rho) &= \eta(t,\rho,\hbar(t,\rho)), \quad t \in J := \left[0, \frac{2}{5}\right], \\ \mathcal{I}^{1-\mu;t}\mathfrak{h}(0,\rho) &= \rho\end{aligned}$$

Denote $\nu = \frac{2}{3}$, $\beta = \frac{3}{4}$ and choose $\mu = \frac{11}{12}$. Set $\eta(t,\rho,\hbar(t,\rho)) = \frac{\rho^2\hbar(t,\rho)}{e^{10(1+\rho^2)}}$. Moreover the hypothesis [H1] is satisfied for $L(\rho) = \frac{1}{e^{10}}$. Finally, hypothesis [H3] is also satisfied with

$$\varphi(t,\rho) = \rho t$$

and

$$\lambda_\varphi = \frac{1}{\Gamma(\nu+2)}.$$

We can easily prove that the solutions fulfill the conditions of various stabilities like UH, g-UH, UHR and g-UHR stability.

5 Conclusion

In this research work we have considered a class of RFODEs by taking ψ -HFD under boundary conditions. We have investigated existence theory as well as various kinds of Ulam stability results for the solutions of the considered problem. We claim that such type of BVPs of the aforesaid differential equations have been very rarely studied earlier. The obtained results have been investigated via an example.

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