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An extension of the Bessel–Wright transform in the class of Boehmians

Received: 6 August 2018 / Accepted: 5 March 2019 / Published online: 14 March 2019
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Abstract In this paper, we first construct a suitable Boehmian space on which the Bessel–Wright transform can be defined and some desired properties are obtained in the class of Boehmians. Some convergence results are also established.

Mathematics Subject Classification 42B35 · 44A35 · 42A38

1 Introduction

The space of Boehmians is constructed using an algebraic approach that utilizes convolution and approximate identities or delta sequences. If the construction is applied to a function space and the multiplication is interpreted as convolution, the construction yields a space of generalized functions. Those spaces provide a natural setting for extensions of the Bessel–Wright transform newly introduced by Fitouhi et al. [6]. We cite here, as briefly as possible, some facts about harmonic analysis related to the Bessel–Wright operator $\Delta_{\alpha,\beta}$. For more details, we refer to [6].

We consider, on $(0, \infty)$ the difference differential operator indexed by two parameters α and β

$$\Delta_{\alpha,\beta} f(x) = \frac{d^2 f}{dx^2}(x) + \frac{2(\alpha + \beta) + 1}{x} \frac{df}{dx}(x) + \frac{4\alpha\beta}{x^2} [f(x) - f(0)]. \quad (1.1)$$

These operators are very important in pure mathematics and especially in special functions and harmonic analysis. The Bessel–Wright functions $j(\alpha, \beta)(\lambda x)$ are eigenfunctions of the Bessel–Wright operator $\Delta_{\alpha,\beta}$ with the eigenvalues $-\lambda^2$

$$j_{(\alpha,\beta)}(\lambda x) = \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{\Gamma(\alpha + 1 + n)\Gamma(\beta + 1 + n)} \left(\frac{\lambda x}{2}\right)^{2n}, \quad (\lambda \in \mathbb{C} \text{ the space of complex numbers}),$$

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which is even and symmetric in α and β and coincides when $\alpha = 0$ or $\beta = 0$ with the normalized Bessel function given by

$$j_\alpha(\lambda x) = \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} \left(\frac{\lambda x}{2}\right)^{2n}, \quad (\lambda \in \mathbb{C}).$$

Let $L_\alpha^p = L_\alpha^p(0, \infty)$ denote the class of measurable functions f on $(0, \infty)$ for which $\|f\|_\alpha^p < \infty$, where

$$\|f\|_\alpha^p = \left(\int_0^\infty |f(x)|^p d\mu_\alpha(x) \right)^{\frac{1}{p}}, \quad \text{if } p < \infty,$$

$$\|f\|_{\infty, \alpha} = \|f\|_\infty = \text{ess sup}_{x \in (0, \infty)} |f(x)|,$$

and $d\mu_\alpha(x) = x^{2\alpha+1} dx$.

The Bessel–Wright transform for $f \in L_\alpha^p$ is defined by

$$\mathcal{F}_{(\alpha, \beta)}(f)(\lambda) = c_\alpha \int_0^\infty f(x) j_{\alpha, \beta}(\lambda x) d\mu_\alpha(x), \quad (1.2)$$

where $c_\alpha = \frac{1}{2^\alpha \Gamma(\alpha+1)}$

The importance of the Bessel–Wright Fourier transform lies in the fact that it generalizes many integral transforms. In fact, given $\alpha > 1$, the Bessel–Wright Fourier transform reduces essentially to those in [15]:

- The Hankel transform when $\beta = 0$.
- The $\mathcal{Y}$ transform when $\beta = \frac{1}{2}$.
- The Hartley transform when $\beta = \frac{-1}{4}$.

We denote by:

- $D(\mathbb{R})$ the space of test functions with bounded support over $\mathbb{R}$.
- $\mathcal{C}_0$ the space of continuous functions on $[0, +\infty[$ having 0 as limit at infinity.

The Bessel–Wright function is related to the Bessel function via the intertwining operator (Riemann Liouville operator)

$$R_\alpha f(x) = 2\alpha \int_0^1 f(xt) (1-t^2)^{\alpha-1} t dt.$$

Its dual is given by

$$R_{\alpha, \beta}^* f(x) = 2\alpha \int_1^\infty f(xt) (t^2-1)^{\alpha-1} t^{2(\beta-\alpha)+1} dt.$$

Proposition 1.1 *The Bessel–Wright transform is related to the Bessel–Fourier transform via*

$$\mathcal{F}_{(\alpha, \beta)} = \mathcal{F}_\alpha \circ R_{(\alpha, \beta)}^*,$$

where

$$\mathcal{F}_\alpha(f)(\lambda) = c_\alpha \int_0^\infty f(x) j_\alpha(\lambda x) x^{2\alpha+1} dx.$$

The following two definitions are needed for our results.

Definition 1.2 The Mellin-type convolution product of first kind is defined by:

$$f \times g(y) = \int_0^\infty f(yx^{-1}) x^{-1} g(x) dx. \quad (1.3)$$

Definition 1.3 Let $\alpha > -\frac{1}{2}$ and $f, g \in L^1(0, \infty)$. Then we define the product $\otimes$ of f and g by the integral

$$f \otimes g(y) = \int_0^\infty f(yt)g(t)d\mu_\alpha(t). \quad (1.4)$$

Using (1.3) and (1.4), we get the following proposition:

Proposition 1.4 Let f, g , and h be integrable functions in $L^1(0, \infty)$ and let $y > 0$. Then

$$f \otimes (g \times h)(y) = (f \otimes g) \otimes h(y).$$

Proposition 1.5 The Bessel–Wright transform $\mathcal{F}_{(\alpha, \beta)}$ is a bounded linear operator from L_α^1 to $\mathcal{C}_0$.

Proof See ([6], Theorem 3) □

2 Generated spaces of Boehmians

The class of Boehmians was introduced to generalize regular operators [5]. The minimal structure necessary for the abstract construction of Boehmian spaces consists of the following elements:

- i. A topological vector space $\mathfrak{a}$
- ii. A commutative semigroup $(\mathfrak{b}, \bullet)$
- iii. An operation $\star : \mathfrak{a} \times \mathfrak{b} \longrightarrow \mathfrak{a}$ such that, for each $x \in \mathfrak{a}$ and $s_1, s_2 \in \mathfrak{b}$,

$$x \star (s_1 \bullet s_2) = (x \star s_1) \star s_2.$$

- iv. A collection $\Delta \subset \mathfrak{b}^\mathbb{N}$ such that:
 - a. if $x, y \in \mathfrak{a}$, $(s_n) \in \Delta$, $x \bullet s_n = y \bullet s_n$ for all n , then $x = y$;
 - b. if $(s_n), (t_n) \in \Delta$, then $(s_n \bullet t_n) \in \Delta$.

The elements of Δ are called delta sequences. Denote by $\mathcal{Q}$ the set

$$\mathcal{Q} = \{(x_n, s_n) : x_n \in \mathfrak{a}, (s_n) \in \Delta, x_n \star s_m = x_m \star s_n \forall m, n \in \mathbb{N}\}.$$

If $(x_n, s_n), (y_n, t_n) \in \mathcal{Q}$, $x_n \star t_m = y_m \star s_n \forall m, n \in \mathbb{N}$, then we say that $(x_n, s_n) \sim (y_n, t_n)$. The relation $\sim$ is an equivalence relation in $\mathcal{Q}$. The space of equivalence classes in $\mathcal{Q}$ is denoted by $\mathfrak{B}$. The elements of $\mathfrak{B}$ are called Boehmians.

Between $\mathfrak{a}$ and $\mathfrak{B}$, there is a canonical embedding expressed as

$$x \rightarrow \frac{x \star s_n}{s_n}.$$

The relationship between the notion of convergence and the product $\star$ is specified as follows:

- i. if $f_n \rightarrow f$ as $n \rightarrow \infty$ in $\mathfrak{a}$ and $\phi \in \mathfrak{b}$ is any fixed element, then $f_n \star \phi \rightarrow f \star \phi$ in $\mathfrak{a}$ (as $n \rightarrow \infty$);
- ii. $f_n \rightarrow f$ as $n \rightarrow \infty$ in $\mathfrak{a}$ and $(\delta_n) \in \Delta$, then $f_n \star \delta_n \rightarrow f$ in $\mathfrak{a}$ (as $n \rightarrow \infty$).

The operation $\star$ is extended to $\mathfrak{B} \times \mathfrak{b}$ as follows:

$$\text{If } \left[\frac{(f_n)}{(s_n)} \right] \in \mathfrak{B} \text{ and } \phi \in \mathfrak{b}, \text{ then } \left[\frac{(f_n)}{(s_n)} \right] \star \phi = \left[\frac{(f_n) \star \phi}{s_n} \right].$$

The convergence in $\mathfrak{B}$ is defined as follows:

A sequence (h_n) in $\mathfrak{B}$ is said to be δ convergent to h in $\mathfrak{B}$, $h_n \xrightarrow{\delta} h$, if there is a sequence $(s_n) \in \Delta$ such that $(h_n \star s_n), (h \star s_n) \in \mathfrak{a} \forall k, n \in \mathbb{N}$ and $(h_n \star s_k) \rightarrow (h \star s_k)$ as $n \rightarrow \infty$, in $\mathfrak{a}$ for every $k \in \mathbb{N}$.

A sequence (h_n) in $\mathfrak{B}$ is said to be Δ convergent to h in $\mathfrak{B}$, $h_n \xrightarrow{\Delta} h$, if there is a sequence $(s_n) \in \Delta$ such that $(h_n - h) \star s_n \in \mathfrak{a} \forall n \in \mathbb{N}$ and $(h_n - h) \star s_n \rightarrow 0$ as $n \rightarrow \infty$, in $\mathfrak{a}$.

Several integral transforms were extended to various spaces of Boehmians by numerous authors, we refer to Karunakaran and Roopkumar [7], Karunakaran and Vembu [8], Mikusinski [11], Al-Omari [1–3], Al-Omari and Kilicman [4], Nemzer [13], among others. For a general construction of Boehmians, we refer to [9, 10, 12].

We establish the following technical result.

Lemma 2.1 Let $f \in L^1_\alpha(0, \infty)$ and $\psi \in D(0, \infty)$. Then,

$$\mathcal{F}_{(\alpha, \beta)}(f \times \psi)(\lambda) = (\mathcal{F}_{(\alpha, \beta)} f \otimes \psi)(\lambda).$$

Proof Assume that the hypothesis of the theorem is satisfied. Now, on the basis of (1.2) and (1.3), we can write

$$\mathcal{F}_{(\alpha, \beta)}(f \times \psi)(\lambda) = c_\alpha \int_0^\infty \int_0^\infty f(xt^{-1})t^{-1}\psi(t)dt j_{(\alpha, \beta)}(\lambda x) d\mu_\alpha(x).$$

By Fubini's theorem, this can be rewritten as

$$\mathcal{F}_{(\alpha, \beta)}(f \times \psi)(\lambda) = \int_0^\infty \psi(t) \int_0^\infty c_\alpha f(xt^{-1})t^{-1} j_{(\alpha, \beta)}(\lambda x) d\mu_\alpha(x) dt.$$

Changing the variables $z = xt^{-1}$, we get

$$\begin{aligned} \mathcal{F}_{(\alpha, \beta)}(f \times \psi)(\lambda) &= \int_0^\infty \psi(t) \int_0^\infty c_\alpha f(z)t^{-1} dt j_{(\alpha, \beta)}(\lambda tz) t^{2\alpha+1} z^{2\alpha+1} t dz dt, \\ &= \int_0^\infty \psi(t) \left[c_\alpha \int_0^\infty f(z) j_{(\alpha, \beta)}(\lambda tz) d\mu_\alpha(z) \right] d\mu_\alpha(t), \\ &= \int_0^\infty \mathcal{F}_{(\alpha, \beta)}(f)(\lambda t) \psi(t) d\mu_\alpha(t), \\ &= (\mathcal{F}_{(\alpha, \beta)} f \otimes \psi)(\lambda). \end{aligned}$$

□

The spaces generated here are the space $\mathfrak{B}_1 = \mathfrak{B}_1(L^1_\alpha(k, \times), \times, \Delta)$ and the space $\mathfrak{B}_2 = \mathfrak{B}_2(L^1_\alpha(k, \times), \otimes, \Delta)$. We denote by Δ , the set of delta sequences (δ_n) of $D(0, \infty)$ with the following properties:

$$\int_0^\infty \delta_n(x) dx = 1, \quad (2.1)$$

$$\int_0^\infty |\delta_n(x)| dx < m, \quad (2.2)$$

where m is a positive real number

$$\text{supp } \delta_n(x) \rightarrow 1, \text{ as } n \rightarrow \infty. \quad (2.3)$$

The general properties of $\times$ are given (see [14, 16]) as follows:

$$\begin{aligned} f \times g &= g \times f, \\ f \times (g + h) &= f \times g + f \times h, \\ f \times (g \times h) &= (f \times g) \times h, \\ (\alpha g) \times g &= \alpha(f \times g) = f \times (\alpha g), \quad \alpha \in \mathbb{C}. \end{aligned} \quad (2.4)$$

Let us now establish that $\mathfrak{B}_1$ is a Boehmian space. We prefer to omit the proof for $\mathfrak{B}_2$ as its details are similar.

Theorem 2.2 Let $f \in L^1_\alpha(0, \infty)$, $\psi \in D(0, \infty)$ and $\alpha > -\frac{1}{2}$. Then $f \times \psi \in L^1_\alpha(0, \infty)$

Proof Let $f \in L^1_\alpha(0, \infty)$ and $\psi \in D(0, \infty)$. Assume that $K = [a, b]$, $0 < a < b$, is a compact subset of $(0, \infty)$ such that $\text{supp } \psi \subseteq K$. Then, we obtain

$$\int_0^\infty |f \times \psi(y)| d\mu_\alpha(y) = \int_0^\infty \left| \int_0^\infty f(yt^{-1})t^{-1}\psi(t) dt \right| d\mu_\alpha(y) \leq \int_a^b |\psi(t)|t^{-1} \int_0^\infty |f(yt^{-1})| d\mu_\alpha(y) dt.$$

By the change of variables $z = yt^{-1}$, we get

$$\int_0^\infty |f \times \psi(y)| d\mu_\alpha(y) \leq \int_a^b |\psi(t)| t^{2\alpha} dt \int_0^\infty |f(z)| d\mu_\alpha(z).$$

Hence,

$$\|f \times \psi(y)\|_\alpha^1 \leq v \|f\|_\alpha^1, \quad (2.5)$$

where

$$v = \int_a^b |\psi(t)| t^{2\alpha} dt.$$

□

Theorem 2.3 Let $f \in L_\alpha^1(0, \infty)$ and $\psi_1, \psi_2 \in D(0, \infty)$, $\alpha > -\frac{1}{2}$. Then

- i. $f \times (\psi_1 + \psi_2) = f \times \psi_1 + f \times \psi_2$,
- ii. $f \times (\psi_1 \times \psi_2) = (f \times \psi_1) \times (\psi_2)$,
- iii. $(\lambda f) \times \psi_1 = \lambda(f \times \psi_1) = f \times (\lambda \psi_1)$, $\lambda \in \mathbb{C}$

Proof The proofs of identities (i) and (iii) follow from simple integral calculus. Identity (ii) directly follows from (2.4). □

Theorem 2.4 Let $f_n \rightarrow f \in L_\alpha^1(0, \infty)$ as $n \rightarrow \infty$ and let $\psi \in D(0, \infty)$, $\alpha > -\frac{1}{2}$. Then

$$f_n \times \psi \rightarrow f \times \psi \text{ as } n \rightarrow \infty$$

in $L_\alpha^1(0, \infty)$.

Proof This theorem can be proved by simple results of integration. Hence, details are omitted. □

Theorem 2.5 Let $f \in L_\alpha^1(0, \infty)$ and let $(\delta_n) \in \Delta$, $\alpha > -\frac{1}{2}$. Then

$$f \times \delta_n \rightarrow f \text{ as } n \rightarrow \infty$$

in $L_\alpha^1(0, \infty)$.

Proof Assume that $f \in L_\alpha^1(0, \infty)$ and $(\delta_n) \in \Delta$ are given. Since the space $D(0, \infty)$ is dense in $L_\alpha^1(0, \infty)$, we can find $\psi \in D(0, \infty)$ such that

$$\|f - \psi\|_\alpha^1 < \varepsilon \quad (2.6)$$

for $\varepsilon > 0$.

Moreover, by virtue of (2.5) and the fact that $(\delta_n) \in D(0, \infty)$, we obtain

$$\|(f - \psi) \times \delta_n\|_\alpha^1 \leq v \|f - \psi\|_\alpha^1$$

for a real number v .

Hence, inserting (2.6) in above equation, we get

$$\|(f - \psi) \times \delta_n\|_\alpha^1 \leq \varepsilon v. \quad (2.7)$$

Thus, we get

$$\begin{aligned} \|\psi \times \delta_n - \psi\|_\alpha^1 &= \int_0^\infty |(\psi \times \delta_n - \psi)(y)| d\mu_\alpha(y), \\ &= \int_0^\infty \left| \int_0^\infty \psi(yt^{-1}) t^{-1} \delta_n(t) dt - \psi(y) \int_0^\infty \delta_n(t) dt \right| d\mu_\alpha(y), \\ &\leq \int_0^\infty \int_0^\infty |\psi(yt^{-1}) t^{-1} - \psi(y)| |\delta_n(t)| dt d\mu_\alpha(y) \end{aligned} \quad (2.8)$$

Now, let $g_y(t) = \psi(yt^{-1})t^{-1}$. Then, $g_y(t)$ is a uniformly continuous function in $D(0, \infty)$. Therefore, we can find $\delta > 0$ such that

$$|g_y(t) - g(1)| < \varepsilon \text{ whenever } |y - 1| < \delta.$$

Thus, using (2.1) in (2.8), we conclude that

$$\|\psi \times \delta_n - \psi\|_\alpha^1 \leq \int_0^\infty \int_0^\infty |g_y(t) - g_y(1)| |\delta_n(t)| dt d\mu_\alpha(y) \leq \varepsilon \int_a^b d\mu_\alpha(y), \quad (2.9)$$

where $[a, b]$ is an interval containing the support of g_y .

Therefore, inequality (2.9) implies that

$$\|\psi \times \delta_n - \psi\|_\alpha^1 \leq A\varepsilon, \quad (2.10)$$

where

$$A = \int_a^b d\mu_\alpha(y).$$

From (2.6), (2.7) and (2.10), we get

$$\|f \times \delta_n - f\|_\alpha^1 \leq \|(f - \psi) \times \delta_n\|_\alpha^1 \|f - \psi\|_\alpha^1 \leq \varepsilon v + A\varepsilon + \varepsilon.$$

Hence, the equation presented above gives

$$\|f \times \delta_n - f\|_\alpha^1 \leq B\varepsilon,$$

where $B = v + A + 1$.

Hence the result. $\square$

We define the sum and multiplication by a scalar of two Boehmians in $\mathfrak{B}_1$ in the natural way as:

$$\left[\frac{(f_n)}{(\delta_n)} \right] + \left[\frac{(g_n)}{(\psi_n)} \right] = \left[\frac{(f_n) \times \psi_n + (g_n) \times (\delta_n)}{(\delta_n) \times (\psi_n)} \right]$$

and

$$\gamma \left[\frac{(f_n)}{(\delta_n)} \right] = \left[\gamma \frac{(f_n)}{(\delta_n)} \right],$$

where $\gamma \in \mathbb{C}$.

The operation $\times$ and the operation of differentiation are defined as follows:

$$\left[\frac{(f_n)}{(\delta_n)} \right] \times \left[\frac{(g_n)}{(\psi_n)} \right] = \left[\frac{(f_n) \times (g_n)}{(\delta_n) \times (\psi_n)} \right]$$

and

$$\mathcal{D}^\gamma \left[\frac{(f_n)}{(\delta_n)} \right] = \left[\frac{\mathcal{D}^\gamma(f_n)}{(\delta_n)} \right],$$

respectively.

A sequence of Boehmians (ζ_n) in $\mathfrak{B}_1$ is said to be δ convergent to a Boehmian ζ in $\mathfrak{B}_1$ denoted by $\zeta_n \xrightarrow{\delta} \zeta$, if there exists a delta sequence (δ_n) such that

$$(\zeta_n \times \delta_k), (\zeta \times \delta_k) \in L_\alpha^1 \quad \forall k, n \in \mathbb{N},$$

and

$$(\zeta_n \times \delta_k) \rightarrow (\zeta \times \delta_k) \text{ as } n \rightarrow \infty, \text{ in } L_\alpha^1, \quad \forall k \in \mathbb{N}.$$

The equivalent statement for the δ convergence has the following form:

$\zeta_n \xrightarrow{\delta} \zeta$ ($n \rightarrow \infty$) in $\mathfrak{B}_1$ if and only if there exist $(\varphi_{n,k}), (\varphi_k) \in L^1_\alpha$ and $(\delta_k) \in \Delta$ such that

$$\zeta_n = \left[\frac{(\varphi_{n,k})}{(\delta_k)} \right], \quad \zeta = \left[\frac{(\varphi_k)}{(\delta_k)} \right],$$

and

$$\varphi_{n,k} \rightarrow \varphi_k \text{ as } n \rightarrow \infty$$

in L^1_α for each $k \in \mathbb{N}$.

A sequence of Boehmians (ζ_n) in $\mathfrak{B}_1$ is said to be Δ convergent to a Boehmian ζ in $\mathfrak{B}_1$ denoted by $\zeta_n \xrightarrow{\Delta} \zeta$, if there exists a delta sequence $(\delta_n) \in \Delta$ such that $(\beta_n - \beta) \times \delta_n \in L^1_\alpha \forall n \in \mathbb{N}$ and $(\zeta_n - \zeta) \times \delta_n \rightarrow 0$ as $n \rightarrow \infty$ in L^1_α .

Similarly, the following theorems generate the Boehmian space $\mathfrak{B}_2$.

Theorem 2.6 Let $f \in L^1_\alpha(0, \infty)$ and $\psi \in D(0, \infty)$. Then $f \otimes \psi \in L^1_\alpha(0, \infty)$.

Theorem 2.7 Let $f \in L^1_\alpha(0, \infty)$ and $\psi_1, \psi_2 \in D(0, \infty)$. Then,

- i. $f \otimes (\psi_1 + \psi_2) = f \otimes \psi_1 + f \otimes \psi_2$,
- ii. $(\lambda f) \otimes \psi_1 = \lambda(f \otimes \psi_1) = f \otimes (\lambda \psi_1)$, $\lambda \in \mathbb{C}$.

Theorem 2.8 For $f \in L^1_\alpha(0, \infty)$ and $\psi_1, \psi_2 \in D(0, \infty)$, the following relation is true:

$$f \otimes (\psi_1 \times \psi_2) = (f \otimes \psi_1) \otimes \psi_2.$$

The proofs of Theorems 2.6 and 2.7 are similar to the proofs of Theorems 2.2 and 2.3, respectively. The proof of Theorem 2.8 follows from Proposition 1.4.

Theorem 2.9 i. Let $f_n \rightarrow f$ in $L^1_\alpha(0, \infty)$ as $n \rightarrow \infty$ and let $\psi \in D(0, \infty)$. Then $f_n \otimes \psi \rightarrow f \otimes \psi$ as $n \rightarrow \infty$.
 ii. Let $f_n \rightarrow f$ in $L^1_\alpha(0, \infty)$ and let $(\delta_n) \in \Delta$. Then $f_n \otimes \delta_n \rightarrow f$ as $n \rightarrow \infty$.

Proof The proof of Part (i) of the theorem can be obtained by simple techniques of integration, whereas proof of the second part is analogous to the proof of Theorem 2.4. $\square$

The sum of two Boehmians in $\mathfrak{B}_2$ and the operation of multiplication by a scalar can be also defined as follows:

$$\left[\frac{(f_n)}{(\delta_n)} \right] + \left[\frac{(g_n)}{(\varepsilon_n)} \right] = \left[\frac{(f_n) \otimes \varepsilon_n + (g_n) \otimes (\delta_n)}{(\delta_n) \times (\varepsilon_n)} \right]$$

and

$$\gamma \left[\frac{(f_n)}{(\delta_n)} \right] = \left[\gamma \frac{(f_n)}{(\delta_n)} \right] = \left[\frac{\gamma(f_n)}{(\delta_n)} \right],$$

where $\gamma \in \mathbb{C}$.

The operation $\otimes$ and the operation of differentiation are, respectively, defined as:

$$\left[\frac{(f_n)}{(\delta_n)} \right] \otimes \left[\frac{(g_n)}{(\varepsilon_n)} \right] = \left[\frac{(f_n) \otimes (g_n)}{(\delta_n) \times (\varepsilon_n)} \right]$$

and

$$\mathcal{D}^k \left[\frac{(f_n)}{(\delta_n)} \right] = \left[\frac{\mathcal{D}^k(f_n)}{(\delta_n)} \right]$$

The notions of δ -convergence and Δ -convergence in $\mathfrak{B}_1$ and $\mathfrak{B}_2$ can be defined in a natural way as above.

3 The Bessel–Wright transform of a Boehmian

Let $\zeta \in \mathfrak{B}_1$ and $\zeta = [\frac{(f_n)}{(\delta_n)}]$. Then, for every $\alpha > -\frac{1}{2}$ we define the generalized Bessel–Wright transform of ζ as follows:

$$\mathcal{F}_{\alpha,\beta}^{ge} \left(\left[\frac{(f_n)}{(\delta_n)} \right] \right) = \left[\frac{(\mathcal{F}_{(\alpha,\beta)} f_n)}{(\delta_n)} \right] \quad (3.1)$$

The right-hand side of (3.1) belongs to $\mathfrak{B}_2$ by virtue of Proposition 1.5. The definition presented above is indeed well defined. Let

$$\left[\frac{(f_n)}{(\omega_n)} \right] = \left[\frac{(g_n)}{(\varepsilon_n)} \right] \in \mathfrak{B}_1.$$

Then, by the notion of equivalence classes in $\mathfrak{B}_1$, we find

$$f_n \times \varepsilon_m = g_m \times \omega_n.$$

Relation (3.1) and the notion of equivalence classes in $\mathfrak{B}_2$ yield

$$\mathcal{F}_{(\alpha,\beta)} f_n \otimes \varepsilon_m = \mathcal{F}_{(\alpha,\beta)} g_m \otimes \omega_n.$$

Hence, we conclude that

$$\frac{(\mathcal{F}_{(\alpha,\beta)} f_n)}{(\omega_n)} \sim \frac{(\mathcal{F}_{(\alpha,\beta)} g_n)}{(\varepsilon_n)} \text{ in } \mathfrak{B}_2$$

Thus, we get

$$\left[\frac{(\mathcal{F}_{(\alpha,\beta)} f_n)}{(\omega_n)} \right] = \left[\frac{(\mathcal{F}_{(\alpha,\beta)} g_n)}{(\varepsilon_n)} \right]$$

This proves the claim.

Theorem 3.1 $\mathcal{F}_{\alpha,\beta}^{ge}$ is an isomorphism from $\mathfrak{B}_1$ into $\mathfrak{B}_2$.

Proof We first establish that $\mathcal{F}_{\alpha,\beta}^{ge}$ is injective. Given

$$\mathcal{F}_{\alpha,\beta}^{ge} \left(\left[\frac{(f_n)}{(\omega_n)} \right] \right) = \mathcal{F}_{\alpha,\beta}^{ge} \left(\left[\frac{(g_n)}{(\varepsilon_n)} \right] \right),$$

by virtue of Lemma 2.1 and the notion of equivalent classes of $\mathfrak{B}_2$, we conclude that

$$\mathcal{F}_{(\alpha,\beta)} f_n \otimes \varepsilon_m = \mathcal{F}_{(\alpha,\beta)} g_m \otimes \omega_n.$$

Therefore, Lemma 2.1 implies that $\mathcal{F}_{(\alpha,\beta)}(f_n \times \varepsilon_m) = \mathcal{F}_{(\alpha,\beta)}(g_m \times \omega_n)$. Employing $\mathcal{F}_{(\alpha,\beta)}$ gives

$$f_n \times \varepsilon_m = g_m \times \omega_n.$$

On the other hand, the notion of equivalent classes in $\mathfrak{B}_1$ yields

$$\left[\frac{(f_n)}{(\omega_n)} \right] = \left[\frac{(g_n)}{(\varepsilon_n)} \right] \in \mathfrak{B}_1.$$

We now show that $\mathcal{F}_{\alpha,\beta}^{ge}$ is a surjective mapping. Let $\mathcal{F}_{(\alpha,\beta)}$ is a surjective mapping. Let

$$\left[\frac{(\mathcal{F}_{(\alpha,\beta)} f_n)}{(\omega_n)} \right] \in \mathfrak{B}_2$$

be arbitrary. Then

$$\mathcal{F}_{(\alpha,\beta)} f_n \otimes \omega_m = \mathcal{F}_{(\alpha,\beta)} f_m \otimes \omega_n$$

for any choice of $m, n \in \mathbb{N}$. Hence, for every $m, n \in \mathbb{N}$, $f_n, f_m \in L^1_\alpha(0, \infty)$, satisfy the relation

$$\mathcal{F}_{(\alpha, \beta)}(f_n \times \omega_m) = \mathcal{F}_{(\alpha, \beta)}(f_m \times \omega_n).$$

This means that

$$\left[\frac{(f_n)}{(\omega_n)} \right] \in \mathfrak{B}_1$$

is such that

$$\mathcal{F}_{\alpha, \beta}^{ge} \left(\left[\frac{(f_n)}{(\omega_n)} \right] \right) = \left[\frac{(\mathcal{F}_{\alpha, \beta}^{ge} f_n)}{(\omega_n)} \right]$$

□

In addition, we now deduce the formula of extension of $\times$ to $\mathfrak{B}_1$ as follows:

$$\mathcal{F}_{\alpha, \beta}^{ge} \left(\left[\frac{(f_n)}{(\omega_n)} \right] \times \phi \right) = \mathcal{F}_{\alpha, \beta}^{ge} \left(\left[\frac{(f_n)}{(\omega_n)} \right] \right) \otimes \phi.$$

It can be proved as follows: By virtue of (3.1), we can write

$$\mathcal{F}_{\alpha, \beta}^{ge} \left(\left[\frac{(f_n)}{(\omega_n)} \right] \times \phi \right) = \left(\left[\frac{(\mathcal{F}_{\alpha, \beta}^{ge} (f_n \times \phi))}{(\omega_n)} \right] \right).$$

Hence, Lemma 2.1 gives

$$\mathcal{F}_{\alpha, \beta}^{ge} \left(\left[\frac{(f_n)}{(\omega_n)} \right] \times \phi \right) = \left(\left[\frac{(\mathcal{F}_{\alpha, \beta}^{ge} f_n \otimes \phi)}{(\omega_n)} \right] \right).$$

The definition of the product $\times$ implies that

$$\mathcal{F}_{\alpha, \beta}^{ge} \left(\left[\frac{(f_n)}{(\omega_n)} \right] \times \phi \right) = \left(\left[\frac{(\mathcal{F}_{\alpha, \beta}^{ge} f_n)}{(\omega_n)} \right] \right) \times \phi.$$

Thus, it follows from relation (3.1) that

$$\mathcal{F}_{\alpha, \beta}^{ge} \left(\left[\frac{(f_n)}{(\omega_n)} \right] \times \phi \right) = \mathcal{F}_{\alpha, \beta}^{ge} \left(\left[\frac{(f_n)}{(\omega_n)} \right] \right) \otimes \phi.$$

Hence, it is now possible to conclude that

$$\mathcal{F}_{\alpha, \beta}^{ge} \left(\left[\frac{(f_n)}{(\omega_n)} \right] \times \phi \right) = \mathcal{F}_{\alpha, \beta}^{ge} \left(\left[\frac{(f_n)}{(\omega_n)} \right] \right) \otimes \phi.$$

Theorem 3.2 $\mathcal{F}_{\alpha, \beta}^{ge} : \mathfrak{B}_1 \rightarrow \mathfrak{B}_2$ is continuous with respect to the δ -convergence and Δ -convergence.

Proof This theorem can be proved using a technique similar to that presented in [1, 2].

□

Acknowledgements The authors would like to thank the referees for their valuable comments and suggestions.

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