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New Riemann–Liouville fractional Hermite–Hadamard type inequalities for harmonically convex functions

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Abstract In this paper, we proved two new Riemann–Liouville fractional Hermite–Hadamard type inequalities for harmonically convex functions using the left and right fractional integrals independently. Also, we have two new Riemann–Liouville fractional trapezoidal type identities for differentiable functions. Using these identities, we obtained some new trapezoidal type inequalities for harmonically convex functions. Our results generalize the results given by İşcan (Hacet J Math Stat 46(6):935–942, 2014).

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1 Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is well known in the literature as Hermite–Hadamard's inequality. There are so many generalizations and extensions of inequalities (1.1) for various classes of functions. One of these classes of functions is harmonically convex functions defined by İşcan.

In [4], İşcan gave the definition of harmonically convex functions as follows.

Definition 1.1 [4] Let $I \subset \mathbb{R}\backslash\{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if

$$f\left(\frac{xy}{tx+(1-t)\,y}\right) \leq tf\,(y) + (1-t)\,f\,(x) \tag{1.2}$$

for all $x, y \in I$ and $t \in [0, 1]$. If the inequality in (1.2) is reversed, then f is said to be harmonically concave.

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Remark 1.2 Let $[a,b] \subset I \subseteq (0,\infty)$, if the function $g : [\frac{1}{b}, \frac{1}{a}] \to \mathbb{R}$ defined $g(x) = f(\frac{1}{x})$, then f is harmonically convex on $[a,b]$ if and only if g is convex on $[\frac{1}{b}, \frac{1}{a}]$ (see [3]).

In [4], İşcan gave Hermite–Hadamard type inequalities for harmonically convex functions as follows.

Theorem 1.3 [4] *Let $f : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a,b]$, then the following inequalities hold:*

$$f\left(\frac{2ab}{a+b}\right) \le \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \le \frac{f(a)+f(b)}{2}. \tag{1.3}$$

For some similar studies with this work about harmonically convex functions, readers can see [1–6,8–10,13,14] and references therein.

The following definitions of the left- and right-side Riemann–Liouville fractional integrals are well known in the literature.

Definition 1.4 Let $a, b \in \mathbb{R}$ with $a < b$ and $f \in L[a,b]$. The left and right Riemann–Liouville fractional integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\mathrm{d}t, \ x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\mathrm{d}t, \ x < b$$

respectively, here $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}\mathrm{d}t$ (see [7, page 69]).

Because of the wide application of Hermite–Hadamard type inequalities and fractional integrals, researchers extend their studies to Hermite–Hadamard type inequalities involving fractional integrals. The papers [1,5,6,8–10,14] are based on Hermite–Hadamard type inequalities involving fractional integrals for harmonically convex functions.

In [6], İşcan and Wu presented Hermite–Hadamard type inequalities for harmonically convex functions in fractional integral form as follows.

Theorem 1.5 *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a function such that $f \in L[a,b]$, where $a, b \in I$ with $a < b$. If f is a harmonically convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{2ab}{a+b}\right) \le \frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left[J^{\alpha}_{1/a-}(f\circ h)(1/b) + J^{\alpha}_{1/b+}(f\circ h)(1/a)\right] \le \frac{f(a)+f(b)}{2} \tag{1.4}$$

with $\alpha > 0$ and $h(x) = 1/x$.

We recall the following inequality and special function which are known as hypergeometric function:

$$ {}_2F_1(a,b;c;z) = \frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,\mathrm{d}t,$$
$$c > b > 0, |z| < 1 \quad \text{(see [7])},$$

where $\beta(x,y)$ is the beta function defined by $\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t$, for $x, y > 0$.

The following properties of convex functions are used in the forward results.

Definition 1.6 [15, page 12] A function f defined on I has a support at $x_0 \in I$ if there exists an affine functions $A(x) = f(x_0) + m(x - x_0)$ such that $A(x) \le f(x)$ for all $x \in I$. The graph of the support function A is called a line of support for f at x_0.

Theorem 1.7 [15, page 12] *$f : (a,b) \to \mathbb{R}$ is a convex function if and only if there is at least one line of support for f at each $x_0 \in (a,b)$.*

As much as we know, there are so many studies in the literature for Hermite–Hadamard type inequalities using the left and right fractional integrals together (such as Riemann–Liouville fractional integrals, Hadamard fractional integrals and conformable fractional integrals). In all of them, the left and right fractional integrals are used together. As much as we know, the studies [11,12] are the first two works using only the right fractional integrals or the left fractional integrals.

In this paper, our aim is to obtain new Riemann–Liouville fractional Hermite–Hadamard type inequalities using only the right or the left fractional integrals separately for harmonically convex functions. Also, we improve the fractional Hermite–Hadamard type inequalities for harmonically convex functions (1.4).

2 Fractional Hermite–Hadamard type inequalities for harmonically convex functions

Theorem 2.1 *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a,b]$, then the following inequality for the right Riemann–Liouville fractional integral holds:*

$$f\left(\frac{(\alpha+1)\,ab}{a+\alpha b}\right) \le \Gamma(\alpha+1)\left(\frac{ab}{b-a}\right)^{\alpha} J^{\alpha}_{\frac{1}{b}-}(f\circ h)\left(\frac{1}{a}\right) \le \frac{\alpha f(a)+f(b)}{\alpha+1}, \tag{2.1}$$

where $h(x)=\frac{1}{x}$ and $\alpha>0$.

Proof Let $\alpha > 0$. Since f is harmonically convex on $[a,b]$ using Remark 1.2, $g(x) = f(\frac{1}{x})$ is convex on $[\frac{1}{b},\frac{1}{a}]$. Hence, using Theorem 1.7, there is at least one line of support

$$A(x) = g\left(\frac{a+\alpha b}{(\alpha+1)\,ab}\right) + m\left(x - \frac{a+\alpha b}{(\alpha+1)\,ab}\right) \le g(x) \tag{2.2}$$

for all $x \in [\frac{1}{b},\frac{1}{a}]$ and $m \in [g'_-(\frac{a+\alpha b}{(\alpha+1)ab}), g'_+(\frac{a+\alpha b}{(\alpha+1)ab})]$. From (2.2) and harmonically convexity of f, we have

$$f\left(\frac{(\alpha+1)\,ab}{a+\alpha b}\right) + m\left(\frac{tb+(1-t)\,a}{ab} - \frac{a+\alpha b}{(\alpha+1)\,ab}\right) \le f\left(\frac{ab}{tb+(1-t)\,a}\right) \le tf(a) + (1-t)\,f(b) \tag{2.3}$$

for all $t \in [0,1]$. Multiplying all sides of (2.3) with $\alpha t^{\alpha-1}$ and integrating over $[0,1]$ respect to t, we have

$$\begin{aligned}
&\int_0^1 \alpha t^{\alpha-1}\left[f\left(\frac{(\alpha+1)\,ab}{a+\alpha b}\right) + m\left(\frac{tb+(1-t)\,a}{ab} - \frac{a+\alpha b}{(\alpha+1)\,ab}\right)\right]\mathrm{d}t \\
&= f\left(\frac{(\alpha+1)\,ab}{a+\alpha b}\right)\alpha\int_0^1 t^{\alpha-1}\mathrm{d}t + m\left[\alpha\int_0^1\left[t^{\alpha}\frac{1}{a} + \left(t^{\alpha-1}-t^{\alpha}\right)\frac{1}{b}\right]\mathrm{d}t - \frac{a+\alpha b}{(\alpha+1)\,ab}\alpha\int_0^1 t^{\alpha-1}\mathrm{d}t\right] \\
&= f\left(\frac{(\alpha+1)\,ab}{a+\alpha b}\right) + m\left[\frac{a+\alpha b}{(\alpha+1)\,ab} - \frac{a+\alpha b}{(\alpha+1)\,ab}\right] \\
&= f\left(\frac{(\alpha+1)\,ab}{a+\alpha b}\right).
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
&\alpha\int_0^1 t^{\alpha-1} f\left(\frac{ab}{tb+(1-t)\,a}\right)\mathrm{d}t = \alpha\int_{\frac{1}{b}}^{\frac{1}{a}} \frac{\left(t-\frac{1}{b}\right)^{\alpha-1}}{\left(\frac{1}{a}-\frac{1}{b}\right)^{\alpha-1}} f\left(\frac{1}{t}\right)\frac{\mathrm{d}t}{\left(\frac{1}{a}-\frac{1}{b}\right)} \\
&= \Gamma(\alpha+1)\left(\frac{ab}{b-a}\right)^{\alpha} J^{\alpha}_{\frac{1}{b}-}(f\circ h)\left(\frac{1}{a}\right).
\end{aligned} \tag{2.5}$$

$$f(a)\,\alpha\int_0^1 t^{\alpha}\mathrm{d}t + f(b)\,\alpha\int_0^1\left(t^{\alpha-1}-t^{\alpha}\right)\mathrm{d}t = \frac{\alpha f(a)+f(b)}{\alpha+1}. \tag{2.6}$$

With a combination of (2.3), (2.4), (2.5) and (2.6), we have (2.2). This completes the proof. □

Remark 2.2 In Theorem 2.1, if one takes $\alpha = 1$, one has the inequality (1.3).

Theorem 2.3 *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a harmonically convex function and $a,b \in I$ with $a < b$. If $f \in L[a,b]$, then the following inequality for the left Riemann–Liouville fractional integral holds:*

$$f\left(\frac{(\alpha+1)\,ab}{\alpha a+b}\right) \le \Gamma(\alpha+1)\left(\frac{ab}{b-a}\right)^{\alpha} J^{\alpha}_{\frac{1}{a}+}(f\circ h)\left(\frac{1}{b}\right) \le \frac{f(a)+\alpha f(b)}{\alpha+1}, \tag{2.7}$$

where $h(x)=\frac{1}{x}$ and $\alpha>0$.

Proof Similar to the proof of Theorem 2.1, there is at least one line of support

$$A(x) = g\left(\frac{\alpha a+b}{(\alpha+1)\,ab}\right) + m\left(x - \frac{\alpha a+b}{(\alpha+1)\,ab}\right) \le g(x) \tag{2.8}$$

for all $x \in [\frac{1}{b}, \frac{1}{a}]$ and $m \in [g'_{-}(\frac{\alpha a+b}{(\alpha+1)ab}), g'_{+}(\frac{\alpha a+b}{(\alpha+1)ab})]$. From (2.8) and harmonically convexity of f, we have

$$f\left(\frac{(\alpha+1)\,ab}{\alpha a+b}\right) + m\left(\frac{ta+(1-t)\,b}{ab} - \frac{\alpha a+b}{(\alpha+1)\,ab}\right) \le f\left(\frac{ab}{ta+(1-t)\,b}\right) \le tf(b) + (1-t)\,f(a) \tag{2.9}$$

for all $t \in [0,1]$. Multiplying all sides of (2.9) with $\alpha t^{\alpha-1}$ and integrating over $[0,1]$ respect to t, similarly we have (2.7) and we omit the details. □

Remark 2.4 In Theorem 2.3, if one takes $\alpha = 1$, one has the inequality (1.3).

Theorem 2.5 *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a harmonically convex function and $a,b \in I$ with $a < b$. If $f \in L[a,b]$, then the following inequality for the Riemann–Liouville fractional integrals holds:*

$$\begin{aligned}\frac{f\left(\frac{(\alpha+1)ab}{a+\alpha b}\right) + f\left(\frac{(\alpha+1)ab}{\alpha a+b}\right)}{2} &\le \frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left[J^{\alpha}_{\frac{1}{b}-}(f\circ h)\left(\frac{1}{a}\right) + J^{\alpha}_{\frac{1}{a}+}(f\circ h)\left(\frac{1}{b}\right)\right] \\ &\le \frac{f(a)+f(b)}{2},\end{aligned} \tag{2.10}$$

where $h(x)=\frac{1}{x}$ and $\alpha>0$.

Proof Adding the inequalities (2.1) and (2.7) side by side, then multiplying the resulting inequalities by $\frac{1}{2}$, we have the inequalities (2.10). □

Remark 2.6 In Theorem 2.5, if one takes $\alpha = 1$, one has the inequality (1.3).

Corollary 2.7 *The left-hand side of (2.10) is better than the left-hand side of (1.4).*

Proof Since f is harmonically convex on $[a,b]$, it is clear from

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &= f\left(\frac{1}{\frac{a+b}{2ab}}\right) = f\left(\frac{1}{\frac{(\alpha+1)(a+b)}{2ab(\alpha+1)}}\right) = f\left(\frac{1}{\frac{a+\alpha b}{2ab(\alpha+1)} + \frac{\alpha a+b}{2ab(\alpha+1)}}\right) \\ &= f\left(\frac{\frac{ab(\alpha+1)}{a+\alpha b}\,\frac{ab(\alpha+1)}{\alpha a+b}}{\frac{ab(\alpha+1)}{\alpha a+b}\,\frac{1}{2} + \frac{ab(\alpha+1)}{a+\alpha b}\,\frac{1}{2}}\right) \le \frac{f\left(\frac{(\alpha+1)ab}{a+\alpha b}\right) + f\left(\frac{(\alpha+1)ab}{\alpha a+b}\right)}{2}. \end{aligned}$$

□

3 Lemmas

In this section, we will prove two new identities used in the forward results.

Lemma 3.1 *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I° (the interior of the interval I) such that $f' \in L[a,b]$, where $a, b \in I^\circ$ with $a < b$. Then the following equality for the right Riemann–Liouville fractional integral holds:*

$$\frac{\alpha f(a)+f(b)}{\alpha+1}-\Gamma(\alpha+1)\left(\frac{ab}{b-a}\right)^{\alpha} J^{\alpha}_{\frac{1}{b}-}(f\circ h)\left(\frac{1}{a}\right) = \frac{ab(b-a)}{\alpha+1}\int_0^1 \frac{1-(\alpha+1)t^{\alpha}}{(tb+(1-t)a)^2} f'\left(\frac{ab}{tb+(1-t)a}\right)\mathrm{d}t, \tag{3.1}$$

where $h(x)=\frac{1}{x}$ and $\alpha > 0$.

Proof It can be proved directly by applying the partial integration to the right-hand side of Eq. (3.1) as follows:

$$\begin{aligned}
&\frac{ab(b-a)}{\alpha+1}\int_0^1 \frac{1-(\alpha+1)t^{\alpha}}{(tb+(1-t)a)^2} f'\left(\frac{ab}{tb+(1-t)a}\right)\mathrm{d}t\\
&=\frac{1}{\alpha+1}\int_0^1 \frac{ab(b-a)}{(tb+(1-t)a)^2} f'\left(\frac{ab}{tb+(1-t)a}\right)\mathrm{d}t+\int_0^1 t^{\alpha}\frac{-ab(b-a)}{(tb+(1-t)a)^2} f'\left(\frac{ab}{tb+(1-t)a}\right)\mathrm{d}t\\
&=-\frac{1}{\alpha+1} f\left(\frac{ab}{tb+(1-t)a}\right)\Bigg|_0^1+t^{\alpha} f\left(\frac{ab}{tb+(1-t)a}\right)\Bigg|_0^1-\int_0^1 \alpha t^{\alpha-1} f\left(\frac{ab}{tb+(1-t)a}\right)\mathrm{d}t\\
&=\frac{f(b)-f(a)}{\alpha+1}+f(a)-\alpha\int_0^1 t^{\alpha-1} f\left(\frac{ab}{tb+(1-t)a}\right)\mathrm{d}t\\
&=\frac{\alpha f(a)+f(b)}{\alpha+1}-\alpha\int_{\frac{1}{b}}^{\frac{1}{a}} \frac{\left(t-\frac{1}{b}\right)^{\alpha-1}}{\left(\frac{1}{a}-\frac{1}{b}\right)^{\alpha-1}} f\left(\frac{1}{t}\right)\frac{\mathrm{d}t}{\left(\frac{1}{a}-\frac{1}{b}\right)}\\
&=\frac{\alpha f(a)+f(b)}{\alpha+1}-\Gamma(\alpha+1)\left(\frac{ab}{b-a}\right)^{\alpha} J^{\alpha}_{\frac{1}{b}-}(f\circ h)\left(\frac{1}{a}\right).
\end{aligned}$$

This completes the proof. □

Remark 3.2 In Lemma 3.1, if one takes $\alpha = 1$, one has [4, Lemma 2.5].

Lemma 3.3 *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I° (the interior of the interval I) such that $f' \in L[a,b]$, where $a, b \in I^\circ$ with $a < b$. Then the following equality for the left Riemann–Liouville fractional integral holds:*

$$\frac{f(a)+\alpha f(b)}{\alpha+1}-\Gamma(\alpha+1)\left(\frac{ab}{b-a}\right)^{\alpha} J^{\alpha}_{\frac{1}{a}+}(f\circ h)\left(\frac{1}{b}\right) = \frac{ab(b-a)}{\alpha+1}\int_0^1 \frac{(\alpha+1)(1-t)^{\alpha}-1}{(tb+(1-t)a)^2} f'\left(\frac{ab}{tb+(1-t)a}\right)\mathrm{d}t, \tag{3.2}$$

where $h(x)=\frac{1}{x}$ and $\alpha > 0$.

Proof Similar to the proof of Lemma 3.1, it can be proved directly by applying the partial integration to the right-hand side of Eq. (3.3) and we omit the details. □

Remark 3.4 In Lemma 3.3, if one takes $\alpha = 1$, one has [4, Lemma 2.5].

4 Some new trapezoid type inequalities for harmonically convex functions

Theorem 4.1 *Let $f : I \subseteq (0, \infty) \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ and $a < b$. If $f' \in L[a, b]$ and $|f^{'}|^q$ harmonically convex on $[a, b]$ for $q \geq 1$, then we have the following inequalities:*

$$\left|\frac{\alpha f(a)+f(b)}{\alpha+1}-\Gamma(\alpha+1)\left(\frac{ab}{b-a}\right)^{\alpha} J^{\alpha}_{\frac{1}{b}-}(f\circ h)\left(\frac{1}{a}\right)\right| \leq \frac{ab(b-a)}{\alpha+1} Z_1^{1-\frac{1}{q}}(a,b,\alpha)\left(\left|f'(a)\right|^q Z_2(a,b,\alpha)+\left|f'(b)\right|^q Z_3(a,b,\alpha)\right)^{\frac{1}{q}}, \tag{4.1}$$

where

$$Z_1(a,b,\alpha)=\left[\begin{array}{c} 2\sqrt[\alpha]{\frac{1}{\alpha+1}}\left[\sqrt[\alpha]{\frac{1}{\alpha+1}}(b-a)+a\right]^{-2} {}_2F_1\left(2,1;2;1-\frac{a}{\sqrt[\alpha]{\frac{1}{\alpha+1}}(b-a)+a}\right) \\ -b^{-2}\,{}_2F_1\left(2,1;2;1-\frac{a}{b}\right) \\ -\frac{2}{\alpha+1}\sqrt[\alpha]{\frac{1}{\alpha+1}}\left[\sqrt[\alpha]{\frac{1}{\alpha+1}}(b-a)+a\right]^{-2} {}_2F_1\left(2,1;\alpha+2;1-\frac{a}{\sqrt[\alpha]{\frac{1}{\alpha+1}}(b-a)+a}\right) \\ -b^{-2}\,{}_2F_1\left(2,1;\alpha+2;1-\frac{a}{b}\right) \end{array}\right],$$

$$Z_2(a,b,\alpha)=\left[\begin{array}{c} \left(\frac{1}{\alpha+1}\right)^{\frac{2}{\alpha}}\left[\sqrt[\alpha]{\frac{1}{\alpha+1}}(b-a)+a\right]^{-2} {}_2F_1\left(2,1;3;1-\frac{a}{\sqrt[\alpha]{\frac{1}{\alpha+1}}(b-a)+a}\right) \\ -\frac{1}{2}b^{-2}\,{}_2F_1\left(2,1;3;1-\frac{a}{b}\right) \\ -\left(\frac{1}{\alpha+1}\right)^{\frac{2}{\alpha}}\frac{2}{\alpha+2}\left[\sqrt[\alpha]{\frac{1}{\alpha+1}}(b-a)+a\right]^{-2} {}_2F_1\left(2,1;\alpha+3;1-\frac{a}{\sqrt[\alpha]{\frac{1}{\alpha+1}}(b-a)+a}\right) \\ -\frac{\alpha+1}{\alpha+2}b^{-2}\,{}_2F_1\left(2,1;\alpha+3;1-\frac{a}{b}\right) \end{array}\right],$$

$$Z_3(a,b,\alpha)=Z_1(a,b,\alpha)-Z_2(a,b,\alpha),$$

and $\alpha > 0$.

Proof Using Lemma 3.1, power mean inequality and harmonically convexity of $|f^{'}|^q$, we have

$$\begin{aligned} &\left|\frac{\alpha f(a)+f(b)}{\alpha+1}-\Gamma(\alpha+1)\left(\frac{ab}{b-a}\right)^{\alpha} J^{\alpha}_{\frac{1}{b}-}(f\circ h)\left(\frac{1}{a}\right)\right| \\ &\leq \frac{ab(b-a)}{\alpha+1}\int_0^1 \frac{\left|1-(\alpha+1)t^{\alpha}\right|}{(tb+(1-t)a)^2}\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right| dt \\ &\leq \frac{ab(b-a)}{\alpha+1}\left(\int_0^1 \frac{\left|1-(\alpha+1)t^{\alpha}\right|}{(tb+(1-t)a)^2}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 \frac{\left|1-(\alpha+1)t^{\alpha}\right|}{(tb+(1-t)a)^2}\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|^q dt\right)^{\frac{1}{q}} \\ &\leq \frac{ab(b-a)}{\alpha+1}\left(\int_0^1 \frac{\left|1-(\alpha+1)t^{\alpha}\right|}{(tb+(1-t)a)^2}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 \frac{\left|1-(\alpha+1)t^{\alpha}\right|}{(tb+(1-t)a)^2}\left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}} \\ &\leq \frac{ab(b-a)}{\alpha+1}\left(\int_0^1 \frac{\left|1-(\alpha+1)t^{\alpha}\right|}{(tb+(1-t)a)^2}dt\right)^{1-\frac{1}{q}}\left(\begin{array}{c}\left|f'(a)\right|^q\int_0^1\frac{|1-(\alpha+1)t^{\alpha}|}{(tb+(1-t)a)^2}t\,dt \\ +\left|f'(b)\right|^q\int_0^1\frac{|1-(\alpha+1)t^{\alpha}|}{(tb+(1-t)a)^2}(1-t)\,dt\end{array}\right)^{\frac{1}{q}}. \end{aligned} \tag{4.2}$$

Calculating the appearing integrals in (4.2), we have

$$\begin{aligned} \int_0^1 \frac{\left|1-(\alpha+1)t^{\alpha}\right|}{(tb+(1-t)a)^2}dt &= \int_0^{\sqrt[\alpha]{\frac{1}{\alpha+1}}} \frac{1-(\alpha+1)t^{\alpha}}{(tb+(1-t)a)^2}dt+\int_{\sqrt[\alpha]{\frac{1}{\alpha+1}}}^1 \frac{(\alpha+1)t^{\alpha}-1}{(tb+(1-t)a)^2}dt \\ &= \int_0^{\sqrt[\alpha]{\frac{1}{\alpha+1}}} \frac{1}{(tb+(1-t)a)^2}dt-\int_{\sqrt[\alpha]{\frac{1}{\alpha+1}}}^1 \frac{1}{(tb+(1-t)a)^2}dt \end{aligned}$$

$$
\begin{aligned}
&-(\alpha+1)\left[\int_0^{\sqrt[\alpha]{\frac{1}{\alpha+1}}}\frac{t^{\alpha}}{(tb+(1-t)a)^2}\mathrm{d}t-\int_{\sqrt[\alpha]{\frac{1}{\alpha+1}}}^{1}\frac{t^{\alpha}}{(tb+(1-t)a)^2}\mathrm{d}t\right]\\
&=2\int_0^{\sqrt[\alpha]{\frac{1}{\alpha+1}}}\frac{1}{(tb+(1-t)a)^2}\mathrm{d}t-\int_0^1\frac{1}{(tb+(1-t)a)^2}\mathrm{d}t\\
&\quad-(\alpha+1)\left[2\int_0^{\sqrt[\alpha]{\frac{1}{\alpha+1}}}\frac{t^{\alpha}}{(tb+(1-t)a)^2}\mathrm{d}t-\int_0^1\frac{t^{\alpha}}{(tb+(1-t)a)^2}\mathrm{d}t\right]\\
&=2\sqrt[\alpha]{\frac{1}{\alpha+1}}\int_0^1\frac{1}{\left(\sqrt[\alpha]{\frac{1}{\alpha+1}}ub+\left(1-\sqrt[\alpha]{\frac{1}{\alpha+1}}u\right)a\right)^2}\mathrm{d}u-\int_0^1\frac{1}{(ua+(1-u)b)^2}\mathrm{d}u\\
&\quad-\left[2\sqrt[\alpha]{\frac{1}{\alpha+1}}\int_0^1\frac{u^{\alpha}}{\left(\sqrt[\alpha]{\frac{1}{\alpha+1}}ub+\left(1-\sqrt[\alpha]{\frac{1}{\alpha+1}}u\right)a\right)^2}\mathrm{d}u-(\alpha+1)\int_0^1\frac{(1-u)^{\alpha}}{(ua+(1-u)b)^2}\mathrm{d}u\right]\\
&=2\sqrt[\alpha]{\frac{1}{\alpha+1}}\int_0^1\frac{1}{\left(\sqrt[\alpha]{\frac{1}{\alpha+1}}(1-v)b+\left(1-\sqrt[\alpha]{\frac{1}{\alpha+1}}(1-v)\right)a\right)^2}\mathrm{d}v-\int_0^1\frac{1}{(ua+(1-u)b)^2}\mathrm{d}u\\
&\quad-\left[2\sqrt[\alpha]{\frac{1}{\alpha+1}}\int_0^1\frac{(1-v)^{\alpha}}{\left(\sqrt[\alpha]{\frac{1}{\alpha+1}}(1-v)b+\left(1-\sqrt[\alpha]{\frac{1}{\alpha+1}}(1-v)\right)a\right)^2}\mathrm{d}v-(\alpha+1)\int_0^1\frac{(1-u)^{\alpha}}{(ua+(1-u)b)^2}\mathrm{d}u\right]\\
&=2\sqrt[\alpha]{\frac{1}{\alpha+1}}\left[\sqrt[\alpha]{\frac{1}{\alpha+1}}(b-a)+a\right]^{-2}\int_0^1\left(1-v\left[1-\frac{a}{\sqrt[\alpha]{\frac{1}{\alpha+1}}(b-a)+a}\right]\right)^{-2}\mathrm{d}v\\
&\quad-b^{-2}\int_0^1\left(1-u\left(1-\frac{a}{b}\right)\right)^{-2}\mathrm{d}u\\
&\quad-\left[\begin{array}{c}2\sqrt[\alpha]{\frac{1}{\alpha+1}}\left[\sqrt[\alpha]{\frac{1}{\alpha+1}}(b-a)+a\right]^{-2}\int_0^1(1-v)^{\alpha}\left(1-v\left[1-\frac{a}{\sqrt[\alpha]{\frac{1}{\alpha+1}}(b-a)+a}\right]\right)^{-2}\mathrm{d}v\\-(\alpha+1)b^{-2}\int_0^1(1-u)^{\alpha}\left(1-u\left(1-\frac{a}{b}\right)\right)^{-2}\mathrm{d}u\end{array}\right]\\
&=\left[\begin{array}{c}2\sqrt[\alpha]{\frac{1}{\alpha+1}}\left[\sqrt[\alpha]{\frac{1}{\alpha+1}}(b-a)+a\right]^{-2}{}_2F_1\left(2,1;2;1-\frac{a}{\sqrt[\alpha]{\frac{1}{\alpha+1}}(b-a)+a}\right)\\-b^{-2}{}_2F_1\left(2,1;2;1-\frac{a}{b}\right)\\-\frac{2}{\alpha+1}\sqrt[\alpha]{\frac{1}{\alpha+1}}\left[\sqrt[\alpha]{\frac{1}{\alpha+1}}(b-a)+a\right]^{-2}{}_2F_1\left(2,1;\alpha+2;1-\frac{a}{\sqrt[\alpha]{\frac{1}{\alpha+1}}(b-a)+a}\right)\\-b^{-2}{}_2F_1\left(2,1;\alpha+2;1-\frac{a}{b}\right)\end{array}\right]\\
&=Z_1(a,b,\alpha),
\end{aligned}
\tag{4.3}
$$

and

$$
\begin{aligned}
&\int_0^1\frac{\left|1-(\alpha+1)t^{\alpha}\right|}{(tb+(1-t)a)^2}t\mathrm{d}t=\int_0^{\sqrt[\alpha]{\frac{1}{\alpha+1}}}\frac{1-(\alpha+1)t^{\alpha}}{(tb+(1-t)a)^2}t\mathrm{d}t+\int_{\sqrt[\alpha]{\frac{1}{\alpha+1}}}^1\frac{(\alpha+1)t^{\alpha}-1}{(tb+(1-t)a)^2}t\mathrm{d}t\\
&=\int_0^{\sqrt[\alpha]{\frac{1}{\alpha+1}}}\frac{t}{(tb+(1-t)a)^2}\mathrm{d}t-\int_{\sqrt[\alpha]{\frac{1}{\alpha+1}}}^1\frac{t}{(tb+(1-t)a)^2}\mathrm{d}t\\
&\quad-(\alpha+1)\left[\int_0^{\sqrt[\alpha]{\frac{1}{\alpha+1}}}\frac{t^{\alpha+1}}{(tb+(1-t)a)^2}\mathrm{d}t-\int_{\sqrt[\alpha]{\frac{1}{\alpha+1}}}^1\frac{t^{\alpha+1}}{(tb+(1-t)a)^2}\mathrm{d}t\right]\\
&=2\int_0^{\sqrt[\alpha]{\frac{1}{\alpha+1}}}\frac{t}{(tb+(1-t)a)^2}\mathrm{d}t-\int_0^1\frac{t}{(tb+(1-t)a)^2}\mathrm{d}t
\end{aligned}
$$

$$-(\alpha+1)\left[2\int_0^{\sqrt[\alpha]{\frac{1}{\alpha+1}}}\frac{t^{\alpha+1}}{(tb+(1-t)a)^2}\mathrm{d}t-\int_0^1\frac{t^{\alpha+1}}{(tb+(1-t)a)^2}\mathrm{d}t\right]$$

$$=2\left(\frac{1}{\alpha+1}\right)^{\frac{2}{\alpha}}\int_0^1\frac{u}{\left(\sqrt[\alpha]{\frac{1}{\alpha+1}}ub+\left(1-\sqrt[\alpha]{\frac{1}{\alpha+1}}u\right)a\right)^2}\mathrm{d}u-\int_0^1\frac{1-u}{(ua+(1-u)b)^2}\mathrm{d}u$$

$$-\left[2\left(\frac{1}{\alpha+1}\right)^{\frac{2}{\alpha}}\int_0^1\frac{u^{\alpha+1}}{\left(\sqrt[\alpha]{\frac{1}{\alpha+1}}ub+\left(1-\sqrt[\alpha]{\frac{1}{\alpha+1}}u\right)a\right)^2}\mathrm{d}u-(\alpha+1)\int_0^1\frac{(1-u)^{\alpha+1}}{(ua+(1-u)b)^2}\mathrm{d}u\right]$$

$$=2\left(\frac{1}{\alpha+1}\right)^{\frac{2}{\alpha}}\int_0^1\frac{1-v}{\left(\sqrt[\alpha]{\frac{1}{\alpha+1}}(1-v)b+\left(1-\sqrt[\alpha]{\frac{1}{\alpha+1}}(1-v)\right)a\right)^2}\mathrm{d}v-\int_0^1\frac{1-u}{(ua+(1-u)b)^2}\mathrm{d}u$$

$$-\left[2\left(\frac{1}{\alpha+1}\right)^{\frac{2}{\alpha}}\int_0^1\frac{(1-v)^{\alpha+1}}{\left(\sqrt[\alpha]{\frac{1}{\alpha+1}}(1-v)b+\left(1-\sqrt[\alpha]{\frac{1}{\alpha+1}}(1-v)\right)a\right)^2}\mathrm{d}v-(\alpha+1)\int_0^1\frac{(1-u)^{\alpha+1}}{(ua+(1-u)b)^2}\mathrm{d}u\right]$$

$$=2\left(\frac{1}{\alpha+1}\right)^{\frac{2}{\alpha}}\left[\sqrt[\alpha]{\frac{1}{\alpha+1}}(b-a)+a\right]^{-2}\int_0^1\left(1-v\left[1-\frac{a}{\sqrt[\alpha]{\frac{1}{\alpha+1}}(b-a)+a}\right]\right)^{-2}(1-v)\,\mathrm{d}v$$

$$-b^{-2q}\int_0^1\left(1-u\left(1-\frac{a}{b}\right)\right)^{-2}(1-u)\,\mathrm{d}u$$

$$-\left[\begin{array}{c}2\left(\frac{1}{\alpha+1}\right)^{\frac{2}{\alpha}}\left[\sqrt[\alpha]{\frac{1}{\alpha+1}}(b-a)+a\right]^{-2}\int_0^1(1-v)^{\alpha+1}\left(1-v\left[1-\frac{a}{\sqrt[\alpha]{\frac{1}{\alpha+1}}(b-a)+a}\right]\right)^{-2}\mathrm{d}v\\-(\alpha+1)b^{-2}\int_0^1(1-u)^{\alpha+1}\left(1-u\left(1-\frac{a}{b}\right)\right)^{-2}\mathrm{d}u\end{array}\right]$$

$$=\left[\begin{array}{c}\left(\frac{1}{\alpha+1}\right)^{\frac{2}{\alpha}}\left[\sqrt[\alpha]{\frac{1}{\alpha+1}}(b-a)+a\right]^{-2}{}_2F_1\left(2q,1;3;1-\frac{a}{\sqrt[\alpha]{\frac{1}{\alpha+1}}(b-a)+a}\right)\\-\frac{1}{2}b^{-2}{}_2F_1\left(2q,1;3;1-\frac{a}{b}\right)\\-\left(\frac{1}{\alpha+1}\right)^{\frac{2}{\alpha}}\frac{2}{\alpha+2}\left[\sqrt[\alpha]{\frac{1}{\alpha+1}}(b-a)+a\right]^{-2}{}_2F_1\left(2q,1;\alpha+3;1-\frac{a}{\sqrt[\alpha]{\frac{1}{\alpha+1}}(b-a)+a}\right)\\-\frac{\alpha+1}{\alpha+2}b^{-2}{}_2F_1\left(2q,1;\alpha+3;1-\frac{a}{b}\right)\end{array}\right]$$

$$=Z_2(a,b,\alpha), \tag{4.4}$$

and

$$\int_0^1\frac{|1-(\alpha+1)t^\alpha|}{(tb+(1-t)a)^{2q}}(1-t)\,\mathrm{d}t=\int_0^1\frac{|1-(\alpha+1)t^\alpha|}{(tb+(1-t)a)^{2q}}\mathrm{d}t-\int_0^1\frac{|1-(\alpha+1)t^\alpha|}{(tb+(1-t)a)^{2q}}t\,\mathrm{d}t$$
$$=Z_1(a,b,\alpha)-Z_2(a,b,\alpha)=Z_3(a,b,\alpha). \tag{4.5}$$

If we use (4.3)–(4.5) in (4.2), we have (4.1). This completes the proof. □

Remark 4.2 In Theorem 4.1, if one takes $\alpha=1$, one has [4, Theorem 2.6].

Theorem 4.3 *Let $f: I\subseteq(0,\infty)\to\mathbb{R}$ be a differentiable function on I°, $a,b\in I^\circ$ and $a<b$. If $f'\in L[a,b]$ and $|f'|^q$ harmonically convex on $[a,b]$ for $q>1$ and $\frac{1}{q}+\frac{1}{p}=1$, then we have the following inequalities:*

$$\left|\frac{\alpha f(a)+f(b)}{\alpha+1}-\Gamma(\alpha+1)\left(\frac{ab}{b-a}\right)^\alpha J^\alpha_{\frac{1}{b}-}(f\circ h)\left(\frac{1}{a}\right)\right|$$
$$\leq\frac{ab(b-a)}{\alpha+1}Z_4^{\frac{1}{p}}(a,b,\alpha)\left(\left|f'(a)\right|^q Z_5(a,b)+\left|f'(b)\right|^q Z_6(a,b)\right)^{\frac{1}{q}}, \tag{4.6}$$

where

$$Z_4(a,b,\alpha)=\frac{1}{\sqrt[\alpha]{\alpha+1}\,(\alpha p+1)}+\frac{\left(\sqrt[\alpha]{\alpha+1}-1\right)^{\alpha p+1}}{\sqrt[\alpha]{\alpha+1}\,(\alpha p+1)},$$
$$Z_5(a,b)=\frac{b^{-2q}}{2}\,{}_2F_1\left(2q,1;3;1-\frac{a}{b}\right),$$
$$Z_6(a,b)=\frac{b^{-2q}}{2}\,{}_2F_1\left(2q,2;3;1-\frac{a}{b}\right),$$

and $0<\alpha\leq 1$.

Proof Using Lemma 3.1, Hölder inequality and harmonically convexity of $\left|f'\right|^{q}$, we have

$$\begin{aligned}
&\left|\frac{\alpha f(a)+f(b)}{\alpha+1}-\Gamma(\alpha+1)\left(\frac{ab}{b-a}\right)^{\alpha}J^{\alpha}_{\frac{1}{b}-}(f\circ h)\left(\frac{1}{a}\right)\right|\\
&\leq\frac{ab(b-a)}{\alpha+1}\int_0^1\frac{|1-(\alpha+1)t^{\alpha}|}{(tb+(1-t)a)^2}\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|\mathrm{d}t\\
&\leq\frac{ab(b-a)}{\alpha+1}\left(\int_0^1\left|1-(\alpha+1)t^{\alpha}\right|^{p}\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1\frac{1}{(tb+(1-t)a)^{2q}}\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|^{q}\mathrm{d}t\right)^{\frac{1}{q}}\\
&\leq\frac{ab(b-a)}{\alpha+1}\left(\int_0^1\left|1-(\alpha+1)t^{\alpha}\right|^{p}\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1\frac{1}{(tb+(1-t)a)^{2q}}\left[t\left|f'(a)\right|^{q}+(1-t)\left|f'(b)\right|^{q}\right]\mathrm{d}t\right)^{\frac{1}{q}}\\
&\leq\frac{ab(b-a)}{\alpha+1}\left(\int_0^1\left|1-(\alpha+1)t^{\alpha}\right|^{p}\mathrm{d}t\right)^{\frac{1}{p}}\left(\begin{array}{l}\left|f'(a)\right|^{q}\int_0^1\frac{t}{(tb+(1-t)a)^{2q}}\mathrm{d}t\\+\left|f'(b)\right|^{q}\int_0^1\frac{1-t}{(tb+(1-t)a)^{2q}}\mathrm{d}t\end{array}\right)^{\frac{1}{q}}.
\end{aligned}\tag{4.7}$$

Calculating the appearing integrals in (4.7) , we have

$$\begin{aligned}
\int_0^1\frac{t}{(tb+(1-t)a)^{2q}}\mathrm{d}t&=\int_0^1\frac{1-t}{(ta+(1-t)b)^{2q}}\mathrm{d}t\\
&=b^{-2q}\int_0^1(1-t)\left(1-t\left(1-\frac{a}{b}\right)\right)^{-2q}\mathrm{d}t\\
&=\frac{b^{-2q}}{2}\,{}_2F_1\left(2q,1;3;1-\tfrac{a}{b}\right)\\
&=Z_5(a,b)
\end{aligned}\tag{4.8}$$

and

$$\begin{aligned}
\int_0^1\frac{1-t}{(tb+(1-t)a)^{2q}}\mathrm{d}t&=\int_0^1\frac{t}{(ta+(1-t)b)^{2q}}\mathrm{d}t\\
&=b^{-2q}\int_0^1 t\left(1-t\left(1-\frac{a}{b}\right)\right)^{-2q}\mathrm{d}t\\
&=\frac{b^{-2q}}{2}\,{}_2F_1\left(2q,2;3;1-\tfrac{a}{b}\right)\\
&=Z_6(a,b).
\end{aligned}\tag{4.9}$$

For the following integral, if we use the fact that $|x^{\alpha}-y^{\alpha}|\leq(x-y)^{\alpha}$ for $0<\alpha\leq 1$ and $0\leq x<y$, we have

$$\int_0^1\left|1-(\alpha+1)t^{\alpha}\right|^{p}\mathrm{d}t=\int_0^{\sqrt[\alpha]{\frac{1}{\alpha+1}}}\left(1-(\alpha+1)t^{\alpha}\right)^{p}\mathrm{d}t+\int_{\sqrt[\alpha]{\frac{1}{\alpha+1}}}^1\left((\alpha+1)t^{\alpha}-1\right)^{p}\mathrm{d}t$$

$$\leq \int_0^{\sqrt[\alpha]{\frac{1}{\alpha+1}}} \left(1-\sqrt[\alpha]{\alpha+1}t\right)^{\alpha p} \mathrm{d}t + \int_{\sqrt[\alpha]{\frac{1}{\alpha+1}}}^1 \left(\sqrt[\alpha]{\alpha+1}t-1\right)^{\alpha p} \mathrm{d}t$$

$$= \frac{\left(1-\sqrt[\alpha]{\alpha+1}t\right)^{\alpha p+1}}{-\sqrt[\alpha]{\alpha+1}\,(\alpha p+1)}\Bigg|_0^{\sqrt[\alpha]{\frac{1}{\alpha+1}}} + \frac{\left(\sqrt[\alpha]{\alpha+1}t-1\right)^{\alpha p+1}}{\sqrt[\alpha]{\alpha+1}\,(\alpha p+1)}\Bigg|_{\sqrt[\alpha]{\frac{1}{\alpha+1}}}^1$$

$$= \frac{1}{\sqrt[\alpha]{\alpha+1}\,(\alpha p+1)} + \frac{\left(\sqrt[\alpha]{\alpha+1}-1\right)^{\alpha p+1}}{\sqrt[\alpha]{\alpha+1}\,(\alpha p+1)}$$

$$= Z_4(a,b,\alpha). \tag{4.10}$$

If we use (4.8)–(4.10) in (4.7), we have (4.6). This completes the proof. □

Remark 4.4 In Theorem 4.3, if one takes $\alpha = 1$, one has [4, Theorem 2.7].

Theorem 4.5 *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ and $a < b$. If $f' \in L[a,b]$ and $|f'|^q$ harmonically convex on $[a,b]$ for $q \geq 1$, then we have the following inequalities:*

$$\left|\frac{f(a)+\alpha f(b)}{\alpha+1} - \Gamma(\alpha+1)\left(\frac{ab}{b-a}\right)^\alpha J^\alpha_{\frac{1}{a}+}(f\circ h)\left(\frac{1}{b}\right)\right|$$
$$\leq \frac{ab(b-a)}{\alpha+1} Z_7^{1-\frac{1}{q}}(a,b,\alpha)\left(\left|f'(a)\right|^q Z_8(a,b,\alpha) + \left|f'(b)\right|^q Z_9(a,b,\alpha)\right)^{\frac{1}{q}}, \tag{4.11}$$

where

$$Z_7(a,b,\alpha) = \left[\begin{array}{c} 2\sqrt[\alpha]{\frac{1}{\alpha+1}}\left[\sqrt[\alpha]{\frac{1}{\alpha+1}}(b-a)+a\right]^{-2} {}_2F_1\left(2,1;2;1-\frac{a}{\sqrt[\alpha]{\frac{1}{\alpha+1}}(b-a)+a}\right) \\ -b^{-2}\,{}_2F_1\left(2,1;2;1-\frac{a}{b}\right) \\ -\frac{2}{\alpha+1}\sqrt[\alpha]{\frac{1}{\alpha+1}}\left[\sqrt[\alpha]{\frac{1}{\alpha+1}}(b-a)+a\right]^{-2} {}_2F_1\left(2,\alpha+1;\alpha+2;1-\frac{a}{\sqrt[\alpha]{\frac{1}{\alpha+1}}(b-a)+a}\right) \\ -b^{-2}\,{}_2F_1\left(2,\alpha+1;\alpha+2;1-\frac{a}{b}\right) \end{array}\right],$$

$$Z_8(a,b,\alpha) = \left[\begin{array}{c} \left(\frac{1}{\alpha+1}\right)^{\frac{2}{\alpha}}\left[\sqrt[\alpha]{\frac{1}{\alpha+1}}(b-a)+a\right]^{-2} {}_2F_1\left(2,2;3;1-\frac{a}{\sqrt[\alpha]{\frac{1}{\alpha+1}}(b-a)+a}\right) \\ -\frac{1}{2}b^{-2}\,{}_2F_1\left(2,2;3;1-\frac{a}{b}\right) \\ -\left(\frac{1}{\alpha+1}\right)^{\frac{2}{\alpha}}\frac{2}{\alpha+2}\left[\sqrt[\alpha]{\frac{1}{\alpha+1}}(b-a)+a\right]^{-2} {}_2F_1\left(2,\alpha+2;\alpha+3;1-\frac{a}{\sqrt[\alpha]{\frac{1}{\alpha+1}}(b-a)+a}\right) \\ -\frac{\alpha+1}{\alpha+2}b^{-2}\,{}_2F_1\left(2,\alpha+2;\alpha+3;1-\frac{a}{b}\right) \end{array}\right],$$

$$Z_9(a,b,\alpha) = Z_7(a,b,\alpha) - Z_8(a,b,\alpha),$$

and $\alpha > 0$.

Proof Similar to the proof of Theorem 4.1, using Lemma 3.3, power mean inequality and harmonically convexity of $|f'|^q$, we have (4.11). □

Remark 4.6 In Theorem 4.5, if one takes $\alpha = 1$, one has [4, Theorem 2.6].

Theorem 4.7 *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ and $a < b$. If $f' \in L[a,b]$ and $|f'|^q$ harmonically convex on $[a,b]$ for $q > 1$ and $\frac{1}{q}+\frac{1}{p} = 1$, then we have the following inequalities:*

$$\left|\frac{f(a)+\alpha f(b)}{\alpha+1} - \Gamma(\alpha+1)\left(\frac{ab}{b-a}\right)^\alpha J^\alpha_{\frac{1}{a}+}(f\circ h)\left(\frac{1}{b}\right)\right|$$
$$\leq \frac{ab(b-a)}{\alpha+1} Z_4^{\frac{1}{p}}(a,b,\alpha)\left(\left|f'(a)\right|^q Z_5(a,b) + \left|f'(b)\right|^q Z_6(a,b)\right)^{\frac{1}{q}}, \tag{4.12}$$

where $Z_4(a,b,\alpha)$, $Z_5(a,b)$ and $Z_6(a,b)$ are same as in Theorem 4.3 and $0 < \alpha \leq 1$.

Proof Similar to the proof of Theorem 4.3, using Lemma 3.3, H ölder inequality and harmonically convexity of $|f^{'}|^{q}$, we have (4.12). □

Remark 4.8 In Theorem 4.3, if one takes $\alpha = 1$, one has [4, Theorem 2.7].

Compliance with ethical standards

Conflict of interest The authors declare that they have no competing interests.

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